Supertwistor formulation for the first-order superstring action

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Abstract

The Lagrangian formulation of the $D = 4$ bosonic string and superstring in terms of the (super)twistors is considered. It is shown that for the proposed superstring action fixing covariantly the $\kappa-$symmetry gauge results in the action quadratic in the (super)twistor variables.

1. Introduction

Despite well-known achievements of twistor theory [1] in mathematical physics until recently its application to the tensile string theory had got relatively poor attention (see, however, some earlier contributions to that subject [2]-[5]). Situation changed after Witten's proposal [6] of the topological string model in projective supertwistor space dual to certain sectors of $N = 4$ superYang-Mills and conformal supergravity theories in 4 dimensions. Since then considerable progress has been achieved in understanding perturbative properties of those and other supersymmetric theories using spinor methods (see [7] for review), as well as, other twistor string models were constructed [8]-[11]. Our activity is in a sense complementary to the above mentioned in that it deals with (super)twistor theory application to the description of conventional (super)string models [12]. The main goal of this contribution is to describe how starting from known Polyakov or Green-Schwarz (super)string formulations it is possible to derive (super)string actions quadratic in terms of (super)twistor variables [13].

2. Some twistor basics

We start by considering some very elementary features of twistors that will be needed for our purposes. In this note we will deal only with the flat twistor space $\mathbb{T}$ isomorphic to $\mathbb{C}^4$ and parametrized by coordinates Z^a $(a = 1, ..., 4)$, transforming in the **4** representation of $PGL(4, \mathbb{C}) = SL(4, \mathbb{C})/\mathbb{Z}_4$ the complexified $4d$ conformal group, that are conventionally split into $SL(2, \mathbb{C})$ 2-component spinors as

$$Z^a = (\mu^\alpha, \bar{u}_{\dot{\alpha}}), \tag{1}$$

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where, according to Penrose [1], μ^α is called the primary spinor part of the twistor Z^a and $\bar{u}_{\dot\alpha}$ its projectional part.

A point in 4-(complex)dimensional space-time $\mathbb{CM}$ with coordinates $y^{\dot\alpha\alpha}$ is defined to be incident to the twistor Z^a if

$$\mu^\alpha = \bar{u}_{\dot\alpha} y^{\dot\alpha\alpha}. \tag{2}$$

This incidence relation is homogeneous w.r.t. complex scalings of the twistor components hence it appears to be useful to introduce projective twistor space $\mathbb{PT} = \mathbb{CP}^3$, each point of which corresponds to equivalence class $[Z^a] = \{\lambda Z^a,\ \lambda \in \mathbb{C} - 0\}$. Incidence relation (2) implies that the space-time point corresponds to 2-dimensional linear subspace in $\mathbb{T}$ or a complex projective line $\mathbb{CP}^1$ in projective twistor space $\mathbb{PT}$. Alternatively, a point in twistor space is represented by 2-(complex)dimensional $\alpha-$plane, that can be shown to be null and self-dual, in the complexified space-time $\mathbb{CM}$. Since two nonproportional twistors Z^a and $W^a = (\nu^\alpha, \bar{v}_{\dot\alpha})$ define projective line in $\mathbb{PT}$, coordinates of the corresponding space-time point can be expressed in terms of their components as

$$y^{\dot\alpha\alpha} = \frac{1}{\bar{u}^{\dot\beta}\bar{v}_{\dot\beta}}\,(\mu^\alpha \bar{v}^{\dot\alpha} - \nu^\alpha \bar{u}^{\dot\alpha}). \tag{3}$$

This relation holds provided $\bar{u}^{\dot\alpha}$ and $\bar{v}^{\dot\alpha}$ are nonproportional to each other that yields the space-time point with finite coordinates.

Important operation is the complex conjugation $\mathcal{C}$ in space-time $\mathbb{CM}$. Points in $\mathbb{CM}$ invariant under $\mathcal{C}$ are identified as points of the real space-time $\mathbb{M}$. This real space-time can be of various signature depending on the definition of the $\mathcal{C}$ action on 2×2 matrix $y^{\dot\alpha\beta}$ that represents the point in $\mathbb{CM}$. For the space-time of Lorentzian signature $(1,3)$ $\mathcal{C}$ acts as Hermitian conjugation and the matrix $x^{\dot\alpha\beta}$ corresponding to a point in $\mathbb{M}$ is Hermitian. Complex conjugation in $\mathbb{CM}$ also induces conjugation in the twistor space that for Lorentzian signature space-time acts as the antiholomorphic map from $\mathbb{T}$ to the dual twistor space $\mathbb{T}^*$:

$$Z^a \to \bar{Z}_a = G_{a\bar{b}}\overline{Z^b} = (u_\alpha, \bar{\mu}^{\dot\alpha}) \tag{4}$$

with Hermitian metric

$$G_{a\bar{b}} = \begin{pmatrix} 0 & \delta_\alpha^\beta \\ \delta_{\dot\beta}^{\dot\alpha} & 0 \end{pmatrix} \tag{5}$$

invariant under $SU(2,2)$ that is the real form of $SL(4,\mathbb{C})$. With the above definition of $\mathcal{C}$ twistors associated with Lorentzian signature space-time necessarily have complex components.

