Brane conservation approach to dimensional reduction and low-energy limit of M theory

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Abstract

We propose the approach to deriving lower-dimensional limit of modern high-energy theory which does not make explicit use of the Kaluza-Klein scheme and predefined compactification manifolds. The approach is based on the selection principle in which a crucial role is played by p-brane solutions and their preservation, in a certain sense, under dimensional reduction. Then we engage a previously developed method of reconstruction of a theory from a given solution which eventually leads to some model acting in the space of field couplings. Thus, our approach focuses on those general features of effective 4D theories which are independent of how the decomposition of spacetime dimensions into "observable" and "unobservable" ones could be done. As an example, we exactly derive the simplified Abelian sector of the effective low-energy M-theory together with its fundamental 0-brane solution describing the family of charged black holes with scalar hair in asymptotically flat, de Sitter or anti-de Sitter spacetime.
As another example, we derive the general form of the cosmological scalar field potential which is compatible both with the existence of black holes and p-branes related to string/M theory and with multidimensional inflationary cosmology. It is shown that the scalar potential alters non-trivially from dimension to dimension yet always obeys one single equation where the number of spacetime dimensions is a free parameter. Using this equation we formulate an eigenvalue problem for the dimensionality parameter. It turns out that in the low-energy regime of sub-Planckian values of the inflaton field, i.e., when the Universe has cooled and expanded sufficiently enough, the value four arises as the largest admissible (eigen)value of this parameter. 11.25.Mj, 04.70.Bw, 11.25.Yb

1. Introduction

Nowadays it is believed that the most promising candidate for a unified theory of interactions is no longer ten-dimensional (10D) string theory but rather 11D M-theory. Its explicit formulation is still pending but one knows some low-energy limits, the 11D supergravity (SUGRA) and 10D superstring theories. Yet, the M-theory remains so far an abstract theory whose predictions can not be experimentally tested in foreseen future. The main reason is that one does not know yet how the low-energy 4D limit of M-theory should look like: in the standard approach to dimensional reduction, the Kaluza-Klein (KK) compactification scheme, the ambiguity arises due to huge variety of possible manifolds to compactify on to get down to 4D [1]. This ambiguity is not surprising though - in its recent form, the KK approach is more a mathematical procedure than a physical law and thus need not to be unique.

Among other issues, the KK ambiguity leads to the well-known "cosmological constant problem" - why this important parameter has a non-zero value (despite being exactly zero on the supermoduli space), and how to determine this value. In string/M-theory the problem arises due to the non-uniqueness of the corresponding 4-form field strengthes [2]: consider a compactification of M-theory from 11D to 4D using, e.g., a 7-torus. The torus has a number of moduli representing the sizes and angles between seven 1-cycles. The 4-form fields have as their origin a fundamental 7-form field strength of M-theory. These 7-forms can be chosen such that three of their indices are identified with compact dimensions, and this can be done in $7!/(3!\,4!) = 35$ ways which produce that many distinct 4-form fields in the uncompactified space. More generally, in the kinds of compact manifolds used to reproduce the Standard Model there can be many independent ways of wrapping three compact directions with flux and thus producing a great number of four-dimensional 4-form fields.

To illustrate how all aforesaid affects four-dimensional physics, let us consider gravity $g_{\mu\nu}$ coupled to electromagnetic $\mathcal{A}_\mu$ and scalar ϕ fields given by the action (in the Einstein frame):

$$S = \int \mathrm{d}^4 x \sqrt{-g}\left[R - \frac{1}{2}(\partial\phi)^2 + \Xi(\phi)\, F^2 + \Lambda(\phi)\right], \tag{1}$$

where $F_{\mu\nu} = \nabla_\mu \mathcal{A}_\nu - \nabla_\nu \mathcal{A}_\mu$; we use the units where $16\pi G = c = 1$, where G is the Newton gravitational constant. The particle content of this theory (graviton, photon and scalar) is capable of describing the bosonic part of many physical phenomena, from electromagnetism to cosmology. It is thus not surprising that (29) appears in the abelian sector of the effective 4D low-energy limit of M-theory (EFMT). The open question here is what are explicit values of the Maxwell-dilaton coupling Ξ and the scalar field self-coupling Λ [3]. In the tree-level approximation (spherical topology of intermediate worldsheets) string theory suggests that

$$\Xi^{(0)} = -e^{-a\phi}, \quad \Lambda^{(0)} = 0, \tag{2}$$

at the leading order in the inverse string tension [4, 5]. Obviously, a model with such couplings is not phenomenologically satisfactory - one can not describe, for instance, cosmological phenomena. The realistic $\Xi(\phi)$ and $\Lambda(\phi)$ are different from these for at least two reasons. First, the EFMT being an effective 4D theory is supposed to hold information about higher dimensions and fields, therefore, in the case of the model (29) this information must be stored in Ξ and Λ thus affecting their form. Second, the couplings will get modified also by quantum corrections to the initial 10, 11D Lagrangians, see, for example, Refs. [6, 7]. In short, to construct the genuine EFMT, ideally one should perform two steps: (i) construct the ultimate M-theory, and then (ii) dimensionally reduce it down to 4D. Both steps are obviously of enormous difficulty.

On the other hand, vast experimental data gave so far no evidence for higher-dimensional physics to a high degree of precision. It is thus a good question how the Nature "chooses" that very unique way of dimensional reduction which determines our essentially 4D Universe. Is there any selection principle (apart from anthropic arguments) by which one could make the unique choice?

2. Branes and selection principle

Here we try to formulate one. Suppose, an observer wants to describe a certain physical phenomenon or object (black hole, for example). According to an M-theorist the complete description of the object is given by a high-D solution, call it $Sol(M)$, of a certain high-D theory M. However, our observer can experimentally operate only in four dimensions - his apparatus (including organs of sense) always gives essentially (3+1)-dimensional output, hence, he never perceives extra dimensions directly but only as additional "forces" in the 4D framework. The main aim of EFMT thus is to incorporate these corrections while staying compatible with the four-dimensionality of experimental data.

In other words, one needs the dimensionally reduced description of the object, i.e., the one in terms of the most appropriate solution $Sol(M_4)$ arising from a corresponding effective 4D theory M_4. The question is how to derive such EFMT M_4, especially assuming the situation like the one we have recently, i.e., when M is not explicitly known? Our solution to the problem would be to use the known fundamental solutions of high-D theories as a guiding thread. Indeed, real phenomena are described by $Sol(M)$'s thus if one wishes to preserve (partially, at least) such description in the dimensionally reduced theory one must make sure that the latter does not disallow $Sol(M)$-like solutions in principle.

Considering aforesaid, we formulate the following selection criterion: "Suppose we have a higher-dimensional theory, call it M. This theory has a certain physically relevant solution, $Sol(M)$ which is unique to M. This solution has certain distinctive property, $Prop(Sol(M))$, which is preserved under reduction of dimensions. On the other hand, we have 4D theory M_4 whose nontrivial solutions we denote by $Sol(M_4)$. Then the necessary condition for M_4 to be a lower-dimensional limit of M is that at least one of $Sol(M_4)$ must have $Prop(Sol(M))$." This criterion can be regarded as the correspondence principle for solutions, henceforth we call it the *Solution Correspondence Principle* (SCP). Now, to determine what solution is physically relevant and which property of it is suitable for SCP, we will make use of the following two facts from the string/M-theory.

