# Recent Developments in the $N$-Extended Supersymmetric Quantum Mechanics*

**Francesco Toppan**†
CBPF (TEO), Rio de Janeiro (RJ), BRAZIL

Abstract

In this paper we review some recent developments in the understanding of the supersymmetric quantum mechanics for large-$N$ values of the extended supersymmetries. A list of the topics here covered includes the new available classification of the finite linear irreducible representations, the construction of manifestly off-shell invariant actions without introducing a superfield formalism, the notion of the "fusion algebra" of the irreducible representations, the connection (for $N = 8$) with the octonionic structure constants, etc. The results presented are based on the work of the author and his collaborators.

## 1. Introduction

The supersymmetric quantum mechanics is a more than twentyfive years old topic [1] with fascinating mathematical (Morse theory, index theorems) and physical (nuclear physics, condensed matter [2, 3]) applications. In the recent years several groups have investigated, see e.g. [4, 5, 6, 7, 8, 9, 10, 11, 12], with different methods and focusing on various interrelated aspects, one-dimensional large-$N$ supersymmetric quantum mechanical systems ($N$ denotes the number of the extended supersymmetries). The main motivation behind this activity is traced on the problem of understanding the supersymmetric unification of the interactions. Indeed, the 11-dimensional maximal supergravity (the low-energy limit of the conjectured $M$-theory), when dimensionally reduced to $D = 1$, produces an $N = 32$ supersymmetric quantum mechanical system. It seems unlikely that any progress towards the understanding of the $M$-theory could be made if we do not comprehend the features of its one-dimensional setting. An example for all: the construction of an off-shell invariant action (in comparison with the on-shell action). In this paper we review the results obtained by the author and his collaborators in a very recent series of works [13, 14, 15, 16].

* Work supported by CNPq and FAPERJ.
† e-mail address: toppan@cbpf.br

They deal with very fundamental properties of the algebra of the supersymmetric quantum mechanics and its representation theory. The list of the topics here discussed includes, in order, the introduction of the algebra of the one-dimensional $N$-extended supersymmetric quantum mechanic as a fundamental mathematical structure allowing to "interpolate" between Clifford and Grassmann algebras. Next, the finite linear irreducible representations will be classified, both in terms of the dimensionality of the fields (bosonic and fermionic) entering the irreducible multiplets, as well as the graphical properties of the supersymmetry transformations. The results will be applied to construct manifestly supersymmetric off-shell invariant actions without introducing the superfield formalism. The $N = 4$ cases and a non-trivial $N = 8$ example where the octonionic structure constants enter the action as coupling constants will be presented. The fusion algebra of the irreducible representations of the supersymmetric quantum mechanics will be introduced and explicitly computed for $N = 2$. It contains information concerning the construction of off-shell invariant actions. For what concerns various other important aspects of the present activity on supersymmetric quantum mechanis (for instance, the investigations concerning its non-linear realizations) which are not covered here, reviews are available (see e.g. [17, 18]).

## 2. The $D = 1$ $N$-extended supersymmetry algebra

The algebra of the one-dimensional $N$-extended supersymmetry (from now on the "$N$-susy algebra") is a $\mathbf{Z}_2$-graded algebra presenting a total number of $N$ odd generators $Q_i$ ($i = 1, \ldots, N$) and a single even generator, a central extension $z$. The $N$-susy algebra is defined by the (anti-)commutation relations

$$\{Q_i, Q_j\} = \delta_{ij} z, \qquad [Q_i, z] = 0. \tag{1}$$

The central extension $z$ plays an important role. In physics it is usually denoted with "$H$" and called the hamiltonian.

The mathematical importance of the above algebra (which is, technically, not a simple super-Lie algebra due to the presence of the central extension) can be understood by the following reasoning. In formulating for the hamiltonian $H$ an eigenvalue problem, we are led with two possibilities. Either the eigenvalue is zero, in the case of a vacuum solution, or it is a positive real number. In the first case we are reduced with the Grassmann algebra, which is the enveloping algebra generated by the $N$ generators $\theta_i$ ($i = 1, \ldots, N$) satisfying the relations

$$\theta_i\theta_j + \theta_j\theta_i = 0 \tag{2}$$

for any $i, j$ pair.

In the second case, for a fixed $z > 0$, after a suitable rescaling of the odd generators $Q_i$'s, we are led to the fundamental relation among the generators $\gamma_i$ ($i = 1, \ldots, N$) of the $N$-dimensional Euclidean Clifford algebra, namely

$$\gamma_i\gamma_j + \gamma_j\gamma_i = 2\delta_{ij}\mathbf{1}. \tag{3}$$

In a loose sense we can say that the supersymmetric quantum mechanics interpolates between the Grassmann algebra and the Clifford algebra. The above remark makes transparent the deep connection between the supersymmetric quantum mechanics and the irreducible representations of the Clifford algebra. It is not surprising that the linear finite irreducible representation of the $N$-susy algebra are classified with the help of the Clifford irreps. On the other hand, the $N$-susy irreps contain more information. The hamiltonian $H$ acts as a time-derivative($H \equiv i\frac{d}{dt}$). The finite linear irreps of (1) consist of an equal finite number $n$ of bosonic and fermionic fields (depending on a single coordinate $t$, the time) upon which the supersymmetry operators act linearly.