Two nonproportional twistors Z^a and W^a define the space-time point with real coordinates provided they satisfy four $SU(2,2)$ invariant constraints

$$\bar{Z}_a Z^a = \bar{W}_a W^a = \bar{Z}_a W^a = \bar{W}_a Z^a = 0 \tag{6}$$

that, when written in components, read

$$\begin{aligned} \mu^\alpha u_\alpha + \bar u_{\dot\alpha}\bar\mu^{\dot\alpha} &= 0, & \nu^\alpha v_\alpha + \bar v_{\dot\alpha}\bar\nu^{\dot\alpha} &= 0, \\ \mu^\alpha v_\alpha + \bar u_{\dot\alpha}\bar\nu^{\dot\alpha} &= 0, & \nu^\alpha u_\alpha + \bar v_{\dot\alpha}\bar\mu^{\dot\alpha} &= 0. \end{aligned} \tag{7}$$

Conditions $\bar Z_a Z^a = \bar W_a W^a = 0$ imply that both twistors have zero norm and they are orthogonal $\bar Z_a W^a = \bar W_a Z^a = 0$. Note that the twistor space $\mathbb{T}$ (and also $\mathbb{PT}$) can be divided into 3 subspaces according the twistor norm value $\mathbb{T}^+$ ($\mathbb{PT}^+$): $\bar Z_a Z^a > 0$, $\mathbb{T}^-$ ($\mathbb{PT}^-$): $\bar Z_a Z^a < 0$ and $\mathbb{N}$ ($\mathbb{PN}$): $\bar Z_a Z^a = 0$. So that twistor lines in $\mathbb{PT}$ corresponding to real space-time points are constrained to lie entirely in the $\mathbb{PN}$ subspace. Another facet of twistor geometry for isotropic twistors is that their $\alpha-$planes in $\mathbb{CM}$ contain real null lines that can be parametrized as

$$x^{\dot\alpha\alpha} = x_0^{\dot\alpha\alpha} + \tau \bar u^{\dot\alpha} u^\alpha, \qquad x_0^{\dot\alpha\alpha}, \tau \in R. \tag{8}$$

3. (Super)twistors and massless (super)particle

Let us consider how twistors can be incorporated into the massless particle model. The trajectory of the particle freely moving in 4-dimensional real space-time of $(1,3)$ signature is given by the light-like line with the parameter τ

$$x^{\dot\alpha\alpha} = x_0^{\dot\alpha\alpha} + \tau p^{\dot\alpha\alpha}, \tag{9}$$

where $p^{\dot\alpha\alpha}$ represents particle's momentum that is real and light-like

$$p^{\dot\alpha\alpha} p_{\alpha\dot\alpha} = 0. \tag{10}$$

Identification of the real light-like line in $\mathbb{M}$ corresponding to the isotropic twistor Z^a with particle's trajectory implies that

$$p^{\dot\alpha\alpha} = \bar u^{\dot\alpha} u^\alpha \tag{11}$$

up to an overall sign corresponding to future or past directed momentum. Twistorially speaking $p^{\dot\alpha\alpha}$ is identified with the flagpole associated to the spin-vector u^α. Above momentum representation identically resolves the mass-shell constraint (10). The relation between $p^{\dot\alpha\alpha}$ and u^α is ambiguous in that the choice of real $p^{\dot\alpha\alpha}$ fixes u^α only up to $U(1)$ rotation.

It is now straightforward to incorporate twistor components into the massless particle action written in the first-order form

$$S_{s-t} = \int d\tau (p^{\dot\alpha\alpha} \dot x_{\alpha\dot\alpha} - \frac{e}{2} p^{\dot\alpha\alpha} p_{\alpha\dot\alpha}) \tag{12}$$

using the bridge relations between the space-time and twistor descriptions

$$p^{\dot\alpha\alpha} = \bar u^{\dot\alpha} u^\alpha, \qquad u^\alpha \dot x_{\alpha\dot\alpha} \bar u^{\dot\alpha} = \frac{i}{2}(\dot{\bar Z}_a Z^a - \bar Z_a \dot Z^a), \tag{13}$$

where

$$Z^a = (\mu^\alpha, \bar{u}_{\dot\alpha}): \ \mu^\alpha = i\bar{u}_{\dot\alpha}x^{\dot\alpha\alpha}, \quad \bar{Z}_a = (u_\alpha, \bar{\mu}^{\dot\alpha}): \ \bar{\mu}^{\dot\alpha} = -ix^{\dot\alpha\alpha}u_\alpha. \tag{14}$$

In the resulting twistor action [14]

$$S_{tw} = \frac{i}{2}\int d\tau(\dot{\bar{Z}}_a Z^a - \bar{Z}_a \dot{Z}^a) + \int d\tau e \bar{Z}_a Z^a \tag{15}$$

Lagrange multiplier e enforces zero norm constraint that is of the first class being the generator of the above mentioned local $U(1)$ rotation of twistor components.

