Gerbes in unoriented WZW Models*

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ABSTRACT

We explain how the Wess-Zumino term in two-dimensional conformal field theory can be understood as the surface holonomy of a gerbe and exhibit some of the advantages of this point of view. In particular, we explain how the choice of additional data – a so-called Jandl structure on the gerbe – admits the definition of the Wess-Zumino term for unoriented conformal field theories.

WZW Models and the Wess-Zumino Term

Wess-Zumino-Witten (WZW) models [Wit84] are one of the most important classes of (two-dimensional) rational conformal field theories. They describe physical systems with (non-abelian) current symmetries, provide gauge sectors in heterotic string compactifications and are the starting point for other constructions of conformal field theories, e.g. the coset construction. Moreover, they have played a crucial role as a bridge between Lie theory and conformal field theory.

A WZW Model is an example of a non-linear sigma model, where the fields are smooth maps $\phi : \Sigma \to G$ from a conformal oriented surface Σ (the worldsheet) to a Lie group G (the target space). Here we assume Σ to be closed, i.e. without boundary, and G to be simple, compact and connected. Then, the target space G is equipped with a metric $\langle -, - \rangle$, usually a multiple of the Killing form. The action functional has the form

$$S(\phi) = \int_\Sigma \langle \mathrm{d}\phi \wedge \star \mathrm{d}\phi \rangle + S_{\mathrm{WZ}}(\phi), \tag{1}$$

composed of a kinetic term, which involves the metric on G and the conformal structure of Σ, and of the Wess-Zumino term $S_{\mathrm{WZ}}(\phi)$. The latter is the main subject of interest in this contribution.

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The presence of the Wess-Zumino term restores the conformal symmetry when the above action is quantized [Wit84]. It requires a detailed definition, which was first explained in [Wit84] for simply-connected groups G as follows: for a given field $\phi : \Sigma \to G$ choose a 3-dimensional oriented manifold B with $\partial B = \Sigma$ and an extension $\Phi : B \to G$ of ϕ, i.e. a smooth map such that $\Phi|_{\partial B} = \phi$. The 3-dimensional manifold B exists because Σ is 2-dimensional and oriented, and the extension Φ exists because every compact, simple, connected and simply-connected Lie group has $\pi_2(G) = 0$, which is precisely the obstruction against an extension. Another important ingredient is the canonical 3-form H on the Lie group G: it is defined by

$$H := \langle \theta, [\theta, \theta] \rangle \,, \tag{2}$$

where θ is the left-invariant Maurer-Cartan form on G. In a faithful matrix representation g of G, $\theta = g^{-1}\mathrm{d}g$. The 3-form H is closed and with an appropriate normalization of the metric, it has integer periods, $\frac{1}{2\pi}[H] \in \mathrm{H}^3(G, \mathbb{Z})$. Equipped with the manifold B, the extension Φ and the 3-form H, the Wess-Zumino term is defined by

$$S_{\mathrm{WZ}}(\phi) := k \int_B \Phi^* H \tag{3}$$

for some $k \in \mathbb{R}$ playing the role of a coupling constant. However, this expression is not independent of the choices of B and Φ: its has ambiguities of the form $2\pi k\mathbb{Z}$. In turn, also the action (1) of the WZW model has these ambiguities.

This is not a problem as long as the associated amplitude in Feynman's path integral is unambiguously well-defined. Instead of the Wess-Zumino term itself, the physically relevant quantity is its exponential

$$\mathcal{A}(\phi) := \exp\left(\mathrm{i}S_{\mathrm{WZ}}(\phi)\right). \tag{4}$$

Since $\exp(2\pi\mathrm{i}\mathbb{Z}) = 1$, the well-definedness of the amplitude $\mathcal{A}(\phi)$ requires the quantization condition $k \in \mathbb{Z}$ on the coupling constant k. As explained in [GR02], this can be understood analogously to Dirac's quantization condition on the electric charge, the coupling constant for electrodynamics. The integer k is called the level of the WZW model.

Summarizing, the definition of the Wess-Zumino term we just gave depends seriously on two conditions: the target space G has to be 2-connected ($\pi_0(G) = \pi_1(G) = \pi_2(G) = 0$) to guarantee the existence of the extension Φ, and the worldsheet Σ has to be oriented, to give sense to the integrals in (1) and (3). The aim of my talk was to describe ideas to overcome both conditions. Especially the conditions on the target space evoked the by now well-established use of gerbes in conformal field theory, which we shall describe in the next section.