The time-derivative can now be used to introduce a grading, corresponding to the mass-dimension, to the fields entering the irreps. This is the crucial difference between irreps of the $N$-susy algebra and the Clifford irreps. In [13] it was proven that all (1) irreps fall into classes of equivalence determined by the irreps of an associated Clifford algebra. As one of the corollaries, a relation between $n$ (the total number of bosonic, or fermionic, fields entering the irrep) and the value $N$ of the extended supersymmetry was established.

A dimensionality $d_i = d_1 + \frac{i-1}{2}$ ($d_1$ is an arbitrary constant) can be assigned to the fields entering an irrep. The difference in dimensionality between a given bosonic and a given fermionic field is a half-integer number. The fields content of an irrep is the set of integers $(n_1, n_2, \ldots, n_l)$ specifying the number $n_i$ of fields of dimension $d_i$ entering the irrep. Physically, the $n_l$ fields of highest dimension are the auxiliary fields which transform as a time-derivative under any supersymmetry generator. The maximal value $l$ (corresponding to the maximal dimensionality $d_l$) is known as the length of the irrep. Either $n_1, n_3, \ldots$ correspond to the bosonic fields (therefore $n_2, n_4, \ldots$ specify the fermionic fields) or viceversa. In both cases the equality $n_1 + n_3 + \ldots = n_2 + n_4 + \ldots = n$ is guaranteed. A multiplet is bosonic (fermionic) if its $n_1$ component fields of lower dimensions are bosonic (fermionic). The representation theory does not discriminate the overall bosonic or fermionic nature of the multiplet.

## 3. Irreducible representations: the classification based on the fields-dimensions

There is a well-known relation between extended supersymmetries (for the values $N = 1, 2, 4, 8$) and the division algebras of the real, complex, quaternionic and octonionic numbers.

For the one-dimensional supersymmetry this relation can be understood in terms of the connection of the (1) $N$-susy algebra with the Clifford algebras. Clifford algebras irreps are infact classified in terms of division algebras [19, 20, 21] and, some of their specific properties, like the Bott's 8 periodicity, are in consequence of the octonions. The finite linear irreps of the (1) algebra are given by multiplets of fields discussed in the previous section.

A fundamental problem in the classification of the irreducible representations consists in determining, for any given $N$, the set of admissible ordered integers

$$(n_1, n_2, n_3, \ldots, n_l)$$

which correspond to irreducible multiplets with $n_i$ fields of dimension $d_i$. An equivalence relation can be introduced s.t. all such multiplets specify one and only one irrep in the given class [16].

This classification was presented in [14]. For $N \leq 10$ the computations were explicitly carried on. The admissible multiplets, for a given $N$, are recovered from the "root multiplets" of type $(n, n)$, which carry a representation of the $N$-susy algebra expressed by the generators

$$Q_i = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & \sigma_i \\ \widetilde{\sigma}_i \cdot H & 0 \end{pmatrix}, \tag{4}$$

where the $\sigma_i$ and $\widetilde{\sigma}_i$ are matrices entering a Weyl type (i.e. block antidiagonal) irreducible representation of a $D$-dimensional (with $D = N$) Clifford algebra relation

$$\Gamma_i = \begin{pmatrix} 0 & \sigma_i \\ \widetilde{\sigma}_i & 0 \end{pmatrix}, \qquad \{\Gamma_i, \Gamma_j\} = 2\delta_{ij}\,. \tag{5}$$

The $Q_i$'s in (4) are supermatrices with vanishing bosonic and non-vanishing fermionic blocks. The total number $2n$ of bosonic plus fermionic fields entering a multiplet is given by the size of the corresponding gamma matrices. The remaining multiplets, for $l \geq 3$, are obtained through a "dressing procedure", see [13], obtained by repeated applications of the transformations,

$$Q_i \quad \mapsto \quad \widehat{Q}_i^{(k)} = S^{(k)} Q_i {S^{(k)}}^{-1} \tag{6}$$

realized by diagonal matrices $S^{(k)}$'s $(k = 1, \ldots, 2n)$ with entries ${s^{(k)}}_{ij}$ given by

$${s^{(k)}}_{ij} = \delta_{ij}(1 - \delta_{jk} + \delta_{jk} H)\,. \tag{7}$$

The "dressed" supersymmetric operators $\widehat{Q}_i$ have entries with integral powers of the hamiltonian $H$. On the other hand, only the regular dressed operators, admitting no entries with poles $\frac{1}{H}$, are genuine supersymmetry operators, linearly acting on a finite multiplet of bosonic and fermionic fields.

For irreps of the $N$-extended supersymmetry the total number of bosonic (fermionic) fields is given by $n$, with $N$ and $n$ linked through

$$N = 8p + q\,, \qquad n = 2^{4p} G(q)\,, \tag{8}$$

where $p = 0, 1, 2, \ldots$ and $q = 1, 2, 3, 4, 5, 6, 7, 8$. $G(q)$ appearing in (8) is the Radon-Hurwitz function [13]

| $q$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| $G(q)$ | 1 | 2 | 4 | 4 | 8 | 8 | 8 | 8 |

(9)

Notice the appearance of the modulo 8 Bott's periodicity.