Introduction of N-extended supersymmetry in the twistor approach following [15] consists in adding to the twistor odd $SO(1,3)$ scalars $\bar{\eta}^i$ in the $\mathbf{N}$ representation of the R-symmetry group $(S)U(N)$: $Z^a \to \mathcal{Z}^A = (\mu^\alpha, \bar{u}_{\dot\alpha}, \bar{\eta}^i)$, $i \in \mathbf{N}$. Supertwistor as the whole transforms in the fundamental representation of the complexified superconformal group $SL(4,N;\mathbb{C})$, whose real form, corresponding to the space-time of Lorentzian signature, is $SU(2,2|N)$. Incidence relations between the N-extended superspace points $(x^{\dot\alpha\alpha}, \theta^\alpha_i, \bar\theta^{\dot\alpha i})$ and supertwistor components are given by

$$\begin{aligned} \mu^\alpha &= i\bar{u}_{\dot\alpha}(x^{\dot\alpha\alpha} + 2i\theta^\alpha_i\bar\theta^{i\dot\alpha}), & \bar\eta^i &= 2\bar{u}^{\dot\alpha}\bar\theta^i_{\dot\alpha}, \\ \bar\mu^{\dot\alpha} &= -i(x^{\dot\alpha\alpha} - 2i\theta^\alpha_i\bar\theta^{i\dot\alpha})u_\alpha, & \eta_i &= 2u^\alpha\theta_{i\alpha}. \end{aligned} \tag{16}$$

Similarly to the previous consideration having two nonproportional supertwistors $\mathcal{Z}^A$ and $\mathcal{W}^A = (\nu^\alpha, \bar{v}_{\dot\alpha}, \bar\zeta^i)$ defines the antichiral superspace point with coordinates

$$x^{\dot\alpha\alpha} + 2i\theta^\alpha_i\bar\theta^{i\dot\alpha} = \frac{1}{\bar{u}^{\dot\beta}\bar{v}_{\dot\beta}}(\mu^\alpha\bar{v}^{\dot\alpha} - \nu^\alpha\bar{u}^{\dot\alpha}), \quad \bar\theta^{i\dot\alpha} = \frac{2}{\bar{u}^{\dot\beta}\bar{v}_{\dot\beta}}(\bar{v}^{\dot\alpha}\bar\eta^i - \bar{u}^{\dot\alpha}\bar\zeta^i). \tag{17}$$

Reality of the superspace body, as in the bosonic case, is equivalent to the Hermiticity of $x^{\dot\alpha\alpha}$ matrix that in turn implies that incident supertwistors lie in the zero norm subspace $\mathbb{P}\mathfrak{N}$ of the projective supertwistor space $\mathbb{P}\mathfrak{T} = \mathbb{CP}^{(3|N)}$

$$\bar{\mathcal{Z}}_A\mathcal{Z}^A = \bar{\mathcal{W}}_A\mathcal{W}^A = 0 \tag{18}$$

and are orthogonal

$$\bar{\mathcal{Z}}_A\mathcal{W}^A = \bar{\mathcal{W}}_A\mathcal{Z}^A = 0, \tag{19}$$

where the dual supertwistor is defined using the antiholomorphic map similarly to bosonic twistors

$$\mathcal{Z}^A \to \bar{\mathcal{Z}}_A = \mathcal{G}_{A\bar{B}}\overline{\mathcal{Z}^B} = (u_\alpha, \bar\mu^{\dot\alpha}, \eta_i) \tag{20}$$

with the $SU(2,2|N)$ invariant metric

$$\mathcal{G}_{A\bar{B}} = \begin{pmatrix} 0 & \delta_\alpha^\beta & 0 \\ \delta_{\dot\beta}^{\dot\alpha} & 0 & 0 \\ 0 & 0 & \delta_j^i \end{pmatrix}. \tag{21}$$

Superparticle action is obtained by supersymmetrizing the massless particle action (12)

$$\dot{x}^{\dot\alpha\alpha} \to \omega_\tau^{\dot\alpha\alpha} \tag{22}$$

that involves the projection onto the world-line of the superspace 1-form

$$\omega^{\dot\alpha\alpha}(d) = dx^{\dot\alpha\alpha} + 2id\theta_i^\alpha\bar\theta^{i\dot\alpha} - 2i\theta_i^\alpha d\bar\theta^{i\dot\alpha} \tag{23}$$

invariant under rigid N-extended supersymmetry transformation

$$\delta_\varepsilon x^{\dot\alpha\alpha} = 2i(\varepsilon_i^\alpha\bar\theta^{i\dot\alpha} - \theta_i^\alpha\bar\varepsilon^{i\dot\alpha}), \qquad \delta_\varepsilon\theta_i^\alpha = \varepsilon_i^\alpha, \qquad \delta_\varepsilon\bar\theta^{i\dot\alpha} = \bar\varepsilon^{i\dot\alpha}. \tag{24}$$

Then massless superparticle action in the first-order formulation acquires the form [15]

$$S_{ss} = \int d\tau(p^{\dot\alpha\alpha}\omega_{\tau\alpha\dot\alpha} - \frac{e}{2}p^{\dot\alpha\alpha}p_{\alpha\dot\alpha}). \tag{25}$$

The bridge relations between superspace and supertwistor formulations now generalize as

$$p^{\dot\alpha\alpha} = \bar{u}^{\dot\alpha}u^\alpha, \qquad u^\alpha\omega_{\tau\alpha\dot\alpha}\bar{u}^{\dot\alpha} = \frac{i}{2}(\dot{\bar{\mathcal{Z}}}_A\mathcal{Z}^A - \bar{\mathcal{Z}}_A\dot{\mathcal{Z}}^A) \tag{26}$$

yielding the supertwistor formulation of the massless superparticle [14]

$$S_{stw} = \frac{i}{2}\int d\tau(\dot{\bar{\mathcal{Z}}}_A\mathcal{Z}^A - \bar{\mathcal{Z}}_A\dot{\mathcal{Z}}^A) + \int d\tau e\bar{\mathcal{Z}}_A\mathcal{Z}^A. \tag{27}$$