First, it is well-known that the branes are known solutions of M theory (one may recall M2 and M5 ones) hence they should be perfectly suitable for a concrete realization of SCP. Moreover, it is known that branes are inevitable for proper describing of black holes (microstates, entropy, etc.) [8], therefore, their absence in a theory would cause serious difficulties with consistent explaining such issues. In turn, absence of black holes would lead to the loss of protection of a theory from ubiquitous appearances of naked singularities [9] and thus to undesirable violations of the Cosmic Censorship principle [10, 11, 12]. In what follows we concentrate solely on the most primitive type of branes, the p-branes, as they are minimal-energy configurations and can be viewed as a kind of "attractor points" in the evolution space of all brane solutions. The metric of a generic p-brane can be schematically described as

$$\mathbf{g}_D = e^A \, \breve{\mathbf{g}}_{\tilde{d}} + \text{ the rest}, \tag{3}$$

where $\breve{\mathbf{g}}_{\tilde{d}}$ is the metric of the $(D-p-2)$-dimensional transverse space, $e^A \equiv R^2_{\text{trans}}$ is the warp factor or effective squared radius of transverse dimensions; "the rest" is usually the $(p+1)$-dimensional Poincaré-invariant metric plus $g_{rr}(r)\mathrm{d}r^2$ with r being the isotropic radial coordinate in the transverse space. The brane solution also has a scalar field (dilaton) part which is represented simply by $\phi = \phi(r)$, and a gauge field part which is of no interest to us here.

Second, we know that the dilaton effectively represents the dynamically varying string coupling constant; from the viewpoint of the 11D M-theory it is related to the size of the 11th dimension. Note that the dilaton is defined as the scalar non-minimally coupled to gravity. Then if one assumes that the spacetime is flat (as in the Standard Model), one can not have the non-minimal term in the SM Lagrangian anyway. What is left is the kinetic and potential terms for the "dilaton", therefore, the dilaton, even if it existed in 10D supergravity, in the flat-spacetime limit becomes a normal (minimally coupled) scalar field (e.g., a long-range background field such as the inflaton). Moreover, in the low-energy limit, even if spacetime curvature does not vanish, one can always conformally rescale the metric such that inside the action any ϕ-dependent factor in front of the Ricci scalar can be absorbed into metric. Of course, it changes the frame we are working in from the string to the Einstein one but here it is not so important as long as phenomenological matter is not involved.

Thus, we have two fundamental quantities: the one of physical origin, ϕ, and the one related to multi-dimensional geometry, e^A. Consequently, the relation $\phi = \phi(R_{\text{trans}}) = \phi(e^A)$ is a scalar one and contains certain information about the theory, for example, about how branes' properties would change at different size of transverse dimensions. Thus, such relation must be a universal property of brane solutions - in particular, it is unlikely that the reducing of dimensions will drastically change it. Therefore, we expect that the function $\phi(e^A)$, or, alternatively, the inverse $A(\phi)$, is *structurally invariant*, i.e., has the same functional dependence (except perhaps for different constant parameters) for any fundamental p-brane and at any physically admissible D. Matching notations with SCP, we associate branes with $Sol(M)$ and $A(\phi)$ with $Prop(Sol(M))$, thus one can restate SCP as: "*the genuine EFMT should necessarily possess at least one solution of the p-brane type*".

To check the structural invariance of $A(\phi)$ and at same time to derive its explicit form, consider the typical theory giving rise to brane solutions, truncated supergravity. Its abelian sector is described by the Lagrangian

$$\mathcal{L} \sim R - \frac{1}{2}(\partial\phi)^2 - (1/2n!)e^{a\phi}F_{[n]}^2, \tag{4}$$

where $F_{[n]}$ is the antisymmetric tensor of rank n [13, 14, 15]. The corresponding p-brane solution respects the $(\text{Poincaré})_d \times \mathrm{SO}(D-d)$ symmetry and is given by (omitting the gauge field part):

$$ds^2 = H^{\frac{-4\tilde{d}}{\Delta(D-2)}} dx^\mu dx^\nu \eta_{\mu\nu} + H^{\frac{4d}{\Delta(D-2)}} dy^m dy^m, \tag{5}$$

$$e^\phi = H^{\frac{2a}{\varsigma\Delta}}, \qquad H \equiv 1 + \frac{\kappa}{r^{\tilde{d}}}, \qquad {}^{\mu=0,1,..,p,}_{m=p+1,..,D-1}\ , \tag{6}$$

where $r = \sqrt{y^m y^m}$, $a^2 = \Delta - \frac{2d\tilde{d}}{D-2}$, $d = p+1$, $\tilde{d} = D - d - 2$, $\varsigma = \pm 1$ for the electric and magnetic branes, respectively; κ is the integration constant which sets the brane's mass scale (its positiveness ensures the absence of naked singularities at finite r). Using $\mathrm{SO}(D-d)$ symmetry, one can rewrite the r.h.s. of Eq. (5) as

$$H^{\frac{-4\tilde{d}}{\Delta(D-2)}} dx^\mu dx^\nu \eta_{\mu\nu} + H^{\frac{4d}{\Delta(D-2)}} dr^2 + r^2 H^{\frac{4d}{\Delta(D-2)}} d\Omega^2_{(\tilde{d}-1)}.$$

It is the last term we are interested in - comparing it to the first term on r.h.s. of Eq. (3) we can identify e^A with $r^2 H^{\frac{4d}{\Delta(D-2)}}$. Eliminating r using Eq. (6) we finally obtain

$$A(\phi) = \frac{2\,d\,\varsigma}{a\,(D-2)}\phi + \frac{2}{\tilde{d}}\ln\kappa - \frac{2}{\tilde{d}}\ln(e^{\frac{\Delta\,\varsigma\,\phi}{2\,a}} - 1), \tag{7}$$

i.e., $A(\phi)$ has the form $c_1\phi + c_2 + c_3 \ln(\mathrm{e}^{c_4\phi} - 1)$, where c_i are certain D-dependent constants, and one can check that such relation holds for any physically admissible (such as those free of naked singularities, etc.) p-brane solution known so far. Thus, $A(\phi)$ has the same structure for any D, the only things to change are the values of c_i's.

Now, according to SCP the necessary condition for the theory to be a genuine EFMT is that at least one of its solutions must have the property (7). If as a skeleton model we take Eq. (29) then our task now is to determine unknown couplings $\Xi(\phi)$ and $\Lambda(\phi)$. Therefore, we demand that its simplest fundamental (static and spherically symmetric) solutions have the property (7) evaluated at $D = 4$. In 4D the only choice is $p = 0$, $d = \tilde{d} = 1$ (0-brane) so Eq. (7) takes the form

$$A(\phi)_{D=4} = (a\varsigma)^{-1}\phi + 2\ln\kappa - 2\ln(e^{\frac{a^2-1}{2a\varsigma}\phi} - 1). \tag{8}$$

The next step thus would be to find a class of 4D theories whose solutions resemble the higher-dimensional p-branes in a sense of having the property expressed by the last formula. One should not confuse this with demanding that the whole higher-dimensional brane's structure (isometries, etc.) should be preserved in 4D - it is unlikely possible in general case. Fortunately, from the viewpoint of SCP such demand would be too strong and redundant (notice that SCP has been defined as a necessary but not sufficient condition). Instead, we accentuate on deriving an effective 4D theory whose solutions mimic the brane in just one property (8) but the theory must be as general as possible (within the initial restrictions made upon its field content, of course). The existence of at least one such solution will be proven subsequently by construction - in fact, in 4D the spherical symmetry and existence of time-like Killing vector guarantee the appearance of the solutions of the form (3) as the latter becomes just the (isotropic) gauge condition for spacetime metric. Incidentally, because of the latter circumstance, $D = 4$ becomes a distinguished number in this sense.

3. Brane class and couplings

Now, how to reconstruct the theory whose solution (one, at least) would possess the property given by Eq. (8)? The related question is how large would the corresponding class of equivalence be, i.e., how many theories exist which have at least one such solution (this is especially crucial for uniqueness of the EFMT defined in such a way)?

More technically: we have the theory defined by Eq. (4) for arbitrary $D > 2$ which has the generic p-brane solutions with the property (7). However, this property can be more "generic" than the theory itself, in a sense that there may exist Lagrangians more general than (4) such that their appropriate solutions (one, at least) still obey Eq. (7). How to reconstruct the bijective preimage (kernel of the homomorphism), i.e., the *most general* form of such Lagrangians assuming $D = 4$ and restricting oneself to a particular skeleton action, e.g., Eq. (29) with the couplings Ξ and Λ to be determined?