We present now the [14] classification of the admissible multiplets. For any $N$, all length-3 multiplets of the type $(n-k, n, k)$ are an irrep of the $N$-susy. On the other hand, length-4 irreps exist for $N = 3, 5, 6, 7$ and $N \geq 9$, while length-5 irreps are present starting from $N \geq 10$.

Up to $N = 8$, the list of length-4 irreps is, e.g., given by the multiplets

| | |
|---|---|
| $N = 3$ | $(1, 3, 3, 1)$ |
| $N = 5$ | $(1, 5, 7, 3)$, $(3, 7, 5, 1)$, $(1, 6, 7, 2)$, $(2, 7, 6, 1)$, $(2, 6, 6, 2)$, $(1, 7, 7, 1)$ |
| $N = 6$ | $(1, 6, 7, 2)$, $(2, 7, 6, 1)$, $(2, 6, 6, 2)$, $(1, 7, 7, 1)$ |
| $N = 7$ | $(1, 7, 7, 1)$ |

(10)

For $N = 9$ there are 28 length-4 irreducible multiplets given by the set of numbers

$$(h, 16 - k, 16 - h, k),$$

with $h, k$ constrained to satisfy

$$h + k \leq 8.$$

For $N = 10$ the length-4 irreducible multiplets are given by the set of values

$$(h, 32 - k, 32 - h, k),$$

where the integers $h$, $k$ are constrained to satisfy

$$h + k + r \leq 24,$$

with $r$ given by

$$r = min(h, k).$$

The classification of the $N = 10$ length-5 irreps and the length-4 irreps of the $N = 11, 12$ extended supersymmetries are found in [14].

Some properties of the classification of the irreps are easily recognized. For instance, a dual multiplet specified by the "reversed" numbers $(n_k, n_{k-1}, \ldots, n_1)$ is an irreducible multiplet iff $(n_1, n_2, \ldots, n_k)$ is an irrep.

## 4. Irreducible representation: the classification of the different connectivities of the supersymmetry transformations

Quite recently it has been pointed out in [11, 12] that certain irreps admitting the same field content can be regarded as inequivalent. These results were obtained by analyzing the "connectivity properties" of certain graphs associated to the irreps. A notion of equivalence class among irreps (spotting their difference in "connectivity") was introduced. In [12], two examples were explicitly presented. They involved a pair of $N = 6$ irreps with $(6, 8, 2)$ fields content and a pair of $N = 5$ irreps with $(6, 8, 2)$ fields content. In [12] the classification of the irreps which differ by connectivity was left as an open problem.

Using the technology developed in [14], in [15] the connectivity properties of the $N \leq 8$ irreps were classified. For length-3 irreps the connectivities can be expressed by the $\psi_g$ symbol defined below. Any given field of dimension $d$ is mapped, under a supersymmetry transformation, either

*a*) to a field of dimension $d+\frac{1}{2}$ belonging to the multiplet (or to its opposite, the sign of the transformation being irrelevant for our purposes) or,

*b*) to the time-derivative of a field of dimension $d - \frac{1}{2}$.

If the given field belongs to an irrep of the $N$-extended (1) supersymmetry algebra, therefore $k \leq N$ of its transformations are of type *a*), while the $N - k$ remaining ones are of type *b*). Let us now specialize our discussion to a length-3 irrep. Its fields content is given by $(n_1, n, n - n_1)$, while the set of its fields is expressed by $(x_i; \psi_j; g_k)$, with $i = 1, \ldots, n_1$, $j = 1, \ldots, n$, $k = 1, \ldots, n-n_1$. The $x_i$'s are 0-dimensional fields (the $\psi_j$ are $\frac{1}{2}$-dimensional and the $g_k$ 1-dimensional fields, respectively). The connectivity associated to the given multiplet is defined in terms of the $\psi_g$ symbol. It encodes the following information. The $n$ $\frac{1}{2}$-dimensional fields $\psi_j$ are partitioned in the subsets of $m_r$ fields admitting $k_r$ supersymmetry transformations of type *a*). We have $\sum_r m_r = n$. Please notice that $k_r$ can take the 0 value. The $\psi_g$ symbol is expressed as

$$\psi_g \quad \equiv \quad m_{1k_1} + m_{2k_2} + \ldots \tag{11}$$

As an example, the $N = 7$ $(6, 8, 2)$ multiplet admits connectivity $\psi_g = 6_2 + 2_1$. It means that there are two types of fields $\psi_j$. 6 of them are mapped, under supersymmetry transformations, in the two auxiliary fields $g_k$. The two remaining fields $\psi_j$ are only mapped into a single auxiliary field.

Please notice that an analogous symbol, $x_\psi$, can be introduced. It describes the supersymmetry transformations of the $x_i$ fields into the $\psi_j$ fields. This symbol is, however, always trivial. An $N$-irrep with $(n_1, n, n - n_1)$ fields content always produce $x_\psi \equiv n_{1N}$.