Gerbes and their Holonomy

In [Gaw88], the definition of the Wess-Zumino-term was generalized, such that the amplitude $\mathcal{A}(\phi)$ from (4) can be defined on target spaces being any smooth manifold M. The main insight was to realize this amplitude as the holonomy of a Deligne hypercohomology class on M around the map $\phi : \Sigma \to M$. More recently, the appropriate differential-geometric object having such holonomies has been identified as a hermitian $U(1)$ bundle gerbe with connection and curving [CJM02]. For the purpose of this talk, we stick to the simplest definition of such gerbes: the one by local data.

Let open subsets V_i cover the smooth manifold M. Local data with respect to the subsets V_i can for instance be used to describe a hermitian line bundle with connection: it has local data $L = (g_{ij}, A_i)$ with smooth functions

$$g_{ij} : U_i \cap U_j \to U(1) \tag{5}$$

and 1-forms $A_i \in \Omega^1(U_i)$, such that the cocycle conditions

$$\begin{aligned} g_{jk} \cdot g_{ik}^{-1} \cdot g_{ij} &= 1 \qquad \text{on} \quad U_i \cap U_j \cap U_k \\ A_j - A_i + g_{ij}^{-1}\mathrm{d}g_{ij} &= 0 \qquad \text{on} \quad U_i \cap U_j \end{aligned} \tag{6}$$

are satisfied. Two collections $L = (g_{ij}, A_i)$ and $L' = (g'_{ij}, A'_i)$ of local data correspond to isomorphic line bundles, if and only if there exist smooth functions $h_i : U_i \to U(1)$ such that

$$\begin{aligned} g'_{ij} &= g_{ij} \cdot h_j \cdot h_i^{-1} \qquad \text{on} \quad U_i \cap U_j \\ A'_i &= A_i + h_i^{-1}\mathrm{d}h_i \qquad \text{on} \quad U_i. \end{aligned} \tag{7}$$

Recall further that a line bundle (g_{ij}, A_i) defines a global 2-form $\mathrm{curv}(L) := \mathrm{d}A_i \in \Omega^2(M)$ (the curvature) and a characteristic class $[g_{ij}] \in \mathrm{H}^2(M, \mathbb{Z})$ (the Chern class).

A gerbe over M with respect to the open cover V_i is defined analogously: we consider collections $\mathcal{G} = (g_{ijk}, A_{ij}, B_i)$ of smooth functions

$$g_{ijk} : U_i \cap U_j \cap U_k \to U(1), \tag{8}$$

1-forms $A_{ij} \in \Omega^1(U_i \cap U_j)$ and 2-forms $B_i \in \Omega^2(U_i)$, such that the cocycle conditions

$$\begin{aligned} g_{jkl} \cdot g_{ikl}^{-1} \cdot g_{ijl} \cdot g_{ijk}^{-1} &= 1 \qquad \text{on} \quad U_i \cap U_j \cap U_k \cap U_l \\ A_{jk} - A_{ik} + A_{ij} - g_{ijk}^{-1}\mathrm{d}g_{ijk} &= 0 \qquad \text{on} \quad U_i \cap U_j \cap U_k \\ B_j - B_i + \mathrm{d}A_{ij} &= 0 \qquad \text{on} \quad U_i \cap U_j \end{aligned} \tag{9}$$

are satisfied. Two gerbes $\mathcal{G} = (g_{ijk}, A_{ij}, B_i)$ and $\mathcal{G}' = (g'_{ijk}, A'_{ij}, B'_i)$ are defined to be isomorphic, if there exist smooth functions $h_{ij} : U_i \cap U_j \to$

$U(1)$ and 1-forms $M_i \in \Omega^1(U_i)$ such that

$$\begin{aligned} g'_{ijk} &= g_{ijk} \cdot h_{jk} \cdot h_{ik}^{-1} \cdot h_{ij} && \text{on} \quad U_i \cap U_j \cap U_k \\ A'_{ij} &= A_{ij} + M_j - M_i + h_{ij}^{-1} \mathrm{d}h_{ij} && \text{on} \quad U_i \cap U_j \\ B'_i &= B_i + \mathrm{d}M_i && \text{on} \quad U_i. \end{aligned} \tag{10}$$