Luckily, we have quite general theory addressing all these issues. In Ref. [16] it has been developed for the abelian theory (29) but definitely the idea should work in more general case as well. It surprisingly turns out that the ambiguity is not big at all: the unknowns $\Xi(\phi)$ and $\Lambda(\phi)$ can be determined uniquely up to two constants each (apart from the freedom to select a frame by conformal rescaling of metric). To see this, consider the field equations following from action (29) in the static spherically symmetric limit (where the metric takes the form (3)). Using them, one can derive the following equation [16]:

$$\frac{\Upsilon'}{A'} + \frac{1}{2}\left(\frac{1}{A'^2} + 1\right)\Upsilon + \frac{e^A}{2}\left(\Lambda + e^{-2A}\hat{\Xi}\right) = -1\,, \tag{9}$$

$$\Upsilon \equiv \frac{1 + e^A(\Lambda + \Lambda' A')/2 + e^{-A}(\hat{\Xi} + \hat{\Xi}' A')/2}{(1/A')'}\,,$$

$$\hat{\Xi} \equiv 2\,(Q^2\,\Xi^{-1} + P^2\,\Xi)\,,$$

where constants Q and P stand for electric and magnetic charges, respectively; throughout the section prime stands for derivative with respect to ϕ. Now, suppose we have fixed $A(\phi)$, then the equation turns into a joint linear second-order ODE with unknown functions $\Lambda(\phi)$ and $\hat{\Xi}(\phi)$. This ODE (called the *class equation*) is, in fact, the constraint for Λ and Ξ which ensures the internal consistency of the theory. Therefore, with every given $A(\phi)$ one can associate the appropriate class of solvability given by a self-consistent $\{\Lambda,\,\Xi\}$ pair. Thus, the space of all possible coupling functions becomes "inhomogeneous" as it can be divided according to the class structure. Henceforth we call the class defined by Eq. (7) the *p-brane class of equivalence*. Theories belonging to this class are equivalent in a sense that their appropriate solutions possess the same dependence $A(\phi)$. For $D = 4$, substituting Eq. (8) into Eq. (9), we obtain:

$$\hat{\Xi}'' + \varsigma\left(a - \frac{1}{a}\right)\hat{\Xi}' - \hat{\Xi} + \left(\frac{\kappa e^{\frac{\phi}{2a\varsigma}}}{e^{\frac{(1+a^2)\phi}{2a}} - 1}\right)^4 \times$$
$$\times\left\{\Lambda'' - \left(a + \frac{1}{a}\right)\coth\left[\frac{(a^2+1)\phi}{4a}\right]\Lambda' + \Lambda\right\} = 0. \tag{10}$$

It contains κ which came from Eqs. (5), (6) where it was a mass scale parameter. Such constants are attributes of solutions and thus should not explicitly appear in the action, in the same manner as, for example, mass is not built into the couplings in a field Lagrangian but appears only inside, say, the Reissner-Nördstrom solution, i.e., as an integration constant. The only way to get rid of κ is to separate the equations:

$$\hat{\Xi}'' + \varsigma\left(a - \frac{1}{a}\right)\hat{\Xi}' - \hat{\Xi} = 0, \tag{11}$$

$$\Lambda'' - \left(a + \frac{1}{a}\right)\coth\left[\frac{(a^2+1)\phi}{4a}\right]\Lambda' + \Lambda = 0, \tag{12}$$

i.e., the physical condition of independence of the skeleton action from the mass scale parameter κ naturally implies that the total coupling space (defined "on top" of the space of fields) factorizes into the direct product of "elementary" coupling spaces: $\{\Lambda,\,\hat{\Xi}\} \longrightarrow \{\Lambda\}\otimes\{\hat{\Xi}\}$. The coupling functions can be easily found by solving these ODE's. For arbitrary a we have:

$$\hat{\Xi}(\phi) = \frac{1}{2}\left(\sigma_1 e^{\phi/a} + \sigma_2\,e^{-a\phi}\right), \tag{13}$$

$$\Lambda(\phi) = \sinh^2(\phi/\phi_1)[C_1 P_2^\mu(z) + C_2 Q_2^\mu(z)], \tag{14}$$

where $P_\nu^\mu(z)$ and $Q_\nu^\mu(z)$ are Legendre functions [35],

$$z = \coth(\phi/\phi_1), \qquad \phi_1 \equiv a(2-\mu) = \frac{4a}{a^2+1}, \tag{15}$$

$$\mu \equiv 2(a^2-1)/(a^2+1), \tag{16}$$

and σ's and C's are integration constants.

Before going further, let us emphasize the generic features of the constructed model, i.e., the one given by Eq. (29) with $\Lambda(\phi)$ and $\Xi(\phi)$ specified above. The first thing to notice is that we actually did not use any specific high-D field action to derive Eqs. (30), (31) - their derivation was based solely on Eq. (7). Thus, what we have derived is the *maximally general* 4D theory of type (29) which can have in principle the brane-like solutions. According to SCP, the corresponding sector of the genuine EFMT must be a subset of this theory: the principle states that the necessary condition to be EFMT is to belong to the p-brane class.

One may wonder, however, whether this brane class is unique: what if there exists other class whose $\phi(e^A)$ is different from Eq. (7) thus leading to the equations different from Eqs. (30), (31) and, therefore, resulting in different coupling functions? The answer is: from what we said after Eq. (7) it follows that the class (7) *is* universal (in fact, we have checked that all known p-brane solutions related to M-theory do belong to this class), but even if this happens to be not true - ambiguity will not increase but, quite the contrary, it will decrease further more. The reason is that if an additional class does exist then the EFMT should lie on the intersection of the new class and the old one (7), therefore, the number of restricting equations would increase. In principle, the only way to correct the theory (apart from introducing new fields into Eq. (29) which is a separate task) would be to generalize it by finding the class characteristic function $A(\phi)$ more general than Eq. (7). However, for that to happen new types of p-branes should be discovered because the known ones do not suggest any necessity for such generalization.

The second point is despite the brane we used was derived from a perturbative low-energy theory (which was used rather as a seed for the reconstruction formalism), it does not mean that the relation (7) is restricted to this brane or to the perturbative theory as such. In particular, it holds also for extremal branes which are known to represent non-perturbative aspects (BPS saturation is protected from quantum corrections [18]). In other words, full M-theory itself should also belong to this class to have brane solutions, therefore, our EFMT provides a certain piece of information about the strong coupling regime.

4. Eigenproblem in coupling space

In previous section we have determined the general functional form of couplings required by SCP. The remaining problem is the value of the constant parameter a. In principle, the extended SUGRA's propose the concrete values of this parameter: $a = 0,\ 1,\ 1/\sqrt{3},\ \sqrt{3}$ [15].

There exists, however, an elegant and all-sufficient way of determining a value of a without involving any external arguments, such as supersymmetry (which is especially helpful if M theory has other physically relevant limits not necessary obeying supersymmetry). In fact, the idea logically follows from what we were doing before. We will show that considering the *eigen*value problem for Eq. (31) inevitably leads to the unique discrete (moreover, finite) spectrum of a's.

Indeed, as long as the self-coupling Λ is given by a solution of a second-order linear homogeneous differential equation, the space of its values (functional, in general case) can be endowed with an orthogonal basis, according to the Sturm-Liouville theory of self-adjoint differential operators. Therefore, the coupling space $\{\Lambda[\phi(x)]\}$ is Hilbert, much alike in ordinary quantum mechanics.