We report here the results of [15]. It was proven that the only values of $N \leq 8$ allowing the existence of multiplets with the same field content but inequivalent connectivities are $N = 5$ and $N = 6$. Moreover, the connectivity can be defined for multiplets of any length, however only length-3 multiplets admit inequivalent connectivities (each given multiplet of length-2 and length-4, for $N \leq 8$, is connected in only one possible way). The following table presents the admissible $\psi_g$ symbols (connectivities) for the $N = 5$ and $N = 6$ length-3 multiplets. We have

| $l = 3$ | $N = 6_A$ | $N = 6$ $\swarrow \searrow$ | $N = 6_B$ | $N = 5_A$ | $N = 5$ $\swarrow \searrow$ | $N = 5_B$ |
|---|---|---|---|---|---|---|
| $(7,8,1)$ | | $6_1 + 2_0$ | | | $5_1 + 3_0$ | |
| $(6,8,2)$ | $6_2 + 2_0$ | $-$ | $4_2 + 4_1$ | $4_2 + 2_1 + 2_0$ | $-$ | $2_2 + 6_1$ |
| $(5,8,3)$ | $4_3 + 2_2 + 2_1$ | $-$ | $2_3 + 6_2$ | $4_3 + 3_1 + 1_0$ | $-$ | $1_3 + 5_2 + 2_1$ |
| $(4,8,4)$ | $4_4 + 4_2$ | $-$ | $8_3$ | $4_4 + 4_1$ | $-$ | $4_3 + 4_2$ |
| $(3,8,5)$ | $2_5 + 2_4 + 4_3$ | $-$ | $6_4 + 2_3$ | $1_5 + 3_4 + 4_2$ | $-$ | $2_4 + 5_3 + 1_2$ |
| $(2,8,6)$ | $2_6 + 6_4$ | $-$ | $4_5 + 4_4$ | $2_5 + 2_4 + 4_3$ | $-$ | $6_4 + 2_3$ |
| $(1,8,7)$ | | $2_6 + 6_5$ | | | $3_5 + 5_4$ | |

(12)

The $(7, 8, 1)$ and $(1, 8, 7)$ length-3 multiplets are connected in one possible way only. In the remaining length-3 cases, the multiplets are connected in two ways, specified by the $\psi_g$ symbol. They are labeled with the subscript $A$ and $B$, respectively.

The above result provides the complete classification of the $N \leq 8$ irreps admitting inequivalent connectivities. The result is also interesting because it produces a counterexample to the [12] claim that different connectivities can be uniquely spotted by (for length-3 multiplets) two sets of three ordered numbers, $S = [s_1, s_2, s_3]$ and $T = [t_1, t_2, t_3]$, known as the "sources" and "targets" respectively. The integer $s_i$ gives the number of fields of dimension $d_i = \frac{i-1}{2}$ which do not result as an
$a$ )-supersymmetry transformation of at least one field of dimension $d_i - \frac{1}{2}$. The integer $t_i$ gives the number of fields of dimension $d_i = \frac{i-1}{2}$ which only admit supersymmetry transformations of type $b$).

Sources and targets can be computed in terms of the $\psi_g$ symbols of the original $(k, n, n-k)$ multiplet and its dually related $(n-k, n, k)$ partner. Sources and targets are given by the following tables. For $N = 6$ we have

| $N = 6:$ | *connectivities* | *sources* | *targets* |
|---|---|---|---|
| $(6,8,2)_A$ | $6_2 + 2_0$ | $S = [6,0,0]$ | $T = [0,2,2]$ |
| $(6,8,2)_B$ | $4_2 + 4_1$ | $S = [6,0,0]$ | $T = [0,0,2]$ |
| $(5,8,3)_A$ | $4_3 + 2_2 + 2_1$ | $S = [5,0,0]$ | $T = [0,0,3]$ |
| $(5,8,3)_B$ | $2_3 + 6_2$ | $S = [5,0,0]$ | $T = [0,0,3]$ |
| $(4,8,4)_A$ | $4_4 + 4_2$ | $S = [4,0,0]$ | $T = [0,0,4]$ |
| $(4,8,4)_B$ | $8_3$ | $S = [4,0,0]$ | $T = [0,0,4]$ |
| $(3,8,5)_A$ | $2_5 + 2_4 + 4_3$ | $S = [3,0,0]$ | $T = [0,0,5]$ |
| $(3,8,5)_B$ | $6_4 + 2_3$ | $S = [3,0,0]$ | $T = [0,0,5]$ |
| $(2,8,6)_A$ | $2_6 + 6_4$ | $S = [2,2,0]$ | $T = [0,0,6]$ |
| $(2,8,6)_B$ | $4_5 + 4_4$ | $S = [2,0,0]$ | $T = [0,0,6]$ |

(13)