Similar to the curvature and the characteristic class of a line bundle, a gerbe $\mathcal{G} = (g_{ijk}, A_{ij}, B_i)$ defines a curvature 3-form $\mathrm{curv}(\mathcal{G}) := \mathrm{d}B_i \in \Omega^3(M)$ and a characteristic class $[g_{ijk}] \in \mathrm{H}^3(M, \mathbb{Z})$ which is called the Dixmier-Douady class of the gerbe. Very much like the curvature of a line bundle, the 3-form $\mathrm{curv}(\mathcal{G})$ is closed and has integer periods, $\frac{1}{2\pi}[\mathrm{curv}(\mathcal{G})] \in \mathrm{H}^3(M, \mathbb{Z})$.

Let us now describe how a gerbe $\mathcal{G} = (g_{ijk}, A_{ij}, B_i)$ over M gives rise to holonomies around maps $\phi : \Sigma \to M$, where Σ is a closed oriented surface. For this purpose, we pull back the gerbe along ϕ and obtain a gerbe

$$\phi^*\mathcal{G} = (\phi^* g_{ijk}, \phi^* A_{ij}, \phi^* B_i) \tag{11}$$

over Σ with respect to the pullback cover consisting of the open subsets $\phi^{-1}(V_i)$. Next, we choose a triangulation of Σ subordinate to the open subsets $\phi^{-1}(V_i)$, so that we have for each face Δ, each edge e and each vertex v of the triangulation indices $i(\Delta)$, $i(e)$ and $i(v)$ labelling open subsets such that $\phi(\Delta) \subset V_{i(\Delta)}$, and so on. Since Σ was assumed to be oriented, every face Δ is also equipped with an orientation, and in turn induces an orientation on each of his boundary edges $e \in \partial\Delta$. Equipped with all this data, we integrate the 2-forms B_i over the faces, the 1-forms A_{ij} over the edges, evaluate the functions g_{ijk} on the vertices, and sum these values all together in the following way:

$$\begin{aligned} \mathrm{hol}_{\mathcal{G}}(\phi) \quad := \quad & \prod_{\Delta} \exp\left(\int_{\Delta} \phi^* B_{i(\Delta)} \right) \\ & \cdot \prod_{e \in \partial\Delta} \exp\left(\int_e \phi^* A_{i(\Delta)i(e)} \right) \cdot \prod_{v \in \partial e} \phi^* g_{i(\Delta)i(e)i(v)}(v). \end{aligned} \tag{12}$$

The result is a number $\mathrm{hol}_{\mathcal{G}}(\phi) \in U(1)$, which happens to be independent of the choice of the triangulation and of the indices. Using Stokes' theorem, one can show that it is also independent under the change of local data to an isomorphic gerbe. It is therefore intrinsically associated to the isomorphism class of the gerbe $\mathcal{G}$ over M and the map $\phi : \Sigma \to M$ and so it is called the holonomy of $\mathcal{G}$ around ϕ.

Among many properties of the surface holonomy (12) such as gluing properties, there is one which relates it to the Wess-Zumino term we discussed in the first section. If B is any oriented 3-dimensional manifold with boundary Σ, and $\Phi : B \to M$ is any smooth map, the surface holonomy of a gerbe $\mathcal{G}$ over M satisfies

$$\mathrm{hol}_{\mathcal{G}}(\Phi|_{\Sigma}) = \exp\left(\mathrm{i} \int_B \Phi^* \mathrm{curv}(\mathcal{G}) \right). \tag{13}$$

Now assume $M = G$ is a simple compact Lie group, and assume that there exists a gerbe $\mathcal{G}$ over G with curvature

$$\mathrm{curv}(\mathcal{G}) = kH, \tag{14}$$

where H is the canonical 3-form from (2), and $k \in \mathbb{Z}$. In this case, the right hand side of (13) coincides with the exponential of the Wess-Zumino-term (3),

$$\mathrm{hol}_{\mathcal{G}}(\phi) = \exp\left(\mathrm{i}S_{WZ}(\phi)\right). \tag{15}$$

Indeed, gerbes over simple compact Lie groups with curvature kH always exist and can even be constructed explicitly [GR02, Mei02, GR03]. Notice that the equality (15) is not just a reformulation of the amplitude $\mathcal{A}(\phi)$: the definition of the Wess-Zumino term $S_{WZ}(\phi)$ was strongly depending on the simply-connectedness of the Lie group G, while the holonomy $\mathrm{hol}_{\mathcal{G}}(\phi)$ is defined for manifolds of arbitrary topology.