There are, however, important differences between this space and that of quantum mechanical state vectors:

(i) wave functions are defined on spacetime whereas couplings are defined on the field space, therefore, some habitual physical concepts, such as energy, ground state, state vector collapse, measurement, etc., may not be directly extrapolated,

(ii) wave functions are additionally normalized as to obey the probability interpretation whereas for coupling functions we usually do not need that,

(iii) in order to have a physically admissible field-theoretical action one has to impose the reality of the couplings themselves, i.e., when doing a "coupling-space quantization" (speaking more precisely, determining the allowed values of the coupling functions' parameters) one should restrict oneself with a "physical" subspace of the Hilbert coupling space,

$$\Lambda(\phi) \subset \text{“physical”} \quad \text{for all} \quad \phi \in \Re\text{e}; \quad a^2 > 0, \tag{17}$$

where "physical" means a set of at least real, single-valued and regular in a finite domain functions of ϕ. The value $a = 0$ has been *ab initio* excluded because in the original p-brane ansatz the scalar would become a constant and thus decouple (in 4D it corresponds to the Reissner-Nördstrom case) so that Eq. (7) is inapplicable. Also, notice that Eqs. (30), (31) are invariant under the replacement $a \to 1/a$ up to the electric-magnetic duality transformation, therefore, in the a parameter space it suffices to consider either the domain $a^2 \leq 1$ with the removed point $a = 0$ or the $a^2 \geq 1$ one.

By transformation (see Eqs. (15), (35) for notations)

$$\Lambda(\phi) \to \Lambda(z) = w(z)/(1 - z^2), \tag{18}$$

the differential equation for Λ reduces to the Legendre one,

$$(1 - z^2)w''(z) - 2zw'(z) + [6 - \mu^2/(1 - z^2)]w(z) = 0, \tag{19}$$

such that

$$w(z) = C_1 P_2^\mu(z) + C_2 Q_2^\mu(z), \tag{20}$$

so the general solution (14) could be eventually reproduced. If we assume the physical range domain of ϕ to be a real axis then via Eq. (15) the range of z is $(-\infty, -1] \cup [1, \infty)$ where the origin in ϕ-space corresponds to infinity in z-space times a sign of a, whereas the infinities in ϕ-space correspond to $z = \pm \operatorname{sgn} a$, respectively.

What about boundary conditions for the last differential equation? Rigorously speaking, in general we do not have any. However, the general solution for $w(z)$ has a cut along the real axis for large values of $|z|$ (which corresponds to small $|\phi|$'s), therefore, when keeping C_i arbitrary the condition (32) can be satisfied by their proper redefinition only if μ takes integer values (see Appendix A for details):

$$\mu = n_\mu = 0,\ \pm 1,\ \pm 2,\ \pm 3, \ldots, \tag{21}$$

though in principle negative values of n_μ can be discarded because they correspond to $1/a$ spectrum, according to the property:

$$\mu \longrightarrow -\mu \iff a \longrightarrow 1/a, \tag{22}$$

see also the comment after Eq. (32). In its turn, the discretization of the μ spectrum implies that the Legendre functions in $w(z)$ become the associated Legendre polynomials, such that, e.g., at $\mu \geq 0$ one can use the formulae:

$$\begin{aligned} P_2^\mu(z) &\longrightarrow \frac{3}{2}\,(z^2-1)^{\mu/2}\,\frac{d^\mu}{dz^\mu}\,(z^2 - 1/3), \\ Q_2^\mu(z) &\longrightarrow \frac{3}{2}\,(z^2-1)^{\mu/2}\,\frac{d^\mu}{dz^\mu}\left[\frac{z^2 - 1/3}{2}\ln\left(\frac{z+1}{z-1}\right) - z\right], \end{aligned}$$

which turn to hyperbolic trigonometric functions upon transforming back to ϕ by virtue of Eq. (15).

The relation between a^2 and μ is given by Eq. (35) so we can draw the Table 2 from which it is clear that under the assumption of reality of a, Eq. (32), the only admissible values of a^2 are 0, 1 and 3 or 1/3 (though, in accordance with Eq. (22), the first and last values are dual to ∞ and 3, respectively). Thus, by Eq. (37) we have derived the relation noticed by Hull and Townsend when dealing with extreme black holes in 4D string compactifications [23]. Thus, we have computed the "magic numbers" of extended supergravity theories from the model acting in the product coupling space $\{\Lambda[\phi(x)]\} \otimes \{\Xi[\phi(x)]\}$.

The corresponding eigenfunctions can be easily found by substituting eigenvalues into the general solution (14). As was already mentioned, for the allowed values of a the Legendre functions become just polynomials that greatly simplifies treatment. For instance, the case $a = 1$ will be performed in the next section, although in string theory it is the $a = \sqrt{3}$ one which

n_μ	a^2	n_μ	a^2
0	1		
1	3	-1	$1/3$
2	∞	-2	0
3	-5	-3	$-1/5$
4	-3	-4	$-1/3$
5	$-7/3$	-5	$-3/7$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
∞	-1	$-\infty$	-1

Table 1: Correspondence between the spectra of μ and a^2.

is usually regarded as describing an "elementary" black hole, in a sense that other states, 1, $1/\sqrt{3}$, 0, can be sequentially created by binding of, respectively, two, three and four "elementary" black holes [15, 24].

Thus, in general case the scalar sector of EFMT is a superposition of six Lagrangians corresponding to the above-mentioned eigenvalues of a. The question which of them corresponds to "ground state", "first excited", etc., depends on whether it is possible to uniquely introduce the notion of energy in the coupling space. If it can be defined as in Appendix C then one can show that if one assumes that ϕ plays a role of "time" in the coupling space then the states with $a = \pm 1$ indeed have larger $k(\phi)$ for all ϕ than others, and thus are less favorable "energetically".

5. EFMT and fundamental solution

Now it is time to explicitly obtain that 4D solution whose high-D analogues are p-branes like the one given by Eqs. (5), (6). Henceforth we will concentrate solely on the $a = 1$ case (solution for arbitrary a is given in the Appendix B). Also, we assume the vanishing magnetic charge, $P = 0$, hence $\hat{\Xi}$ becomes simply Ξ^{-1} up to the constant factor $2Q^2$ which cancels out anyway (then the electric charge also does not appear in the action, as expected). Then Eqs. (33), (14) after substituting $a = 1$ and redefining integration constants yield

$$1/\Xi = \frac{1}{2}\left(\sigma_1 e^{\phi} + \sigma_2 e^{-\phi}\right), \tag{23}$$

$$\Lambda = -2\lambda(\cosh\phi + 2) - 4\chi(3\sinh\phi - \phi(\cosh\phi + 2)), \tag{24}$$

where σ_i, χ and λ are arbitrary integration constants, λ is known as the "cosmological constant" parameter. In principle, if one chooses Q to vanish instead of P then $1/\Xi$ above should be replaced with Ξ as the EM duality suggests (and one should also reverse the sign of ϕ in Ξ).

Notice that the derived model (23), (24) generalizes not only the tree-level case (2) [19, 20] but also the heuristic beyond-tree-level Monni-Cadoni's S-dual model (invariant under $\phi \to -\phi$) [21], for which:

$$1/\Xi^{\varsigma}_{(MC)} = -\cosh a\phi\,, \qquad \Lambda_{(MC)} = 0\,, \tag{25}$$

(they considered the magnetic case where $\varsigma = -1$). Besides, the potential (24) generalizes also the monoscalar gauged $N = 8$, $SO(8)$ SUGRA representing the massless $U(1)^4$ sector of the M-theory's $AdS_4 \times S^7$ vacuum [22, 24] where:

$$\Xi_{(S^7)} = 0\,, \qquad \Lambda_{(S^7)} = -2\lambda(\cosh\phi + 2). \tag{26}$$

In this connection, one could ask whether the χ-term in the potential (24) (odd under inversion of the scalar) is redundant. Clearly, it is not - later on we will show that it is responsible for the compatibility of the scalar field with black holes as it guarantees the existence of event horizon (and thus protects from a naked singularity [9]) even in absence of the electromagnetic field.