For $N=5$ we have

$$
\begin{array}{|c|c|c|c|}
\hline
N=5: & connectivities & sources & targets \\
\hline
(6,8,2)_A & 4_2+2_1+2_0 & S=[6,0,0] & T=[0,2,2] \\
(6,8,2)_B & 2_2+6_1 & S=[6,0,0] & T=[0,0,2] \\
\hline
(5,8,3)_A & 4_3+3_1+1_0 & S=[5,0,0] & T=[0,1,3] \\
(5,8,3)_B & 1_3+5_2+2_1 & S=[5,0,0] & T=[0,0,3] \\
\hline
(4,8,4)_A & 4_4+4_1 & S=[4,0,0] & T=[0,0,4] \\
(4,8,4)_B & 4_3+4_2 & S=[4,0,0] & T=[0,0,4] \\
\hline
(3,8,5)_A & 1_5+3_4+4_2 & S=[3,1,0] & T=[0,0,5] \\
(3,8,5)_B & 2_4+5_3+1_2 & S=[3,0,0] & T=[0,0,5] \\
\hline
(2,8,6)_A & 2_5+2_4+4_3 & S=[2,2,0] & T=[0,0,6] \\
(2,8,6)_B & 6_4+2_3 & S=[2,0,0] & T=[0,0,6] \\
\hline
\end{array}
\qquad (14)
$$

The following irreps differ by connectivity, while admitting the same number of sources and targets:

$$
\begin{array}{llcl}
N=6: & (3,8,5)_A & \longleftrightarrow & (3,8,5)_B \\
N=6: & (4,8,4)_A & \longleftrightarrow & (4,8,4)_B \\
N=6: & (5,8,3)_A & \longleftrightarrow & (5,8,3)_B \\
N=5: & (4,8,4)_A & \longleftrightarrow & (4,8,4)_B\,.
\end{array}
\qquad (15)
$$

It is useful to explicitly present the supersymmetry transformations (depending on the $\varepsilon_i$ global fermionic parameters) in at least one case. We write below the unique pair of $N=5$ irreps (the $(4,8,4)_A$ and the $(4,8,4)_B$ multiplets) differing by connectivity, while admitting the same number of sources and the same number of targets.

The supersymmetry transformations are given by

*i) The* $N=5$ $(4,8,4)_A$ *transformations:*

$$
\begin{array}{rcl}
\delta x_1 &=& \varepsilon_2\psi_3+\varepsilon_4\psi_5+\varepsilon_3\psi_6+\varepsilon_1\psi_7+\varepsilon_5\psi_8 \\
\delta x_2 &=& \varepsilon_2\psi_4+\varepsilon_3\psi_5-\varepsilon_4\psi_6-\varepsilon_5\psi_7+\varepsilon_1\psi_8 \\
\delta x_3 &=& -\varepsilon_2\psi_1-\varepsilon_1\psi_5-\varepsilon_5\psi_6+\varepsilon_4\psi_7+\varepsilon_3\psi_8 \\
\delta x_4 &=& -\varepsilon_2\psi_2+\varepsilon_5\psi_5-\varepsilon_1\psi_6+\varepsilon_3\psi_7-\varepsilon_4\psi_8 \\
\delta\psi_1 &=& -i\varepsilon_2\dot{x}_3-\varepsilon_4 g_1-\varepsilon_3 g_2-\varepsilon_1 g_3-\varepsilon_5 g_4 \\
\delta\psi_2 &=& -i\varepsilon_2\dot{x}_4-\varepsilon_3 g_1+\varepsilon_4 g_2+\varepsilon_5 g_3-\varepsilon_1 g_4 \\
\delta\psi_3 &=& i\varepsilon_2\dot{x}_1+\varepsilon_1 g_1+\varepsilon_5 g_2-\varepsilon_4 g_3-\varepsilon_3 g_4 \\
\delta\psi_4 &=& i\varepsilon_2\dot{x}_2-\varepsilon_5 g_1+\varepsilon_1 g_2-\varepsilon_3 g_3+\varepsilon_4 g_4 \\
\delta\psi_5 &=& i\varepsilon_4\dot{x}_1+i\varepsilon_3\dot{x}_2-i\varepsilon_1\dot{x}_3+i\varepsilon_5\dot{x}_4+\varepsilon_2 g_3 \\
\delta\psi_6 &=& i\varepsilon_3\dot{x}_1-i\varepsilon_4\dot{x}_2-i\varepsilon_5\dot{x}_3-i\varepsilon_1\dot{x}_4+\varepsilon_2 g_4 \\
\delta\psi_7 &=& i\varepsilon_1\dot{x}_1-i\varepsilon_5\dot{x}_2+i\varepsilon_4\dot{x}_3+i\varepsilon_3\dot{x}_4-\varepsilon_2 g_1 \\
\delta\psi_8 &=& i\varepsilon_5\dot{x}_1+i\varepsilon_1\dot{x}_2+i\varepsilon_3\dot{x}_3-i\varepsilon_4\dot{x}_4-\varepsilon_2 g_2 \\
\delta g_1 &=& -i\varepsilon_4\dot{\psi}_1-i\varepsilon_3\dot{\psi}_2+i\varepsilon_1\dot{\psi}_3-i\varepsilon_5\dot{\psi}_4-i\varepsilon_2\dot{\psi}_7 \\
\delta g_2 &=& -i\varepsilon_3\dot{\psi}_1+i\varepsilon_4\dot{\psi}_2+i\varepsilon_5\dot{\psi}_3+i\varepsilon_1\dot{\psi}_4-i\varepsilon_2\dot{\psi}_8
\end{array}
\qquad (16)
$$

