Classical String in Curved Backgrounds*

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Abstract

The Mathisson-Papapetrou method is originally used for derivation of the particle world line equation from the covariant conservation of its stress-energy tensor. We generalize this method to extended objects, such as a string. Without specifying the type of matter the string is made of, we obtain both the equations of motion and boundary conditions of the string. The world sheet equations turn out to be more general than the familiar minimal surface equations. In particular, they depend on the internal structure of the string. The relevant cases are classified by examining canonical forms of the effective 2-dimensional stress-energy tensor. The case of homogeneously distributed matter with the tension that equals its mass density is shown to define the familiar Nambu-Goto dynamics. The other three cases include physically relevant massive and massless strings, and unphysical tachyonic strings.

1. Introduction

The original motivation for introducing strings in particle physics came from the analysis of meson resonances. As it appears, the known resonances follow the Regge trajectories pattern. In order to explain that, the meson resonances are viewed as excited 2-quark bound states. It has been shown that relativistic rotating string with light quarks attached to its ends indeed reproduces the above pattern. The string is characterized by the tension alone, and has no other structure. It was realized later that realistic field configurations with such properties really exist (for example, [1]).

Our motivation for considering stringy shaped matter in curved backgrounds is twofold. First, as we have already explained, realistic strings (like flux tubes) are really believed to exist, and to be relevant for the description of hadronic matter. Second, the basic Nambu-Goto string action [2, 3] is in literature often modified to include interaction with additional background fields. Apart from the target-space metric, the antisymmetric tensor field $B_{\mu\nu}(x)$ and the dilaton field $\Phi(x)$ are considered [4, 5, 6, 7]. While spacetime metric has obvious geometric interpretation, the background fields $B_{\mu\nu}(x)$ and $\Phi(x)$ do not. The attempts have been made to

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interpret $B_{\mu\nu}$ and Φ as originating from the background torsion and nonmetricity, respectively [8, 9, 10, 11, 12]. Thus it seems that string dynamics in target-spaces of general geometry is worth considering.

In this paper, we shall restrict our considerations to the simplest case of purely Riemannian spacetime. Thus, the geometry is given in terms of the metric tensor alone, and the dynamics is governed by the Einstein's equations which imply that the stress-energy tensor of matter fields is symmetric, $T^{\mu\nu} = T^{\nu\mu}$, and covariantly conserved, $\nabla_\nu T^{\mu\nu} = 0$. This can be rewritten in the more suitable form

$$\partial_\nu(\sqrt{-g}\, T^{\mu\nu}) + \Gamma^\mu{}_{\rho\nu}\sqrt{-g}\, T^{\rho\nu} = 0, \tag{1}$$

and will be the starting point in our analysis of motion of extended objects in curved spacetime.

The method we use is a straightforward generalization of the Mathisson-Papapetrou method for pointlike matter [13, 14]. It boils down to the analysis of the covariantly conserved stress-energy tensor of matter fields, without specifying their nature. The basic assumption used is the existence of a stringlike localized kink solution in a curved background. Then, the world sheet effective equations of motion are obtained in the approximation of an infinitely thin string.

The layout of the paper is as follows. In Sec. 2., the point particle is considered as a demonstration of our method and conventions. The known result is reproduced, but the emphasis is put on the fact that the mass parameter transforms as 1-dimensional stress-energy tensor. In Sec. 3., the effective world sheet equations are derived from the covariant conservation law of the stress-energy tensor of matter fields. Instead of the mass parameter in the point particle case, the effective 2-dimensional stress-energy tensor m^{ab} appears to characterize the internal structure of the string, and gives rise to different equations of motion. If the string is open, the world sheet equations also include some boundary conditions. Sec. 4. is devoted to the analysis of possible canonical forms of m^{ab}. In Sec. 5. we give our final remarks.

Our conventions are as follows. Greek indices from the middle of the alphabet, $\mu, \nu, \ldots$, are the target-space indices, and run over $0, 1, 2, 3$. Latin indices $a, b, \ldots$ are the world sheet indices and run over $0, 1$. The target-space and world sheet coordinates are denoted by x^μ and ξ^a, respectively. The target-space and world sheet metric tensors are denoted by $g_{\mu\nu}(x)$ and $\gamma_{ab}(\xi)$, respectively. The signature convention is defined by $\eta_{\mu\nu} = \mathrm{diag}\,(1, -1, \ldots, -1)$, $\eta_{ab} = \mathrm{diag}\,(1, -1)$.

2. Particle dynamics

We begin with the treatment of a point particle in a curved background spacetime. The problem was studied in the early days of relativity by Einstein, Infeld, Hoffman, Mathisson, Papapetrou and others [13, 14, 15, 16, 17, 18]. Here, we formalize the calculations, and adjust the algorithm for the case of a string in the next section.