Finally, utilizing Appendix B we can derive the fundamental 0-brane solution of our EFMT (29), (23), (24). Indeed, it describes a charged black hole with scalar hair and is given by

$$\mathrm{d}s^2 = -N^2 \mathrm{d}t^2 + \frac{\mathrm{d}r^2}{N^2} + R^2(\mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\varphi^2), \tag{27}$$

$$e^{\phi} = H,\ \mathcal{A}_0 = -\frac{Q}{2}\left(\frac{\sigma_1}{r} + \frac{\sigma_2}{r+\kappa}\right), \tag{28}$$

where $N^2 = 1 - 2\chi\left[\kappa(r+\kappa/2) - R^2 \ln H\right] + \frac{Q^2}{\kappa}\left(\frac{\sigma_2}{r+\kappa} - \frac{\sigma_1}{r}\right) - \lambda R^2$, $R = \sqrt{r(r+\kappa)}$ and $H = 1 + \kappa/r$ (it is given by Eq. (6) at $\tilde{d} = 1$). The model also admits another solution which can be deduced from above by the simultaneous transformations $\{\phi \to -\phi, \chi \to -\chi, \sigma_1 \leftrightarrow \sigma_2\}$ because the Lagrangian has corresponding symmetry. One can show that the solution retains an event horizon even if the gauge field is switched off which leads to important phenomenological consequences - the potential (24) has been used for the unified description of cosmological phenomena and black holes [25]. Besides, there it was explained why the cosmological constant (mentioned at the top of paper) should be bound from above by some positive number whose origin and physical meaning are related to black holes.

6. Why do we live in a 4D world: Can cosmology, black holes and branes give an answer?

Despite the continuously growing popularity and successes of higher-dimensional models one should not forget one of the fundamental problems of string/M theory: to explain why at low energies our world is effectively four-dimensional to such a high degree of precision.

Recent efforts to dynamically derive or explain 4D spacetime have been made in the matrix formulation of type IIB superstrings but no definite conclusion has been made so far [26, 27, 28]. Also, attempts have been made in alternative theories of quantum gravity, such as Causal Quantum Gravity [29] but that theory has too little in common with M theory.

Other attempts were also made in several multidimensional models imposing one or another predefined way of splitting a higher-dimensional manifold into the external space (the observable Universe) and an internal one. For instance, working in 11D supergravity quantum cosmology if one assumes the seed instanton of the Universe to be a product of two spheres, then it can be proven that the external spacetime must be 4D [30]. Although, in such models the problem was not entirely solved but rather reduced to the well-known problem of which way of dimensional reduction the Universe uses to descend from 11D to 4D. This is still far from being answered.

Thus, the problem of formation of the Universe's dimensionality still persists. Is it possible to solve it in a way consistent with M theory ideas? In this paper we will try to get an affirmative answer.

In fact, as compared to all other parameters which appear in field theory, the parameter D has a number of distinctive features which, on one hand, create difficulties with solving the above-mentioned problem but, on the other hand, hint about possible clues. First, we know that any field theory in spacetime is defined by virtue of the action functional, $S[\psi(x)] = \int L(\psi_i, \partial\psi_i) d^D x$, where ψ_i is a set of the fields built into the theory, L is the Lagrangian function. Here the spacetime dimensionality D is involved both in the integration measure of the functional, $d^D x$, and in the notion of distance between events in spacetime, thus it is the parameter which must be fixed *before* one starts specifying the Lagrangian itself. Therefore, D can not be directly derived from a theory which acts in spacetime. Second, being an integer rather than a continuous parameter, D hints about the possibility that it could be a discrete eigenvalue of a certain dynamical theory (which acts not in spacetime, according to the aforesaid).

Below we show that both suppositions are indeed very close to the truth. We demonstrate that the values of the parameter D are dynamically derivable from a differential equation defined in a certain space (called the coupling space) whereas the properties of the latter are determined by black hole physics closely related to M theory, supersymmetry (SUSY) and p-branes.

We start by recalling that the global dynamics of the Universe is determined by long-range fields. From the Standard Model we know that Yang-Mills and fermionic fields are short-range thus here we can restrict ourselves to the Abelian sector. The latter includes gravity described by the Lorentzian metric tensor g_{MN} (capital Latin indices run from 1 to D), real scalar field ϕ (often called also the *inflaton* as it has been widely used in cosmology in scalar-driven inflationary models) and several gauge fields described by antisymmetric tensors of different ranks. Thus, our action functional must contain at least the graviton and the inflaton. To this we add also a couple of gauge fields: the electromagnetic one described by the antisymmetric tensor of second rank, F_{MN}, and the one of rank $\tilde{p}$, with the components $F_{(\tilde{p})M...N}$. The gauge fields are added just to check whether they affect the problem we are going to study. The simplest effective action one can thus write is (we work in the Einstein frame):

$$S = \int d^D x \sqrt{-g} \left[R - \frac{1}{2}(\partial\phi)^2 + \Xi(\phi) F^2 + \Psi(\phi) F^2_{(\tilde{p})} - V(\phi) \right], \qquad (29)$$

where $(\partial\phi)^2 = \partial_M \phi \, \partial^M \phi$, $F^2 = F_{MN} F^{MN}$, $F^2_{(\tilde{p})} = F_{(\tilde{p})M..N} F_{(\tilde{p})}{}^{M..N}$, and $g = \det |g_{MN}|$ and R is the scalar curvature constructed out of g_{MN}. By this action we mean an *ensemble* of theories which are equivalent with respect to their dynamical content but act in different D. Then we can regard D as a free parameter from now on. For further, it will be convenient to assume $D \equiv \tilde{p} + 2$ and work in terms of the parameter $\tilde{p}$.

The explicit values of the Maxwell-scalar coupling Ξ, the $\tilde{p}$-tensor-scalar coupling Ψ and the scalar self-coupling (potential) V are largely unknown, even in string/M theory [3], except in the case where they were obtained in the tree-level superstring approximation [19] - but those were too simple to describe any realistic phenomena because SUSY gets broken at early stages of the evolution of the Universe. Thus, our task now is to find the coupling functions without using the supersymmetrical arguments solely but engaging instead some other, physically more general, arguments.

The first thing we require from the theory (29) is that it should not forbid black hole solutions at any physically relevant D (apart from being compatible with inflation and string/M theory, of course). Indeed, the mathematical absence of black hole solutions would lead to a loss of protection of a theory from naked singularities [9] and thus to undesirable violations of the Cosmic Censorship principle [10, 11, 12]. This is especially dangerous in our case: if, according to modern cosmological views, the Universe began as an extremely hot, dense and compact higher-dimensional object then the ubiquitous fluctuations of the spacetime metric which took place at that stage, inevitably led to the appearance of spacetime singularities. As long as it is impossible to set initial data in a singular point, the evolution of physical objects inside the spatial domains which are causally-dependent from singularities is essentially unpredictable. Then during the inflationary era such regions would become extremely large. Black holes protect from this by "dressing" the singularities with event horizons and thus causally disconnecting them from the rest of the Universe. Besides, there exist some arguments that the existence of black holes places an upper bound for the *cosmological constant* parameter responsible for the rate of expansion of the Universe and thus prevents the latter from hyper-accelerated inflation [25].

In view of the aforesaid one may wonder how could it be that black holes, having such a small size and mass comparing to those of the Universe, can nevertheless influence its global properties? One should recall, however, that the notion of event horizon is a non-local one - in general it requires the knowledge of not only the distribution of matter in space but also its future evolution, and depends on the fate of Universe.Thus, the existence of event horizons inevitably affects the large-scale structure of the whole space. In the case of scalar-driven inflationary cosmology black holes act with the aid of the global scalar field - it is well-known that the scalar "no-hair" theorems forbid the appearance of black holes for a large set of scalar potentials, e.g., convex or positive semi-definite [31]. Thus, by far not every inflaton potential, and hence by far not every inflationary scenario, is compatible with the existence of regular horizons (here we call a horizon *regular* if not only the metric but also other fields do not become singular on it).

Further, in turn black holes need for a proper description of their own properties (microstates, entropy, etc.) certain type of M-theory solutions - branes [8]. Therefore, the mathematical absence of brane-like solutions in a theory would cause, apart from incompatibility with the main idea of M theory, serious difficulties with a consistent microscopical description of such phenomena as the black hole thermodynamical laws. Also, speaking more generally, the branes as extended objects are natural generalizations of the notion of a point particle and thus can serve as a better approximation to the quantum-mechanical reality: for instance, one can ask a question to what extent the entangled collective quantum states (e.g., the Cooper pairs) which act as a whole can be regarded as (a plain set of) point-like objects and is there any better effective description of them.