The main distinction with the nonsupersymmetric case is that superparticle's trajectory in superspace is not a light-like line but rather a $(1|2N)$-real dimensional supersurface [16], [14] because the superspace action (25) is invariant under fermionic $\kappa-$symmetry [17] transformation

$$\begin{aligned} &\delta_\kappa\theta_i^\alpha = \bar\kappa_{i\dot\alpha}(\tau)p^{\dot\alpha\alpha}, && \delta_\kappa x^{\dot\alpha\alpha} = 2i(\theta_i^\alpha\delta_\kappa\bar\theta^{i\dot\alpha} - \delta_\kappa\theta_i^\alpha\bar\theta^{i\dot\alpha}), \\ &\delta_\kappa\bar\theta^{i\dot\alpha} = p^{\dot\alpha\alpha}\kappa_\alpha^i(\tau), && \delta_\kappa e = 4i(\dot\theta_i^\alpha\kappa_\alpha^i(\tau) + \dot{\bar\theta}^{i\dot\alpha}\bar\kappa_{i\dot\alpha}(\tau)). \end{aligned} \tag{28}$$

This symmetry is infinitely reducible but can be cast into irreducible form [14], [18] by representing particle's momentum in the spinor form according to the first expression in (26)

$$\delta_\kappa\theta_i^\alpha = \kappa_i(\tau)u^\alpha, \quad \delta_\kappa\bar\theta^{i\dot\alpha} = \bar\kappa^i(\tau)\bar{u}^{\dot\alpha}, \quad \delta_\kappa x^{\dot\alpha\alpha} = 2i(\theta_i^\alpha\delta_\kappa\bar\theta^{i\dot\alpha} - \delta_\kappa\theta_i^\alpha\bar\theta^{i\dot\alpha}). \tag{29}$$

However, in the supertwistor formulation $\kappa-$symmetry is absent in the list of superparticle's gauge symmetries. This implies that upon transition from superspace to supertwistor formulation $\kappa-$symmetry gauge freedom has been fixed without breaking the $4d$ Lorentz covariance. Indeed odd supertwistor components include just the projection of $\theta_{i\alpha}$ coordinates on the u^α 2-spinor that are $\kappa-$invariant.

4. (Super)twistors and (super)strings

Following the steps of twistorizing the massless (super)particle we need to introduce commuting 2-spinors into the superstring action. We consider first the Polyakov string and then the Green-Schwarz superstring.

From the Polyakov's action in flat background [19]

$$S=-\frac{1}{2c\alpha'}\int d^2\xi\sqrt{-g}g^{\mu\nu}\partial_\mu x^m\partial_\nu x_m,\quad \xi^\mu=(\tau,\sigma),\quad m=(0,1,...,D-1), \tag{30}$$

it follows as the consequence of the equations of motion that the pair of linearly independent light-like vectors tangent to the world-sheet can be identified with $e^\mu_{\pm2}\partial_\mu x^m$

$$e^\mu_{+2}\partial_\mu x^m e^\nu_{+2}\partial_\nu x_m=e^\mu_{-2}\partial_\mu x^m e^\nu_{-2}\partial_\nu x_m=0, \tag{31}$$

where there were introduced the world-sheet zweibein $e^f_\mu(\xi)=(e^{+2}_\mu,e^{-2}_\mu)$ and its inverse $e^\mu_f(\xi)=(e^{\mu+2},e^{\mu-2})$ related to the world-sheet metric $g_{\mu\nu}=e^f_\mu\eta_{fh}e^h_\nu$, $g=\det g_{\mu\nu}$ and its inverse $g^{\mu\nu}=e^\mu_f\eta^{fh}e^\nu_h$. $SO(1,1)$ vector indices f,h in the light-cone basis take values ±2, the metric tensor η_{fh} in this basis is characterized by nonzero components $\eta_{+2-2}=\eta_{-2+2}=\frac{1}{2}$, while its inverse η^{fh} by $\eta^{+2-2}=\eta^{-2+2}=2$.

Light-likeness of the vectors $e^\mu_{\pm2}\partial_\mu x^m(\xi)$ suggests that in the 4-dimensional space-time they can be identified with the flagpoles associated with two independent spinor fields $u^\alpha(\xi)$ and $v^\alpha(\xi)$

$$e^\mu_{+2}\partial_\mu x^{\dot\alpha\alpha}=c(\alpha')^{1/2}u^\alpha\bar u^{\dot\alpha},\quad e^\mu_{-2}\partial_\mu x^{\dot\alpha\alpha}=c(\alpha')^{1/2}v^\alpha\bar v^{\dot\alpha}, \tag{32}$$

where $x^{\dot\alpha\alpha}=x^m\tilde\sigma^{\dot\alpha\alpha}_m$ represents the contraction of space-time coordinates with 2×2 relativistic Pauli matrices. From the above relation it appears that $SL(2,\mathbb{C})$ spinors u^α and v^α also carry $SO(1,1)$ chiral spinor indices $\pm$ and so do the (super)twistors associated to them, but we refrain from explicitly writing such labels in order not to overburden the notation. Note that in the Lorentzian signature $D=4$ space-time at each point there exist precisely 2 independent light-like vectors $n^f_m=(n^{+2}_m,n^{-2}_m)$ that together with the single complex vector $n^\perp_m$ form orthonormal Cartan-Penrose tetrade