$$\begin{array}{rcl}\delta g_3 &=& -i\varepsilon_1\dot{\psi}_1 + i\varepsilon_5\dot{\psi}_2 - i\varepsilon_4\dot{\psi}_3 - i\varepsilon_3\dot{\psi}_4 + i\varepsilon_2\dot{\psi}_5\\ \delta g_4 &=& -i\varepsilon_5\dot{\psi}_1 - i\varepsilon_1\dot{\psi}_2 - i\varepsilon_3\dot{\psi}_3 + i\varepsilon_4\dot{\psi}_4 + i\varepsilon_2\dot{\psi}_6\end{array}$$

*ii) The $N = 5$ $(4,8,4)_B$ transformations:*

$$\begin{array}{rcl}\delta x_1 &=& \varepsilon_5\psi_2 + \varepsilon_2\psi_3 + \varepsilon_4\psi_5 + \varepsilon_3\psi_6 + \varepsilon_1\psi_7\\ \delta x_2 &=& -\varepsilon_5\psi_1 + \varepsilon_2\psi_4 + \varepsilon_3\psi_5 - \varepsilon_4\psi_6 + \varepsilon_1\psi_8\\ \delta x_3 &=& -\varepsilon_2\psi_1 - \varepsilon_5\psi_4 - \varepsilon_1\psi_5 + \varepsilon_4\psi_7 + \varepsilon_3\psi_8\\ \delta x_4 &=& -\varepsilon_2\psi_2 + \varepsilon_5\psi_3 - \varepsilon_1\psi_6 + \varepsilon_3\psi_7 - \varepsilon_4\psi_8\\ \delta\psi_1 &=& -i\varepsilon_5\dot{x}_2 - i\varepsilon_2\dot{x}_3 - \varepsilon_4 g_1 - \varepsilon_3 g_2 - \varepsilon_1 g_3\\ \delta\psi_2 &=& i\varepsilon_5\dot{x}_1 - i\varepsilon_2\dot{x}_4 - \varepsilon_3 g_1 + \varepsilon_4 g_2 - \varepsilon_1 g_4\\ \delta\psi_3 &=& i\varepsilon_2\dot{x}_1 + i\varepsilon_5\dot{x}_4 + \varepsilon_1 g_1 - \varepsilon_4 g_3 - \varepsilon_3 g_4\\ \delta\psi_4 &=& i\varepsilon_2\dot{x}_2 - i\varepsilon_5\dot{x}_3 + \varepsilon_1 g_2 - \varepsilon_3 g_3 + \varepsilon_4 g_4\\ \delta\psi_5 &=& i\varepsilon_4\dot{x}_1 + i\varepsilon_3\dot{x}_2 - i\varepsilon_1\dot{x}_3 - \varepsilon_5 g_2 + \varepsilon_2 g_3\\ \delta\psi_6 &=& i\varepsilon_3\dot{x}_1 - i\varepsilon_4\dot{x}_2 - i\varepsilon_1\dot{x}_4 + \varepsilon_5 g_1 + \varepsilon_2 g_4\\ \delta\psi_7 &=& i\varepsilon_1\dot{x}_1 + i\varepsilon_4\dot{x}_3 + i\varepsilon_3\dot{x}_4 - \varepsilon_2 g_1 + \varepsilon_5 g_4\\ \delta\psi_8 &=& i\varepsilon_1\dot{x}_2 + i\varepsilon_3\dot{x}_3 - i\varepsilon_4\dot{x}_4 - \varepsilon_2 g_2 - \varepsilon_5 g_3\\ \delta g_1 &=& -i\varepsilon_4\dot{\psi}_1 - i\varepsilon_3\dot{\psi}_2 + i\varepsilon_1\dot{\psi}_3 + i\varepsilon_5\dot{\psi}_6 - i\varepsilon_2\dot{\psi}_7\\ \delta g_2 &=& -i\varepsilon_3\dot{\psi}_1 + i\varepsilon_4\dot{\psi}_2 + i\varepsilon_1\dot{\psi}_4 - i\varepsilon_5\dot{\psi}_5 - i\varepsilon_2\dot{\psi}_8\\ \delta g_3 &=& -i\varepsilon_1\dot{\psi}_1 - i\varepsilon_4\dot{\psi}_3 - i\varepsilon_3\dot{\psi}_4 + i\varepsilon_2\dot{\psi}_5 - i\varepsilon_5\dot{\psi}_8\\ \delta g_4 &=& -i\varepsilon_1\dot{\psi}_2 - i\varepsilon_3\dot{\psi}_3 + i\varepsilon_4\dot{\psi}_4 + i\varepsilon_2\dot{\psi}_6 + i\varepsilon_5\dot{\psi}_7\,.\end{array} \tag{17}$$

## 5. The $N = 4$ off-shell invariant actions

We show here how to use the knowledge of the irreducible representations of the supersymmetry in order to produce manifestly off-shell invariant actions without introducing the superfield formalism. We discuss the $N = 4$ case, because in this case the lagrangians entering the off-shell invariant actions have the correct $d = 2$ mass-dimension of a kinetic term. We remember that $N = 4$ admits four irreps, given by the $(4,4)$, $(3,4,1)$, $(2,4,2)$ and $(1,4,3)$ multiplets. The lowest-dimensional fields are assumed to have 0 dimension. They correspond, physically, to coordinates of a target manifold whose dimension is given, respectively, by 4, 3, 2 and 1.