Lastly, to ensure good ultraviolet behavior we restrict ourselves to the class of those p-branes which are explicitly related to M theory and tend to supersymmetric BPS states in some limit. We demand thus that our model (29) should be compatible with such branes at any physically admissible D. We accentuate that this is a necessary but, of course, not a sufficient condition for a theory to be physically relevant. Nevertheless, we will see straight away that it significantly decreases the number of allowed coupling functions.

Imposing this requirement for the 0-branes from the above-mentioned class (we are looking for necessary conditions, therefore, we can take 0-branes for simplicity), using the corresponding ansatz from above ref. [32] and applying our approach assuming D arbitrary, we obtain that the most general couplings of (29) which allow such 0-branes are those obeying the following second-order differential equations

$$\hat{\Xi}''(\phi) + \frac{4\eta}{\phi_1}\,\hat{\Xi}'(\phi) - \frac{2(\tilde{p}-1)}{\tilde{p}}\,\hat{\Xi}(\phi) = 0, \tag{30}$$

$$V''(\phi) - \frac{\tilde{p}\coth(\frac{\phi}{\phi_1}) - \eta(\tilde{p}-2)}{\phi_1(\tilde{p}-1)/2}\,V'(\phi) + \frac{2}{\tilde{p}}\,V(\phi) = 0, \tag{31}$$

where $\hat{\Xi}(\phi) \equiv 2Q^2/\Xi(\phi) + \tilde{p}\,!P^2\Psi(\phi)$, with Q and P being, respectively, the electric and magnetic brane charges (up to a numerical coefficient), and

$$\phi_1 \equiv \frac{4a}{\varsigma\Delta}\,, \qquad \eta \equiv 1 - \frac{4(\tilde{p}-1)}{\tilde{p}\Delta}\,, \qquad \Delta = a^2 + 2 - \frac{2}{\tilde{p}}\,,$$

a is some constant parameter related to p-branes [33] (here we assume it free as well), and ς is $+1$ for elementary (electric) branes and -1 for solitonic (magnetic) ones. The fact that the gauge-scalar couplings $\Xi(\phi)$ and $\Psi(\phi)$ appear in eq. (30) only as the combination $\hat{\Xi}(\phi)$ is explained by the existence of electric-magnetic duality between the 1-form $\mathcal{A}$ (such that $F = d\mathcal{A}$) coupled to the worldline of 0-brane and the $(\tilde{p}-1)$-form $\mathcal{A}_{(\tilde{p}-1)}$ (such that $F_{(\tilde{p})} = d\mathcal{A}_{(\tilde{p}-1)}$).

As long as the solutions of eqs. (30), (31) could be, in general, complex-valued functions with cuts and singularities, the equations must be supplemented with the restriction for the couplings to be physically admissible for real values of the inflaton. Otherwise, the action (29) would become ill-defined from the field-theoretical point of view. We impose thus the following "boundary" condition:

$$\Xi(\phi),\ \Psi(\phi),\ V(\phi) \subset \text{"physical"} \quad \text{for all} \quad \phi \in \Re\text{e}, \tag{32}$$

where "physical" means a set of at least real, single-valued and regular in a finite real domain functions of ϕ - as the couplings should be. At first sight, this condition looks too weak to give any physically interesting restrictions. However, it turns out that in some cases it is sufficient to pose a well-defined *eigen*problem.

Further, eq. (30) can be easily solved,

$$\hat{\Xi}(\phi) = \sigma_1 e^{\frac{2\varsigma(\tilde{p}-1)}{a\tilde{p}}\phi} + \sigma_2\, e^{-\varsigma a \phi}, \tag{33}$$

where σ_i's are arbitrary integration constants, and it is of no interest to us here, as eq. (32) can not bring any restrictions for the parameter $\tilde{p}$ but only infers that σ's must be real-valued. Thus, it seems that gauge fields, even those having a long range, have little or no influence on the process of the Universe's dimensionality formation.

The other coupling equation, for the scalar field potential, looks more promising in this connection. In fact, it describes the self-interaction of the inflaton which is known to determine the global dynamics of the Universe, according to the scalar-driven inflationary scenarios. Could it thus be that eq. (31) determines not only the potential itself but also the parameter D?

Due to the complexity of eq. (31) it seems difficult to solve exactly the eigenvalue problem imposed by eq. (32) in the general case. Luckily, one can engage certain physical picture to see things more clearly. Starting from some cosmological epoch, when the characteristic energy became of the order of the string scale, the global dynamics of the Universe can be effectively described by the action (29) at $D > 4$ in the supergravity (SUGRA) approximation [34]. At that stage the existence of p-branes did not imply any restrictions for D as the high symmetry ensured that the model (29) has p-brane solutions at any D provided its couplings are compatible with supergravity [19, 33], i. e., the eigenvalues of D have a "continuous" spectrum - in a sense that they take integer values (restricted by $D \leq 11$ so as to exclude the appearance of an infinite number of fields with spins higher than

2) but otherwise arbitrary. However, as the Universe expands and cools, SUSY gets dynamically broken such that in eq. (29) the SUSY-compatible couplings become physically inadmissible (in particular, $V(\phi)$ cannot be set to zero anymore, otherwise inflation cannot start). Therefore, one should go on computing the general couplings directly from eqs. (30) and (31). Those, of course, will include SUGRA couplings as a special case.

All this can be quantitatively visualized if one clocks the history of the early Universe by means of the cosmological scalar field's scale (of course, starting from the epoch when this field has already appeared). Then one could distinguish the following three regimes of eq. (31) and, hence, of $V(\phi)$:

Top high-energy regime. In the pre- and early-inflationary Universe nothing could prevent the cosmological scalar from having large initial magnitude. This regime thus corresponds to large values of ϕ - such that the cotangent's absolute value in eq. (31) approaches one. Actually, "large" is a way too strong word here: the cotangent's magnitude approaches unity exponentially rapidly so that already at $|\phi/\phi_1| \geq 2$ (in Planck units) it differs from one by less than 4%. The general solution of eq. (31) in this regime is a linear combination of two exponents and no restrictions for D arise from eq. (32). Thus, in this regime the dimensionality parameter is essentially free that confirms aforesaid. Notice that the full supersymmetry can be restored because $V = 0$ is also a solution of this equation.

Bottom high-energy regime. As the cosmological time passes, the inflaton's energy density gets diluted by inflation so the scalar starts decreasing its magnitude towards its vacuum value and eventually at some point $|\phi/\phi_1|$ becomes approximately one in Planck units, $|\phi/\phi_1| \approx (M_{\rm Planck})^{\tilde{p}/2}$. In this regime one should consider eq. (31) in full and study the *eigen*problem by imposing the condition (32). The dimensionality of spacetime is still above four but perhaps some values of D are already forbidden.

Low-energy regime. The Universe continues expanding so that the scalar gradually approaches its recent value, $\phi \to \phi_{\rm now}$, which is very small: $\phi_{\rm now} \ll (M_{\rm Planck})^{\tilde{p}/2}$. Therefore, in this regime the cotangent predominates the constant η-term in eq. (31) so that the general solution can be expressed in terms of the associated Legendre functions [35]:

$$V(\phi)_{|\phi/\phi_1| \ll 1} = (z^2 - 1)^{\frac{\tilde{p}}{2(1-\tilde{p})}} \left(C_1 P^\mu_\nu(z) + C_2 Q^\mu_\nu(z) \right), \tag{34}$$

where $z = \coth(\phi/\phi_1)$, the C_i's are arbitrary constants (complex-valued, in general), and

$$\nu \equiv \frac{1}{1-\tilde{p}} - 2, \qquad \mu \equiv \frac{1}{\tilde{p}-1} \sqrt{\tilde{p}^2 - \frac{32\, a^2 (\tilde{p}-1)^2}{\tilde{p}\Delta^2}}. \tag{35}$$