$$\begin{gathered}n^{\pm2}\cdot n^{\pm2}=0,\quad n^{\pm2}\cdot n^{\mp2}=2,\quad n^{\pm2}\cdot n^\perp=n^{\pm2}\cdot\bar n^\perp=0,\\ n^\perp\cdot\bar n^\perp=-1,\quad n^\perp\cdot n^\perp=\bar n^\perp\cdot\bar n^\perp=0\end{gathered} \tag{33}$$

that admits the spinor realization [1]

$$n^{-2\dot{\alpha}\alpha} = u^{\alpha}\bar{u}^{\dot{\alpha}}, \quad n^{+2\dot{\alpha}\alpha} = v^{\alpha}\bar{v}^{\dot{\alpha}}, \quad n^{\perp\dot{\alpha}\alpha} = u^{\alpha}\bar{v}^{\dot{\alpha}}, \quad \bar{n}^{\perp\dot{\alpha}\alpha} = v^{\alpha}\bar{u}^{\dot{\alpha}}. \tag{34}$$

Orthonormality of the tetrade components implies that

$$u^{\alpha}v_{\alpha} = \bar{u}^{\dot{\alpha}}\bar{v}_{\dot{\alpha}} = 1 \tag{35}$$

so that 2×2 matrix $v^{\alpha}_{(\beta)} = (u^{\alpha}, v^{\alpha})$ is unimodular

$$\det v^{\alpha}_{(\beta)} = 1 \implies v^{\alpha}_{(\beta)} \in SL(2,\mathbb{C}). \tag{36}$$

Such matrix is called the spin system or normalized Newman-Penrose dyad [20]. Note that it also appears helpful to work with unnormalized Newman-Penrose dyad for which the constraint (35), (36) is relaxed.

Considered above realization of the world-sheet tangents through the components of Cartan-Penrose tetrade can be derived from the first-order string action principle [21]

$$S = \int d^2\xi e(-\frac{1}{(\alpha')^{1/2}}e^{\mu}_{f}\partial_{\mu}x^{m}n^{f}_{m} + c) \tag{37}$$

$$= \int d^2\xi e(-\frac{1}{2(\alpha')^{1/2}}(e^{\mu+2}u^{\alpha}\bar{u}^{\dot{\alpha}} + e^{\mu-2}v^{\alpha}\bar{v}^{\dot{\alpha}})\partial_{\mu}x_{\alpha\dot{\alpha}} + c), \qquad e = \sqrt{-g}$$

classically equivalent to Polyakov action (30) as follows from the variation of (37) with respect to zweibein and tetrade components that are considered as dynamical variables

$$c(\alpha')^{1/2}e^{f}_{\mu}(\xi) = \partial_{\mu}x(\xi)\cdot n^{f}(\xi), \qquad \partial_{\mu}x(\xi)\cdot n^{\perp}(\xi) = 0. \tag{38}$$

Taking into account tetrade orthonormality these equations are equivalent to the relations (32) for $\partial_{\mu}x^{m}(\xi)$. Their substitution back into the action (37) recovers Polyakov formulation.

Introducing 2 twistor fields $Z^{a}(\xi) = (\mu^{\alpha}, \bar{u}_{\dot{\alpha}})$ and $W^{a}(\xi) = (\nu^{\alpha}, \bar{v}_{\dot{\alpha}})$ associated to the 2-spinor fields $\bar{u}^{\dot{\alpha}}(\xi)$ and $\bar{v}^{\dot{\alpha}}(\xi)$ it is possible to bring the projections of $\partial_{\mu}x^{m}$ onto tetrade components that enter the action (37) to the form

$$\begin{aligned} u^{\alpha}\partial_{\mu}x_{\alpha\dot{\alpha}}\bar{u}^{\dot{\alpha}} &= \frac{i}{2}\,(\partial_{\mu}\bar{Z}_{a}Z^{a} - \bar{Z}_{a}\partial_{\mu}Z^{a}), \\ v^{\alpha}\partial_{\mu}x_{\alpha\dot{\alpha}}\bar{v}^{\dot{\alpha}} &= \frac{i}{2}\,(\partial_{\mu}\bar{W}_{a}W^{a} - \bar{W}_{a}\partial_{\mu}W^{a}). \end{aligned} \tag{39}$$

These relations constitute the bridge between the space-time and twistor formulations of the string[1]. As the result one obtains the $D = 4$ string

[1]Let us note that substituting (39) into (38) and then back into the action (37) yields the twistor formulation for the bosonic string recently considered in [22].

action [13]

$$S = -\frac{i}{4(\alpha')^{1/2}}\int d^2\xi e[e^{+2\mu}(\partial_\mu \bar{Z}_a Z^a - \bar{Z}_a\partial_\mu Z^a) + e^{-2\mu}(\partial_\mu \bar{W}_a W^a - \bar{W}_a\partial_\mu W^a)] + c\int d^2\xi e \tag{40}$$

written in terms of Penrose twistors. It should be noted that in the similar way as the action (37) has to be supplemented by either tetrade (33) or dyad (35) normalization conditions introduced via the Lagrange multipliers, above derived twistor action functional needs to be supplemented by four algebraic constraints on the twistor variables (6) to maintain the correspondence with real space-time description.