Summarizing, gerbes provide a possibility to generalize the Wess-Zumino term in cases where its definition is not possible due to topological obstructions. While first the integer k seemed to be the only degree of freedom in the definition of $S_{WZ}(\phi)$, it now appears that there is one Wess-Zumino term for each gerbe over G with curvature kH. For the simply-connected groups, this fits together by the fact that there is only *one* such gerbe for each level k. But already the case of non-simply connected Lie groups with non–cyclic fundamental group, such as $G := Spin(4n)/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ shows that gerbes and their holonomy are really indispensable. Here there exist two non-isomorphic gerbes for each level k, which explains the well-established fact that to such a group *two* different rational conformal field theories that differ by “discrete torsion” can be associated.

Unoriented Worldsheets

We have discussed in the previous section how to reformulate the Wess-Zumino term in topologically non-trivial situations by interpreting it as the holonomy of a gerbe over G around a map $\phi : \Sigma \to G$. However, both definitions of the Wess-Zumino term depend on the existence and on the choice of an orientation on the worldsheet Σ. In fact there are surfaces which do not admit orientations at all, for example the Klein bottle or the Möbius strip. Nonetheless, a long series of algebraic results indicate that the WZW model can be consistently considered on such unorientable surfaces. Early results include a detailed study of the abelian case [BPS92] and of $SU(2)$ [PSS95b, PSS95a]. Sewing constraints for unoriented surfaces have been derived in [FPS94]. Already the abelian case [BPS92] shows that not every rational conformal field theory that is well-defined on oriented surfaces can be considered on unoriented surfaces. Moreover, if the theory can be extended to unoriented surfaces, there can be different extensions that yield inequivalent correlation functions. Aspects of these results have been proven in [FRS04] combining topological field theory in three-dimensions with algebra and representation theory in modular tensor categories. As a crucial ingredient, a generalization of the notion of an algebra with involution, i.e. an algebra together with an algebra-isomorphism to the opposed algebra, has been identified in [FRS04].

The success of the algebraic theory leads, for WZW models, to the quest for corresponding geometric structures on the target space. From previous work [BCW01, HSS02, Bru02] it is clear that a map

$$k : M \to M \tag{16}$$

on the target space with the additional property that

$$k^*H = -H \tag{17}$$

will be one ingredient. Examples like the Lie group $SO(3)$, for which four different unoriented WZW models with the same map k are known, already show that this structure does not suffice.

We are thus looking for an additional structure on a gerbe which allows to define a Wess-Zumino term, i.e. which allows to define holonomy for unoriented surfaces. For an arbitrary gerbe, such a structure need not exist; if it exists, it will not be unique. In [SSW05] we have defined such a structure for a gerbe $\mathcal{G}$ and we called it a Jandl structure. To write down the local data of a Jandl structure for a given involution $k : M \to M$ in a succinct manner, we make the simplifying assumption that we have an open cover V_i of M that is invariant under k, i.e. $k(V_i) = V_i$. If the gerbe is given by local data (g_{ijk}, A_{ij}, B_i), a Jandl structure is a collection (j_i, t_{ij}, W_i) consisting of smooth functions

$$j_i : V_i \to U(1) \quad \text{and} \quad t_{ij} : V_i \cap V_j \to U(1), \tag{18}$$

and 1-forms $W_i \in \Omega^1(V_i)$. They have to relate the pullbacks of the gerbe data under k to the gerbe data itself as follows:

$$\begin{aligned} k^*B_i &= -B_i + \mathrm{d}W_i && \text{on} \quad U_i \\ k^*A_{ij} &= -A_{ij} - \mathrm{dlog}(t_{ij}) + W_j - W_i && \text{on} \quad U_i \cap U_j \\ k^*g_{ijk} &= g_{ijk}^{-1} \cdot t_{jk} \cdot t_{ik}^{-1} \cdot t_{ij} && \text{on} \quad U_i \cap U_j \cap U_k. \end{aligned} \tag{19}$$