First, we need a general form of the stress-energy tensor, suitable for the description of a point particle. Let us introduce a timelike curve $x^\mu = z^\mu(\tau)$ in spacetime, with τ an arbitrary parameter. Expand $T^{\mu\nu}(x)$ into the δ function series around the point $\vec{x} = \vec{z}(\tau)$, using the formalism given in the Appendix of [19]:

$$
\begin{aligned}
T^{\mu\nu}(t,\vec{x}) \;=\; & \int d\tau\; b^{\mu\nu}(\tau) \frac{\delta^{(4)}(x - z(\tau))}{\sqrt{-g}} + \\
& + \int d\tau\; b^{\mu\nu\rho}(\tau) \nabla_\rho \frac{\delta^{(4)}(x - z(\tau))}{\sqrt{-g}} + \ldots
\end{aligned}
$$

Now, we introduce the basic assumption about matter. It is localized around the line $z^\mu(\tau)$, ie. the *stress-energy tensor drops exponentially to zero as we move away from the line.* As a consequence of this assumption, each coefficient in the expansion is smaller then previous ones. In the lowest approximation (the so called *single pole* approximation), all b-s except the first are neglected, and we end up with

$$
\sqrt{-g}\, T^{\mu\nu}(x) = \int d\tau\; b^{\mu\nu}(\tau)\, \delta^{(4)}(x - z(\tau)) \,. \tag{2}
$$

Since this equation is covariant, we can infer the transformation properties of $b^{\mu\nu}$. It is a tensor with respect to general coordinate transformations, and scalar with respect to world line reparametrizations.

Equation (2) describes matter localized around the line $z^\mu(\tau)$. Now we look for the solution of (1) in this form, where $b^{\mu\nu}(\tau)$ and $z^\mu(\tau)$ are the unknown functions to be determined. Using the procedure explained in detail in [19], we obtain two resulting equations. One determines $b^{\mu\nu}$ as a function of $z^\mu(\tau)$ and an arbitrary coefficient $m(\tau)$:

$$
b^{\mu\nu} = m\, u^\mu\, u^\nu \,. \tag{3}
$$

The other is a differential equation for $z^\mu(\tau)$:

$$
\frac{d}{d\tau}\,(m u^\mu) + m \Gamma^\mu{}_{\rho\nu} u^\rho u^\nu = 0 \,, \tag{4}
$$

which determines the world line. It contains the undetermined $m(\tau)$, but this coefficient is constrained by the very same equation. Indeed, the projection of (4) on the tangent vector u^μ can straightforwardly be brought to the simple form

$$
\frac{dm}{d\tau} = 0 \,. \tag{5}
$$

We see that m is a constant of motion, and consequently, it can easily be eliminated from the world line equation, which becomes the standard geodesic equation.

The world line equation we have obtained is manifestly covariant with respect to both general coordinate transformations and world line reparametrizations. Thus it is easy to deduce that m, beside being spacetime scalar, transforms as a second rank contravariant tensor with respect to world line reparametrizations. This gives us the idea that m can be viewed as an effective, conserved, one-dimensional stress-energy tensor of the pointlike matter. In this respect, m should be considered the particle mass.

All the above results are obtained in the lowest approximation in the δ function expansion. Keeping the second term (*pole-dipole* approximation), or higher order terms, one finds that the world line equation would give deviations from the geodesic trajectory, as has been studied extensively in the literature (see, for example [14]). Here, we just prepare the setting for the study of string dynamics in the next section.

3. String dynamics

The calculations presented in the previous section are well known, and there are papers [20, 21] which generalize the procedure to include torsion etc. However, this research has been focused on the particle case, and we want to address the problem of finding equations of motion of an extended object, such as string. In this section, we generalize the Papapetrou method to linelike matter.

In contrast to the particle, the string is an extended, one-dimensional object whose trajectory is not a world line, but rather a two-dimensional world sheet $\mathcal{M}$. Let us introduce a two-dimensional surface $x^\mu = z^\mu(\xi^a)$ in spacetime, where ξ^0 and ξ^1 are the surface coordinates. We shall also frequently use the notion of the world sheet coordinate vectors and induced metric tensor:

$$u_a^\mu \equiv \frac{\partial z^\mu}{\partial \xi^a}\,, \qquad \gamma_{ab} = g_{\mu\nu} u_a^\mu u_b^\nu\,.$$

First we expand the stress-energy tensor into a δ function series around the world sheet. In the single-pole approximation, we drop all the terms in the expansion except the leading one, and obtain an expression:

$$\sqrt{-g}\, T^{\mu\nu}(x) = \int d^2\xi \sqrt{-\gamma}\, b^{\mu\nu}(\xi)\, \delta^{(4)}(x - z(\xi))\,. \tag{6}$$

The coefficients $b^{\mu\nu}$ transform covariantly with respect to both target-space and world sheet reparametrizations. Having this, the equation (1) can be solved with respect to the unknowns $b^{\mu\nu}(\xi)$ and $z^\mu(\xi)$. Using the procedure explained in [19], (1) decouples to three equations.