Supersymmetric generalization is straightforward and yields the first order formulation of the $D = 4$ $N = 1$ superstring action [21]

$$S = \int d^2\xi e(-\tfrac{1}{(\alpha')^{1/2}}\, e^\mu_f \omega^m_\mu n^f_m + c) + S_{WZ}, \tag{41}$$

where the Wess-Zumino term equals

$$S_{WZ} = \frac{is}{c\alpha'}\int d^2\xi \varepsilon^{\mu\nu}\omega^{\dot\alpha\alpha}_\mu(\partial_\mu\theta_\alpha\bar\theta_{\dot\alpha} - \theta_\alpha\partial_\mu\bar\theta_{\dot\alpha}). \tag{42}$$

$s = \pm 1$ being the numerical factor, whose values are consistent with the $\kappa-$invariance of the superstring action.

The reformulation in terms of $N = 1$ supertwistors for the kinetic term proceeds by analogy with the bosonic case and the Newman-Penrose dyad components are introduced into the Wess-Zumino term via the orthonormality relation

$$u^\alpha v_\beta - u^\beta v_\alpha = \delta^\alpha_\beta \tag{43}$$

and its c.c. As the result superstring action (41) acquires the form [13]

$$\begin{aligned} S \;=\; & \int d^2\xi e[-\frac{i}{4(\alpha')^{1/2}}\,(e^{+2\mu}\omega_{\mathcal{Z}\mu} + e^{-2\mu}\omega_{\mathcal{W}\mu}) + c] \\ & - \frac{s}{8c\alpha'}\int d^2\xi\varepsilon^{\mu\nu}(\omega_{\mathcal{Z}\mu}\omega_{\zeta\nu} + \omega_{\mathcal{W}\mu}\omega_{\eta\nu} - \omega_{\bar{\mathcal{Z}}\mathcal{W}\mu}\omega_{\bar\eta\zeta\nu} - \omega_{\bar{\mathcal{W}}\mathcal{Z}\mu}\omega_{\bar\zeta\eta\nu}), \end{aligned} \tag{44}$$

where the new notation has been introduced

$$\begin{aligned} &\omega_{\mathcal{Z}\mu} = \partial_\mu\bar{\mathcal{Z}}_A\mathcal{Z}^A - \bar{\mathcal{Z}}_A\partial_\mu\mathcal{Z}^A, && \omega_{\mathcal{W}\mu} = \partial_\mu\bar{\mathcal{W}}_A\mathcal{W}^A - \bar{\mathcal{W}}_A\partial_\mu\mathcal{W}^A, \\ &\omega_{\bar{\mathcal{Z}}\mathcal{W}\mu} = \partial_\mu\bar{\mathcal{Z}}_A\mathcal{W}^A - \bar{\mathcal{Z}}_A\partial_\mu\mathcal{W}^A, && \omega_{\bar{\mathcal{W}}\mathcal{Z}\mu} = -(\omega_{\bar{\mathcal{Z}}\mathcal{W}\mu})^*, \end{aligned} \tag{45}$$

and

$$\begin{aligned} &\omega_{\eta\mu} = \mathcal{D}_\mu\eta\bar\eta - \eta\mathcal{D}_\mu\bar\eta, && \omega_{\zeta\mu} = \mathcal{D}_\mu\zeta\bar\zeta - \zeta\mathcal{D}_\mu\bar\zeta, \\ &\omega_{\bar\eta\zeta\mu} = \mathcal{D}_\mu\zeta\bar\eta - \zeta\mathcal{D}_\mu\bar\eta, && \omega_{\bar\eta\zeta\mu} = -(\omega_{\bar\eta\zeta\mu})^*. \end{aligned} \tag{46}$$

The world-sheet covariant derivatives of odd supertwistor components

$$\mathcal{D}_\mu\eta = \partial_\mu\eta - \omega_\mu\eta - \tilde{\omega}_\mu\zeta, \qquad \mathcal{D}_\mu\zeta = \partial_\mu\zeta + \omega_\mu\zeta - \Omega_\mu\eta \tag{47}$$

include the world-sheet projections of the spin system derivation coefficients

$$\begin{aligned}
\omega(d) &= \frac{1}{2}\,(\mathcal{W}^A\mathcal{I}_{AB}d\mathcal{Z}^B + \mathcal{Z}^A\mathcal{I}_{AB}d\mathcal{W}^B) = \frac{1}{2}\,(d\bar{u}^{\dot\alpha}\bar{v}_{\dot\alpha} + d\bar{v}^{\dot\alpha}\bar{u}_{\dot\alpha}),\\
\tilde{\omega}(d) &= d\mathcal{Z}^A\mathcal{I}_{AB}\mathcal{Z}^B = \bar{u}^{\dot\alpha}d\bar{u}_{\dot\alpha},\\
\Omega(d) &= \mathcal{W}^A\mathcal{I}_{AB}d\mathcal{W}^B = d\bar{v}^{\dot\alpha}\bar{v}_{\dot\alpha}
\end{aligned} \tag{48}$$

involving asymptotic or infinity supertwistor defined as

$$\mathcal{I}_{AB} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \varepsilon^{\dot\alpha\dot\beta} & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{49}$$