Further, the general solution for $V(\phi)_{|\phi/\phi_1| \ll 1}$ is a complex-valued function which has a cut along the real axis for small values of $|\phi/\phi_1|$ (that correspond to large $|z|$). Therefore, using App. A one can show that the

n_ν	$D-2$	n_ν	$D-2$
0	1/2		
1	2/3	-1	0
2	3/4	-2	$\pm\infty$
3	4/5	-3	2
4	5/6	-4	3/2
5	6/7	-5	4/3
$\vdots$	$\vdots$	$\vdots$	$\vdots$
∞	1	$-\infty$	1

Table 2: Correspondence between the spectra of ν and $\tilde{p}$.

condition (32) can be satisfied at arbitrary C_i (by their proper redefinition) if and only if μ and ν both take integer values:

$$\nu = n_\nu = 0,\ \pm 1,\ \pm 2,\ \dots, \tag{36}$$

$$\mu = n_\mu = 0,\ \pm 1,\ \pm 2,\ \dots. \tag{37}$$

Using the first of these spectral formulae we can draw Table 2, from which one can see that the maximal allowed eigenvalue of D is 4. In that case the tensor $F_{(\tilde{p})}$ becomes of a second rank, hence, we arrive at the model previously studied above whereas some of the above-mentioned 0-branes become 4D black holes with regular scalar and gauge hair. At that, when the gauge field is off, the regular horizon survives only if the parameter a obeys the second of the spectral formulae above (incidentally, eq. (37) brings us to the relation noticed by Hull and Townsend when dealing with extremal black holes in 4D string compactifications [23]). Notice also that in 4D both spectral formulae become exact, as one can see from eq. (31). Thus, when the Universe is hot, dense and compact, all the modes of the scalar potential, described by eigenfunctions of $V(\phi)$, are equally permitted. But as the Universe evolves in time, i.e., expands and cools, the potential occupies one of the few (eigen)modes which are allowed by above-mentioned physics. Other modes become dynamically unstable in the low-energy regime. At that, the transition from the "continuous" (i.e., integer but otherwise arbitrary) spectrum of D to $D = 4$ happens not continuously but by jumps, due to the discrete nature of D.

One can find some similarity between this scenario and the recombination of an electron and a stand-alone hydrogen ion. When the electron has sufficiently high energy (above the ionization threshold), its energy spectrum is essentially continuous. As the energy decreases, e.g., due to the spontaneous emission of photons, the electron gets captured by the ion and its energy enters a discrete spectrum. At that, we know that no description of such binding is possible in terms of classical orbits and trajectories - instead, one should think in terms of quantum superpositions and transition probabilities. One can not exclude the possibility that somewhere a similar uncertainty arises in the very early Universe: there may exist no fully deterministic description of the transition from higher dimensions to 4D so that the only essential information about this process that one can have in principle is the set of the allowed coupling (eigen)functions, such as $V(\phi)$. However, the analogy of the above-mentioned quantum-mechanical phenomenon with our case should be treated with great care because it is not perfect. In particular, in the quantum-mechanical case one has a linear superposition of eigenfunctions which correspond to different eigenvalues of energy whereas here we have different regimes of the same function, $V(\phi)$, at different values of its argument. Thus, the interpretation of our *eigen*problem is largely an open question so far. It would be interesting to study it in full, i.e., when the inflaton potential stays in the above-mentioned "bottom high-energy" regime. This should provide more details about how the transition from larger values of D proceeds to $D = 4$. Besides, if one has the exact solutions of eqs. (30), (31) in hand then using the method described above and in ref. [32] one can immediately obtain exact generalized 0-brane solutions which are higher-dimensional generalizations of black holes.

7. Conclusion

To conclude, above we have described how it may be possible to derive a class of effective theories representing the low-energy 4D limit of M-theory. Thereby, our method does not make explicit use of any *predefined* compactification manifolds or other geometrical constructions which presume one or another way of the decomposition of space dimensions into "observable" and "unobservable" ones. The approach does not address the question why our everyday world is essentially four-dimensional (though, $D = 4$ does look distinguishable from a certain point of view, see the bottom of Sec. 2., but the physical meaning of that fact is rather obscure so far). Neither the approach refuses the compactification idea as such. Instead, it focuses on the *generic* features of EFMT, i.e., those which are independent from how the above-mentioned decomposition could be done.

Here we have demonstrated the use of the method (by deriving both a particular sector of EFMT and its corresponding brane-like solution) for the simplest abelian case - when bosonic sector contains graviton, photon and scalar. The future directions of work here would be to further generalize the skeleton model (29) by engaging other fields (Yang-Mills, higher-rank tensors, spinors, etc.) which could be required for describing physics at higher energies and shorter scales of length.

Finally, we have determined the self-coupling function (potential) of a cosmological scalar field which would break supersymmetry in such a way as to be compatible with inflationary cosmology and yet preserve some universal features of the low-energy M-theory's solutions. Assuming the number of spacetime dimensions to be a free parameter, we considered the eigenvalue problem for it. It is shown that in the low-energy regime of small values of the cosmological scalar field $D = 4$ arises as a largest allowed eigenvalue. Hopefully, knowledge of the behaviour of the inflaton potential at different D will advance also our understanding of other phenomena which took place in the early Universe.

A Eigenvalues of μ

Let us demonstrate that imposing the condition (32) immediately leads to discretization of the μ spectrum. So, we first infer that Λ tends to a finite real value when approaching the $z = \infty$ (i.e., $\phi = 0$) point. Using the asymptotics of w for large z, we obtain:

$$\Lambda(z) \propto -\mathrm{e}^{i\pi\mu/2} \, \frac{C_1 + i\pi C_2/2}{\Gamma(3-\mu)/3} + O(1/z^2), \tag{38}$$

such that the constants must be related by

$$C_1 + i\pi C_2/2 = -\mathrm{e}^{-i\pi\mu/2} \, \Gamma(3-\mu) \, \Lambda_0/3, \tag{39}$$

where $\Lambda_0 \equiv \lim\limits_{z\to\infty} \Lambda(z) = \Lambda(\phi)|_{\phi=0}$ has physical meaning of the (effective) cosmological constant, up to a numerical factor, and thus is real-valued.

Further, assuming μ positive for definiteness (see Eq. (22)) and using the asymptotic of w for $z \to 1+0$, we obtain:

$$\Lambda(z) \propto -\frac{C_1 + \pi C_2 \cot(\pi\mu)/2}{2^{1-\mu/2} \, \mathrm{e}^{i\pi\mu/2} \, \Gamma(1-\mu) \, (z-1)^{1+\mu/2}}, \tag{40}$$

such that we arrive at another relation for C's:

$$C_1 + \pi C_2 \cot(\pi\mu)/2 = -2^{1-\mu/2} \, \mathrm{e}^{i\pi\mu/2} \, \Gamma(1-\mu)\Lambda_\infty, \tag{41}$$

where $\Lambda_\infty \equiv \lim\limits_{z\to 1+0} (z-1)^{1+\mu/2}\Lambda(z)$ is another real-valued constant. Thus, with Eqs. (39) and (41) in hand one can always express the constants C_1 and C_2 in terms of the real-valued ones Λ_0 and Λ_∞.

Now, we are left with the imposing the condition (32) in two more limits, at $z \to -\infty$ and $z \to -1-0$. The former yields obviously nothing new as it brings us back to Eq. (39) whereas considering the asymptotical behavior of $\Lambda(z)$ at $z \to -1-0$, we obtain:

$$\Lambda(z) \propto -\frac{\pi C_2 \csc(\pi\,\mu)}{2^{2-\mu/2} \, \mathrm{e}^{i\pi\mu/2} \, \Gamma(1-\mu) \, (-z-1)^{1+\mu/2}}, \tag{42}$$

which means that

$$\frac{\csc(\pi\,\mu) C_2}{\mathrm{e}^{i\pi\mu/2} \, \Gamma(1-\mu)} \tag{43}$$

must be real-valued. Expressing here C_2 in terms of Λ_0 and Λ_∞ by virtue of Eqs. (39) and (41), we conclude that

$$2^{\mu/2}(\mu-1)(\mu-2)\Lambda_0 - 6 \, \mathrm{e}^{i\pi\mu}\Lambda_\infty \tag{44}$$

must be real-valued. This is possible only if μ is integer.