The first one gives a solution for $b^{\mu\nu}(\xi)$ in terms of $z^\mu(\xi)$:

$$b^{\mu\nu} = m^{ab} u_a^\mu u_b^\nu\,. \tag{7}$$

Here, $m^{ab}(\xi)$ are arbitrary coefficients. They transform as scalars with respect to spacetime diffeomorphisms, and as components of a contravariant symmetric second rank tensor with respect to the world sheet reparametrizations.

The second one is the boundary condition:

$$\sqrt{-\gamma}\, m^{ab}\, n_b\, u_a^\mu\Big|_{\partial\mathcal{M}} = 0. \tag{8}$$

Here, n_a is the outward directed normal to the boundary $\partial\mathcal{M}$. The boundary conditions do not appear if the string is closed, because in that case $\partial\mathcal{M} = \emptyset$.

The third is a differential equation for $z^\mu(\xi)$, and determines the world sheet of the string:

$$\nabla_a(m^{ab}u_b^\mu) = 0. \tag{9}$$

Here we make use of the *total covariant derivative* ∇_a, which acts on both spacetime and world sheet indices (for the latter the induced connection $\Gamma^a{}_{bc}$ is used). Viewed as an equation for the string trajectory, this equation contains the unknown coefficients m^{ab}. It can be shown, however, that m^{ab} are not fully arbitrary. Instead, they are constrained by the very same equation. To see this, we contract (9) with u_μ^c, to obtain

$$\nabla_a m^{ac} = 0. \tag{10}$$

Thus, m^{ab} is covariantly conserved, symmetric world sheet tensor. As such, it is seen as the effective two-dimensional stress-energy tensor of the string.

We need to remark that the boundary conditions obtained in this section are naturally associated with the familiar Neumann boundary conditions of the conventional string theory. The alternative Dirichlet boundary conditions are defined by imposing additional constraints on the variation of the string boundary. Precisely, the string ends are attached to an external p-brane, which (partially or fully) fixes their trajectories. However, the interaction of the string with the p-brane violates the covariant conservation of the stress-energy tensor at the string ends. The natural way to incorporate Dirichlet boundary conditions is to consider the p-brane and the attached string as a single object moving in an external gravitational field, but that is out of scope of this paper. Instead, we assume that the string has nothing else to interact with, which in turn explains why the derivation of the equations of motion automatically gives also Neumann boundary conditions.

Next, as opposed to the particle case, the dynamics of a stringy shaped matter generally depends on its internal structure. Indeed, the two-dimensional stress-energy conservation $\nabla_a m^{ab} = 0$ has no unique solution. There is a variety of possibilities to choose m^{ab}, each leading to a different string dynamics.

Let us note, in the end of this section, that it is possible to extend the whole discussion to a very general case of a p-brane moving in a D-dimensional curved spacetime. The equations of motion and boundary conditions are virtually the same, the only difference being bigger sets of values for world sheet and target-space indices.

4. Internal structure of the string

As we have seen in the previous section, the world sheet equations depend on the type of matter the string is made of. To completely characterize the string trajectory, we need the type and distribution of its mass tensor m^{ab}. In this section, we shall classify possible canonical forms of m^{ab}.

Let us analyze the eigenproblem of the two-dimensional mass tensor m^{ab}. The analogous 4-dimensional analysis has been done in [22]. The eigenproblem of m^{ab} in a general world sheet with metric γ_{ab}, is defined by the equation

$$m^{ab}e_b = \lambda e^a,$$

where $e^a \equiv \gamma^{ab}e_b$. The existence of nonvanishing eigenvectors e^a is guaranteed by the condition $\det[m^{ab} - \lambda\gamma^{ab}] = 0$. It is rewritten as the quadratic equation

$$\lambda^2 - m^a{}_a\lambda + \gamma \det[m^{ab}] = 0,$$

with the discriminant

$$\Delta \equiv (m^a{}_a)^2 - 4\gamma \det[m^{ab}].$$

Because of the indefiniteness of the metric, three cases are possible: $\Delta > 0$, $\Delta = 0$, and $\Delta < 0$. The eigenvectors can be either timelike, spacelike or null, and the mass tensor m^{ab} cannot always be diagonalized.