As follows from the preceding discussion the action (44) has to be accompanied by the constraints (18), (19) ensuring incidence with superspace points characterized by real commuting coordinates $x_{\alpha\dot\alpha}$.
Unlike the massless superparticle case [14] the supertwistor representation of the superstring action (44) is still invariant under irreducible $\kappa-$symmetry transformations. Their explicit form depends on the sign of the of the Wess-Zumino term. If $s = 1$ one has

$$\begin{aligned}
&\delta_\kappa\mathcal{Z}^A = \tilde{\omega}(\delta_\kappa)\mathcal{W}^A,\\
&\delta_\kappa\mathcal{W}^A = \Omega(\delta_\kappa)\mathcal{Z}^A - (\bar{\mathcal{W}}_C\mathcal{K}^C)\mathcal{I}^{AB}\bar{\mathcal{Z}}_B + (\bar{\mathcal{Z}}_C\mathcal{K}^C)\mathcal{I}^{AB}\bar{\mathcal{W}}_B + \mathcal{K}^A,\\
&\delta_\kappa e_\mu^{+2} = \frac{i}{c(\alpha')^{1/2}}\,(\mathcal{D}_\mu\bar{\mathcal{W}}_C\mathcal{K}^C - \bar{\mathcal{K}}_C\mathcal{D}_\mu\mathcal{W}^C), \quad \delta_\kappa e_\mu^{-2} = 0,
\end{aligned} \tag{50}$$

where the $\kappa-$variations of derivation coefficients read

$$\begin{aligned}
&\omega(\delta_\kappa) = 0, \qquad\qquad \tilde{\omega}(\delta_\kappa) = \frac{i}{2c(\alpha')^{1/2}}\,\bar{\mathcal{K}}_C e^{-2\mu}\mathcal{D}_\mu\mathcal{Z}^C,\\
&\bar{\Omega}(\delta_\kappa) = -\frac{i}{2c(\alpha')^{1/2}}\,\bar{\mathcal{K}}_C e^{+2\mu}\mathcal{D}_\mu\mathcal{Z}^C.
\end{aligned} \tag{51}$$

Correspondingly for $s = -1$

$$\begin{aligned}
&\delta_\kappa\mathcal{Z}^A = \tilde{\omega}(\delta_\kappa)\mathcal{W}^A - (\bar{\mathcal{W}}_C\mathcal{K}^C)\mathcal{I}^{AB}\bar{\mathcal{Z}}_B + (\bar{\mathcal{Z}}_C\mathcal{K}^C)\mathcal{I}^{AB}\bar{\mathcal{W}}_B + \mathcal{K}^A,\\
&\delta_\kappa\mathcal{W}^A = \Omega(\delta_\kappa)\mathcal{Z}^A,\\
&\delta_\kappa e_\mu^{+2} = 0, \qquad \delta_\kappa e_\mu^{-2} = \frac{i}{c(\alpha')^{1/2}}\,(\mathcal{D}_\mu\bar{\mathcal{Z}}_C\mathcal{K}^C - \bar{\mathcal{K}}_C\mathcal{D}_\mu\mathcal{W}^C)
\end{aligned} \tag{52}$$

with

$$\begin{aligned}&\omega(\delta_\kappa)=0, \qquad\qquad \tilde{\omega}(\delta_\kappa)=\frac{i}{2c(\alpha')^{1/2}}\,e^{-2\mu}\mathcal{D}_\mu\bar{\mathcal{W}}_A\mathcal{K}^A,\\ &\bar{\Omega}(\delta_\kappa)=-\frac{i}{2c(\alpha')^{1/2}}\,e^{+2\mu}\mathcal{D}_\mu\bar{\mathcal{W}}_A\mathcal{K}^A.\end{aligned}\tag{53}$$

The gauge parameter $\kappa(\xi)$ was presented as the odd component of the supertwistor $\mathcal{K}^A=(0,0,\kappa)$.

It is possible to simplify essentially the action (44) by fixing the gauge freedom associated with the $\kappa-$symmetry. This can be accomplished in the Lorentz-covariant way by utilizing nondynamical equations of motion resulting from the supertwistor action variation

$$\omega_{\mathcal{Z}\mu}=-2ic(\alpha')^{1/2}e^{-2}_\mu, \qquad \omega_{\mathcal{W}\mu}=-2ic(\alpha')^{1/2}e^{+2}_\mu, \qquad \omega_{\bar{\mathcal{Z}}\mathcal{W}\mu}=0. \tag{54}$$

These expressions are the supertwistor representation for supersymmetric generalization of equations (38) and emphasize the induced nature of the world-sheet zweibein. Their substitution back into the action (44) yields

$$\begin{aligned}S^{s=1}_{gf}=-\frac{i}{4(\alpha')^{1/2}}\int d^2\xi e[&e^{+2\mu}(\partial_\mu\bar{\mathcal{Z}}_A\mathcal{Z}^A-\bar{\mathcal{Z}}_A\partial_\mu\mathcal{Z}^A)\\ &+e^{-2\mu}(\partial_\mu\bar{W}_aW^a-\bar{W}_a\partial_\mu W^a)]+c\int d^2\xi e,\end{aligned}\tag{55}$$

when $s=1$, and

$$\begin{aligned}S^{s=-1}_{gf}=-\frac{i}{4(\alpha')^{1/2}}\int d^2\xi e[&e^{+2\mu}(\partial_\mu\bar{Z}_aZ^a-\bar{Z}_a\partial_\mu Z^a)\\ &+e^{-2\mu}(\partial_\mu\bar{\mathcal{W}}_A\mathcal{W}^A-\bar{\mathcal{W}}_A\partial_\mu\mathcal{W}^A)]+c\int d^2\xi e\end{aligned}\tag{56}$$

for $s=-1$ modulo changing the numerical factors of the odd supertwistor components $\eta\to\eta=4u^\alpha\theta_\alpha$ or $\zeta\to\zeta=4v^\alpha\theta_\alpha$. Gauge-fixed actions are those of the free $2d$ theory modulo the constraints that ensure the reality of commuting superspace coordinates (6), (18), (19).