We construct the associated invariants using the fact that the supersymmetry generators $Q_i$'s act as graded Leibniz derivatives. Manifestly invariant actions $S$ of the $N$-extended supersymmetry can be obtained by expressing them as

$$S = \int dt\,(Q_1\cdot\ldots\cdot Q_N f(x_1, x_2, \ldots, x_k)) \tag{18}$$

with the supersymmetry transformations applied to an arbitrary function of the 0-dimensional fields $x_i$'s, $i = 1, \ldots, k$ entering an irreducible multiplet. It is only for $N = 4$ that the manifestly supersymmetric lagrangian density has the correct dimension of a kinetic term (the supersymmetry generators, the "square roots" of the hamiltonian, have mass-dimension $d = \frac{1}{2}$).

The $k$ variables $x_i$'s can be regarded as a coordinates of a $k$-dimensional manifold. The corresponding actions can therefore be seen as $N = 4$ supersymmetric one-dimensional sigma models evolving in a $k$-dimensional target manifold. For each $N = 4$ irrep we get the following results. In all cases below the arbitrary $\alpha(x_i)$ function is given by $\alpha = \nabla f(x_i)$. We get

*i*) The $N = 4$ $(4, 4)$ case:

$$\begin{array}{l} Q_i(x, x_j; \psi, \psi_j) = (-\psi_i, \delta_{ij}\psi - \epsilon_{ijk}\psi_k; \dot{x}_i, -\delta_{ij}\dot{x} + \epsilon_{ijk}\dot{x}_k) \\ Q_4(x, x_j; \psi, \psi_j) = (\psi, \psi_j; \dot{x}, \dot{x}_j)\,. \end{array} \tag{19}$$

The most general invariant lagrangian $L$ of dimension $d = 2$ is given by

$$\begin{aligned} L \;=\; & \alpha(\vec{x})[\dot{x}^2 + \dot{x}_j^2 - \psi\dot{\psi} - \psi_j\dot{\psi}_j] + \partial_x\alpha[\psi\psi_j\dot{x}_j - \frac{1}{2}\,\epsilon_{ijk}\psi_i\psi_j\dot{x}_k] \\ & + \partial_l\alpha[\psi_l\psi\dot{x} + \psi_l\psi_j\dot{x}_j + \frac{1}{2}\,\epsilon_{ljk}\psi_j\psi_k\dot{x} - \epsilon_{ljk}\psi_j\dot{x}_k\psi] \\ & - \nabla\alpha\frac{1}{6}\,\epsilon_{ljk}\psi\psi_l\psi_k\psi_k\,. \end{aligned} \tag{20}$$

*ii*) The $N = 4$ $(3, 4, 1)$ case:

$$\begin{array}{l} Q_i(x_j; \psi, \psi_j; g) = (\delta_{ij}\psi - \epsilon_{ijk}\psi_k; \dot{x}_i; -\delta_{ij}g + \epsilon_{ijk}\dot{x}_k; -\dot{\psi}_i) \\ Q_4(x_j; \psi, \psi_j; g_j) = (\psi_j; g, \dot{x}_j; \dot{\psi})\,. \end{array} \tag{21}$$

The most general invariant lagrangian $L$ of dimension $d = 2$ is given by

$$\begin{aligned} L \;=\; & \alpha(\vec{x})[\dot{x}_j^2 + g^2 - \psi\dot{\psi} - \psi_j\dot{\psi}_j] + \partial_i\alpha[\epsilon_{ijk}(\psi\psi_j\dot{x}_k + \frac{1}{2}\,g\psi_j\psi_k) \\ & - g\psi\psi_i + \psi_i\psi_j\dot{x}_j] - \frac{\nabla\alpha}{6}\,\epsilon_{ijk}\psi\psi_i\psi_j\psi_k\,. \end{aligned} \tag{22}$$

*iii*) The $N = 4$ $(2, 4, 2)$ case:

$$\begin{array}{l} Q_1(x, y; \psi_0, \psi_1, \psi_2, \psi_3; g, h) = (\psi_0, \psi_3; \dot{x}, -g, h, -\dot{y}; -\dot{\psi}_1, \dot{\psi}_2) \\ Q_2(x, y; \psi_0, \psi_1, \psi_2, \psi_3; g, h) = (\psi_3, \psi_0; \dot{y}, -h, -g, \dot{x}; -\dot{\psi}_2, -\dot{\psi}_1) \\ Q_3(x, y; \psi_0, \psi_1, \psi_2, \psi_3; g, h) = (-\psi_2, \psi_1; h, \dot{y} - \dot{x}, -g; -\dot{\psi}_3, \dot{\psi}_0) \\ Q_4(x, y; \psi_0, \psi_1\psi_2, \psi_3; g, h) = (\psi_1, \psi_2; g, \dot{x}, \dot{y}, h; \dot{\psi}_0, \dot{\psi}_3)\,. \end{array} \tag{23}$$