Let us analyze the behavior of m^{ab} in the vicinity of a point on the world sheet. We shall use such ξ^a coordinates which ensure $\gamma_{ab} = \eta_{ab}$, and $\Gamma^a{}_{bc} = 0$ in the chosen point. If we write m^{ab} in a matrix form as

$$m^{ab} = \begin{pmatrix} \rho & \pi \\ \pi & p \end{pmatrix},$$

we see that ρ represents energy density along the string, π is the energy flux, and $-p$ is the string tension. The components of the stress-energy tensor are subject to the physical condition that energy flux must not exceed the energy density. Otherwise, matter would travel faster than light [22]. This must be satisfied in every reference frame, which can be shown to imply the general conditions on the components of m^{ab}:

$$\rho + p \geq 2|\pi|, \qquad \rho \geq p. \tag{11}$$

In the case $\Delta > 0$, one can employ a Lorentz transformation that brings m^{ab} to a diagonal form:

$$m^{ab} = \begin{pmatrix} \lambda^{(1)} & 0 \\ 0 & -\lambda^{(2)} \end{pmatrix}, \qquad \lambda^{(1)} \geq |\lambda^{(2)}|, \qquad \lambda^{(1)} \neq \lambda^{(2)},$$

where $\lambda^{(1)}$ and $\lambda^{(2)}$ are the eigenvalues of m^{ab}. This means that there exists a *rest frame*, where the energy flux is zero, $\pi = 0$, and matter does not move. This is the case for the usual massive matter. The energy density ρ is always positive, and exceeds the absolute value of the tension.

In the case $\Delta = 0$, there exists a boost that brings m^{ab} to the form

$$m^{ab} = \begin{pmatrix} \lambda + \mu & \mu \\ \mu & -\lambda + \mu \end{pmatrix}.$$

Here, $\lambda \geq 0$ is the single eigenvalue, and the Lorentz invariant sign of μ defines two subcases: $\mu > 0$ and $\mu = 0$. Thus, every nontrivial m^{ab} is the sum of matrices corresponding to the cases $\lambda = 0$, $\mu > 0$ and $\lambda > 0$, $\mu = 0$. Let us discuss these two situations in turn.

In the case $\lambda = 0$, $\mu > 0$, the only eigenvector of m^{ab} is lightlike, and no rest frame exists. The situation is interpreted as that of a *massless matter*. Energy density is positive, and equal to both energy flux and pressure. The boundary is lightlike.

In the case $\lambda > 0$, $\mu = 0$, the mass tensor is not only diagonal, but proportional to the metric: $m^{ab} = \lambda\eta^{ab}$. This can covariantly be written as $m^{ab} = \lambda\gamma^{ab}$, and defines the known *Nambu-Goto string*. The energy density is positive and equal to the tension. The boundary conditions reduce to well known Neumann boundary conditions which imply that the boundary is lightlike.

The case $\Delta < 0$ is in contradiction with the conditions (11), and one can always find a reference frame where energy flux exceeds the energy density. Thus, the case is unphysical, corresponding to tachyonic matter.

5. Concluding remarks

The analysis in this paper concerns the dynamics of realistic material strings in curved backgrounds. In the simple case we have considered, the background geometry is Riemannian, defined in terms of the metric tensor alone. The dynamics of geometry and matter fields is governed by the Einstein's equations.

In the specific setting considered, we assume the existence of a stable string-like kink configuration. The type of matter fields involved is not specified. We only assume that matter fields are sharply localized around a line, while geometry itself is not.

The method used is, basically, the Mathisson-Papapetrou method for point-like matter [13, 14] generalized to linelike configurations. The world sheet equations are obtained in the lowest order ie. single-pole approximation.

The results of our analysis can be summarized as follows. The dynamics of a stringy shaped matter in torsionless spacetimes generally depends on the internal structure of the string. The coefficients m^{ab} entering the world sheet equations are the components of the covariantly conserved effective 2-dimensional stress-energy tensor of the string. As opposed to the point particle case, m^{ab} can not generally be eliminated by world sheet reparametrizations. The diversity of possible forms of m^{ab} has been analyzed, and various types of matter have been found. Among others, the case of homogeneously distributed matter whose tension equals its mass density gives the known Nambu-Goto string dynamics.

In closing our exposition, let us mention again that our main result can easily be generalized to include arbitrary p-brane distribution of matter. The corresponding world sheet equations are of the same form, but this time $a, b = 0, 1, \ldots, p$, and m^{ab} is the covariantly conserved $(p + 1)$-dimensional energy-momentum tensor of the brane. Obviously, the diversity of possible forms of m^{ab} is bigger than in the string case.

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