In the case of the $N=2$ superstring the Wess-Zumino term includes the 4-fermion contribution as compared to $N=1$ case

$$\begin{aligned}S^{N=2}_{WZ}=&\frac{is}{c\alpha'}\int\varepsilon^{\mu\nu}\omega^{\dot{\alpha}\alpha}_\mu(\partial_\nu\theta^1_\alpha\bar{\theta}^1_{\dot{\alpha}}-\theta^1_\alpha\partial_\nu\bar{\theta}^1_{\dot{\alpha}}-\partial_\nu\theta^2_\alpha\bar{\theta}^2_{\dot{\alpha}}+\theta^2_\alpha\partial_\nu\bar{\theta}^2_{\dot{\alpha}})\\ &+\frac{2s}{c\alpha'}\int\varepsilon^{\mu\nu}(\partial_\mu\theta^{1\alpha}\bar{\theta}^{1\dot{\alpha}}-\theta^{1\alpha}\partial_\mu\bar{\theta}^{1\dot{\alpha}})(\partial_\nu\theta^2_\alpha\bar{\theta}^2_{\dot{\alpha}}-\theta^2_\alpha\partial_\nu\bar{\theta}^2_{\dot{\alpha}}).\end{aligned}\tag{57}$$

It is, however, still possible to bring the $N=2$ superstring action to the form similar to (55), (56) in terms of $N=1$ supertwistors if the $\kappa-$symmetry is fixed by the conditions

$$\theta^1_\alpha=\theta^2_\alpha, \qquad \bar{\theta}^1_{\dot{\alpha}}=\bar{\theta}^2_{\dot{\alpha}}. \tag{58}$$

Resulting gauge-fixed action acquires the form

$$\begin{aligned} S_{gf}^{N=2} = -\frac{i}{4(\alpha')^{1/2}} \int d^2\xi e[e^{+2\mu}(\partial_\mu \bar{\mathcal{Z}}_A \mathcal{Z}^A - \bar{\mathcal{Z}}_A \partial_\mu \mathcal{Z}^A) \\ + e^{-2\mu}(\partial_\mu \bar{\mathcal{W}}_A \mathcal{W}^A - \bar{\mathcal{W}}_A \partial_\mu \mathcal{W}^A)] + c \int d^2\xi e \end{aligned} \tag{59}$$

modulo rescaling the odd supertwistor components and independently of the sign of the WZ term that vanishes in the above gauge.

So we can summarize that the bosonic string and $\kappa-$symmetry gauge fixed $N = 1$ and $N = 2$ superstring actions can be presented in the following uniform manner

$$\begin{aligned} S = -\frac{i}{4(\alpha')^{1/2}} \int d^2\xi e[e^{\mu+2}(\partial_\mu \bar{\mathbf{Z}}_{\mathcal{A}} \mathbf{Z}^{\mathcal{A}} - \bar{\mathbf{Z}}_{\mathcal{A}} \partial_\mu \mathbf{Z}^{\mathcal{A}}) \\ + e^{\mu-2}(\partial_\mu \bar{\mathbf{W}}_{\mathcal{A}} \mathbf{W}^{\mathcal{A}} - \bar{\mathbf{W}}_{\mathcal{A}} \partial_\mu \mathbf{W}^{\mathcal{A}})] + c \int d^2\xi e, \end{aligned} \tag{60}$$

where $\mathbf{Z}^{\mathcal{A}}$ and $\mathbf{W}^{\mathcal{A}}$ stand for either bosonic twistor or $N = 1$ supertwistor that are isotropic and orthonormal so that (super)string propagates in $\mathbb{P}\mathfrak{N}$ subspace of the (super)twistor space. It should be noted that the generalization to the case of the unnormalized Newman-Penrose dyad $n = \bar{\mathbf{W}}_{\mathcal{A}} I^{\mathcal{AB}} \bar{\mathbf{Z}}_{\mathcal{B}} = u^\alpha v_\alpha \neq 1$, $\bar{n} = \mathbf{W}^{\mathcal{A}} I_{\mathcal{AB}} \bar{\mathbf{Z}}^{\mathcal{B}} = \bar{u}^{\dot{\alpha}} \bar{v}_{\dot{\alpha}} \neq 1$ can be attained by replacing

$$c \int d^2\xi e \to c \int d^2\xi e n \bar{n}. \tag{61}$$

Then the action functional (60) becomes invariant under gauge Weyl symmetry

$$e_\mu^{\pm 2} \to e^{-\lambda} e_\mu^{\pm 2}, \quad e^{\mu \pm 2} \to e^{\lambda} e^{\mu \pm 2}, \quad \mathbf{Z}^{\mathcal{A}} \to e^{\lambda/2} \mathbf{Z}^{\mathcal{A}}, \quad \mathbf{W}^{\mathcal{A}} \to e^{\lambda/2} \mathbf{W}^{\mathcal{A}} \tag{62}$$

allowing one to gauge away all the auxiliary zweibein components. Such generalization is useful for the study of the quantum symmetries [23] and supertwistor formulations for higher dimensional superstrings [24].

5. Acknowledgements

It is a pleasure to thank the Organizers of IV Summer School in Modern Mathematical Physics and especially Prof. B. Dragovich for the warm hospitality in Belgrade and financial support.

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