In fact, one can easily prove a more general result: let $L^\mu_\nu(z) \equiv C_1 P^\mu_\nu(z) + C_2 Q^\mu_\nu(z)$ be a linear combination of the associated Legendre functions on the real axis, whereby C's are arbitrary. Then the function $L^\mu_\nu(z)$ is real, single-valued and regular for all $z \in (-\infty, -1) \cup (1, \infty)$ iff μ and ν are integers. The proof consists in expanding $L^\mu_\nu(z)$ in series in the neighborhood of some conveniently chosen points and imposing the reality of the leading-order term, next-to-leading, etc.

B General static solution

Here we present the exact general spherically symmetric static solution of the Einstein-Maxwell-scalar gravity (29) with the couplings Ξ and Λ given by Eqs. (33) and (14), assuming electric case, i.e., $\hat{\Xi} = \Xi^{-1}$ up to a constant factor. In fact, according to the method of Ref. [16], once a class is established and the class equation is derived and solved, the task of finding expressions for metric, scalar ϕ and electrostatic potential $\mathcal{A}_0$ is straightforward.

So, assuming that we are working in the gauge (27) we respectively find

$$N^2 = (1-\mu/2)^2 \mathrm{e}^{\frac{\mu\phi}{\phi_1}} (1 + N_\sigma/\kappa^2 - N_\Lambda R^2), \tag{45}$$

$$R^2 = \frac{\mathrm{e}^{\phi/a}\,\kappa^2}{(1-\mathrm{e}^{\frac{2\phi}{\phi_1}})^2}\,, \qquad \mathrm{e}^{\frac{2\phi}{\phi_1}} = 1 + \frac{\kappa(1-\mu/2)}{r}\,, \tag{46}$$

$$\mathcal{A}_0 = -\frac{Q(1-\mu/2)^2}{2r}\left[\sigma_1 + \frac{\sigma_2}{1+\frac{\kappa(1-\mu/2)}{r}}\right], \tag{47}$$

$$N_\sigma \equiv \frac{2Q^2\sigma_1}{2+\mu}\left(1-\mathrm{e}^{\frac{2\phi}{\phi_1}}\right) + \frac{2Q^2\,\sigma_2}{2-\mu}\left(1-\mathrm{e}^{-\frac{2\phi}{\phi_1}}\right),$$

and $N_\Lambda \equiv (z + \mu/2)\Lambda'(\phi)/\phi_1 - \Lambda(\phi)/2$ where it is assumed that one substitutes Eq. (14) and its derivative into this formula. Not all of these solutions describe the black holes embedded in the asymptotically flat or dS/AdS spacetime but those with $a^2 = 1,\ 3,\ 1/3$ definitely do [23].

C Measure of energy in coupling space

We have a differential equation for a coupling function which takes values in the coupling space $\{C_i[\phi(x)]\}$ lying "on top" of the space of scalar field ϕ. Its general form is

$$C''(\phi) + w_1(\phi)C'(\phi) + w_2(\phi)C(\phi) = 0, \tag{48}$$

where functions $w_i(\phi)$ are given. This equation can be treated as an evolution equation of some dynamical system in some effective space such that one can introduce a notion of energy.

It seems, however, that the way of choosing the appropriate system is not unique. In particular, it depends upon whether variable ϕ is associated with "space-like" or "time-like" coordinate in that effective space. In former case, one could regard Eq. (48) as the stationary Schrödinger equation for an imaginary particle where ϕ plays the role of the 1D coordinate x, so that one can determine energy by comparison:

$$2m(E - U) \quad \Longleftrightarrow \quad k(\phi), \tag{49}$$

where

$$k(\phi) \equiv w_2(\phi) - \frac{1}{2}\, w_1'(\phi) - \frac{1}{4}\, w_1^2(\phi). \tag{50}$$

However, as long as our ϕ has physical meaning of the inflaton from scalar-driven inflationary cosmology, we know that it is correlated with the cosmological "arrow of time". Then Eq. (48) can be regarded as the Euler-Lagrange equations for a classical-mechanical system where ϕ plays the role of time. Let us make a substitution

$$C(\phi) = X(\phi)\, e^{-\frac{1}{2}\int w_1(\phi) d\phi}, \quad \phi \to t, \tag{51}$$

then Eq. (48) becomes the one for the harmonic oscillator with a time-varying spring constant:

$$\ddot{X}(t) + k(t) X(t) = 0, \tag{52}$$

from which one can sequentially derive the Lagrangian and Hamiltonian functions. Thus, in principle one can use

$$E = \frac{1}{2}\, X'(\phi)^2 + \frac{1}{2}\, k(\phi) X(\phi)^2, \tag{53}$$

as a measure of energy in the coupling space, but as long as $C(\phi)$ and hence $X(\phi)$ are determined up to a constant factor, sometimes it will be more convenient to use the "energy-per-square-distance" value

$$\mathcal{E} \equiv 2E/X^2 = \left[\frac{d \ln C(\phi)}{d\phi} + \frac{1}{2}\, w_1(\phi)\right]^2 + k(\phi), \tag{54}$$

which is $C(\phi)$-scale-invariant. It agrees with the estimation of energy based on the fact that the effective oscillation frequency equals to square root of the spring constant, thus minimum of energy corresponds to minimum of $k(\phi)$. By the way, notice that in the "Schrödinger" interpretation (49) the minimum of energy would correspond to maximum of $k(\phi)$ unless $m < 0$.

D Theories in coupling space and renormgroup approach

The following is mainly a conjecture which lies somewhere outside the main line of paper and thus can be omitted in first reading. Yet, it could be a good starting point for those interested in studying possible relationship between the renormgroup (RG) approach in quantum field theory and dynamical models acting on the space of field couplings.

Indeed, Eqs. (30), (31) are in fact describing the behavior of coupling "constants" (which have been promoted to locally defined values, of course) with respect to the fundamental scalar, ϕ. We can assume that ϕ also defines the energy scale, i.e.:

$$\phi \sim \ln \mu/\mu_0, \tag{55}$$

where μ is a characteristic cut-off energy-momentum scale, and μ_0 is a normalization constant. To see more clearly a motivation, notice first that Eqs. (30), (31) are homogeneous second-order ODE's of type (48) (actually, it is a common feature of the dynamical equations for coupling functions because field-theoretical Lagrangians are obviously linear with respect to couplings as such). It is well-known that from the two integration constants which appear in the solution of such an ODE only one characterizes the details of dynamical behavior whereas other constant just determines scale: $C(\phi) = c_1 C_{(1)}(\phi) + c_2 C_{(2)}(\phi) = c_2[\hat{c}_1 C_{(1)}(\phi) + C_{(2)}(\phi)]$. Thus, all the essential information contained in a second-order homogeneous ODE can be encoded in an appropriately chosen first-order ODE. Indeed, by substitution

$$g(\phi) = d \ln C(\phi)/d\phi \tag{56}$$

the equation can be brought to the RG-like form:

$$\mu dg/d\mu = dg/d\phi = \beta_g(\phi, g) \equiv -g^2 - w_1(\phi) g - w_2(\phi).$$

If one is unhappy with the explicit appearance of ϕ inside $\beta_g(\phi, g)$, one can in principle solve the differential equation for $g(\phi)$, invert the function, and replace ϕ by $\phi(g)$ in the r.h.s. of the equation. Doing that way, one will arrive at the equivalent "μ-free r.h.s." form (as in the habitual RG formalism):

$$\mu dg/d\mu = dg/d\phi = \beta_g(g), \tag{57}$$

where $\beta_g(g) \equiv \beta_g(\phi(g), g)$, i.e., in general case the beta function is not just a quadratic polynomial of g.

Incidentally, one should not think that the conjectured relationship infers that the RG flow equations can be derived solely from the (classical) field equations. Despite one can definitely say that about Eq. (9), to obtain Eq. (10) one needs the additional (external) input - the class characteristic equation, Eq. (8), which comes from the brane physics.

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