Notice that the derivative of the first equation gives for the curvature $H := \mathrm{curv}(\mathcal{G}) := \mathrm{d}B_i$ exactly the property (17). The local data of a Jandl structure are required to be equivariant under k in the sense that

$$\begin{aligned} k^*t_{ij} &= t_{ij} \cdot j_j^{-1} \cdot j_i && \text{on} \quad U_i \cap U_j \\ k^*W_i &= W_i - \mathrm{dlog}(j_i) && \text{on} \quad U_i \cap U_j \\ k^*j_i &= j_i^{-1} && \text{on} \quad U_i \cap U_j. \end{aligned} \tag{20}$$

We say that two Jandl structures (j_i, t_{ij}, W_i) and (j'_i, t'_{ij}, W'_i) are equivalent, if there exists a function $\nu_i : V_i \to U(1)$ such that

$$\begin{aligned} t'_{ij} &= t_{ij} \cdot \nu_j^{-1} \cdot \nu_i && \text{on} \quad U_i \cap U_j \\ W'_i &= W_i - \mathrm{dlog}(\nu_i) && \text{on} \quad U_i \\ j'_i &= j_i \cdot \nu_i^{-1} \cdot k^*\nu_i && \text{on} \quad U_i. \end{aligned} \tag{21}$$

Now we can classify the equivalence classes of Jandl structures. It is a consequence of the conditions (19), that the collection (g_{ij}, A_i) defined by

$$g_{ij} := t'_{ij} \cdot t_{ij}^{-1} \quad \text{and} \quad A_i := W'_i - W_i \tag{22}$$

defines a flat line bundle L over M. Furthermore, equations (20) indicate that the function $j'_i \cdot j_i^{-1}$ defines a k-equivariant structure on this line bundle: an isomorphism

$$\varphi : k^*L \to L \tag{23}$$

of flat line bundles which satisfies $k^*\varphi = \varphi^{-1}$. If the two Jandl structures are equivalent, the line bundle L is trivializable. Thus, equivalence classes of Jandl structures on a gerbe $\mathcal{G}$ with involution k are classified by the group $\mathrm{Pic}_0^k(M)$ of isomorphism classes of flat k-equivariant line bundles over M. This is an important result, because this group is canonically isomorphic to the equivariant cohomology group

$$\mathrm{Pic}_0^k(M) \cong \mathrm{H}_k^1(M, U(1)), \tag{24}$$

which can be calculated in concrete situations.

We have proven in [SSW05] that a gerbe with Jandl structure has a well-defined notion of holonomy around unoriented surfaces: we derive a formula for local data, which generalizes the holonomy formula (12). The basic idea of this generalization is, to triangulate the worldsheet as before, and now – failing which a global orientation – to equip each face with an individual choice of a *local* orientation. Along the edges between faces with different local orientations, we integrate the pullback of the local data of the Jandl structure. Due to the relations (20) and (21), the result is finally independent of the choices of the local orientations.

The notion of a Jandl structure naturally explains algebraic results for specific classes of rational conformal field theories. Let us shortly explain this in the example of the Lie group $SU(2)$ and its quotient $SO(3)$, to each of them we can associate four different unoriented theories. In the case of $SU(2)$, this is explained by the fact that two different involutions are relevant:

$$k : g \mapsto g^{-1} \quad \text{and} \quad k : g \mapsto zg^{-1}, \tag{25}$$

where z is the non-trivial element in the center of $SU(2)$. Now, since $SU(2)$ is simply-connected, we find $\mathrm{H}_k^1(SU(2), U(1)) = \mathbb{Z}_2$ for both involutions and hence for each of the two involution two Jandl structures, that gives all together four theories. But the two involutions of $SU(2)$ descend to the same involution of the quotient $SO(3)$. The latter manifold, however, has fundamental group $\mathbb{Z}_2$ and thus twice as many equivariant flat line bundles as $SU(2)$. The different Jandl structures of $SO(3)$ are therefore not explained by different involutions on the target space but rather by the fact that one involution admits four different Jandl structures.

Obstructions and classification results for Jandl structures on gerbes over all other compact simple Lie groups have recently explicitly been derived

[GSW07]. In this article we reduce the study of Jandl structures on gerbes over compact simple Lie groups to the study of certain equivariant Jandl structures on gerbes over their universal covering groups. For such simply-connected Lie groups, obstruction classes and classifications results can be derived using the theory of finite group cohomology. This gives a systematic and complete overview over all unoriented closed WZW models on simple compact Lie groups.

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