The most general invariant lagrangian $L$ of dimension $d = 2$ is given by

$$\begin{aligned} L \;=\; & \alpha(x, y)[\dot{x}^2 + \dot{y}^2 + g^2 + h^2 - \psi\dot{\psi} - \psi_j\dot{\psi}_j] \\ & + \partial_x\alpha[\dot{y}(\psi_1\psi_2 - \psi_0\psi_3) + g(\psi_2\psi_3 - \psi_0\psi_1) + h(\psi_1\psi_3 + \psi_0\psi_2)] \\ & + \partial_y\alpha[-\dot{x}(\psi_1\psi_2 - \psi_0\psi_3) - g(\psi_1\psi_3 + \psi_0\psi_2) + h(\psi_2\psi_3 - \psi_0\psi_1)] \\ & - \nabla\alpha\psi_0\psi_1\psi_2\psi_3\,. \end{aligned} \tag{24}$$

*iv*) The $N = 4$ $(1, 4, 3)$ case:

$$\begin{aligned} &Q_i(x;\psi,\psi_j,g_j) = (-\psi_i; g_i, -\delta_{ij}\dot{x} + \epsilon_{ijk}g_k; \delta_{ij}\dot{\psi} - \epsilon_{ijk}\dot{\psi}_k),\\ &Q_4(x;\psi,\psi_j;g_j) = (\psi;\dot{x},g_j;\dot{\psi}_j)\,. \end{aligned} \tag{25}$$

The most general invariant lagrangian $L$ of dimension $d = 2$ is given by

$$\begin{aligned} L \;=&\; \alpha(x)[\dot{x}^2 - \psi\dot{\psi} - \psi_i\dot{\psi}_i + {g_i}^2] \\ &+ \alpha'(x)[\psi g_i\psi_i - \frac{1}{2}\,\epsilon_{ijk}g_i\psi_j\psi_k] - \frac{\alpha''(x)}{6}\,[\epsilon_{ijk}\psi\psi_i\psi_j\psi_k]\,. \end{aligned} \tag{26}$$

## 6. An example of an $N = 8$ off-shell invariant action

The $N = 4$ invariant actions for the $(4, 4)$, $(3, 4, 1)$, $(1, 4, 3)$ multiplets presented in the previous section are expressed in terms of the $\delta_{ij}$ and $\epsilon_{ijk}$ quaternionic tensors. This is of course a consequence of the relation between $N = 4$ supersymmetry and quaternions. This information allows us to construct $N = 8$ off-shell invariant actions by exploiting the connection with the octonions. Indeed, according to [14], the $N = 8$ supersymmetry is produced from the lifting of the $Cl(0, 7)$ Clifford algebra to $Cl(9, 0)$. On the other hand, it is well-known [22], that the seven $8\times 8$ antisymmetric gamma matrices of $Cl(0, 7)$ can be recovered by the left-action of the imaginary octonions on the octonionic space. As a result, the entries of the seven antisymmetric gamma-matrices of $Cl(0, 7)$ (and, as a consequence, of the $N = 8$ supersymmetry transformations) can be expressed in terms of the totally antisymmetric octonionic structure constants $C_{ijk}$'s which generalize the quaternionic $\epsilon_{ijk}$ antisymmetric tensors. The non-vanishing $C_{ijk}$'s are given by

$$C_{123} = C_{147} = C_{165} = C_{246} = C_{257} = C_{354} = C_{367} = 1. \tag{27}$$

and are associated with the seven lines of the Fano's projective plane, the smallest example of a finite projective geometry, see [23]. A strategy can be adopted to construct $N = 8$ off-shell invariant actions. We illustrate it in the simplest example, the $N = 8$ $(1, 8, 7)$ multiplet, admitting seven auxiliary fields. This multiplet preserves the octonionic structure since the seven auxiliary fields are related to the seven imaginary octonions. The supersymmetry transformations are given by

$$\begin{aligned} &Q_i(x;\psi,\psi_j,g_j) = (-\psi_i; g_i, -\delta_{ij}\dot{x} + C_{ijk}g_k; \delta_{ij}\dot{\psi} - C_{ijk}\dot{\psi}_k),\\ &Q_8(x;\psi,\psi_j;g_j) = (\psi;\dot{x},g_j;\dot{\psi}_j) \end{aligned} \tag{28}$$

for $i, j, k = 1, \ldots, 7$. We construct the most general $N = 8$ off-shell invariant action with the dimension of a kinetic term for the $(1, 8, 7)$ multiplet by requiring an octonionic covariantization principle. When restricted to an $N = 4$ subalgebra, the invariant action should have the form of the $N = 4$

$(1, 4, 3)$ action (26). This restriction can be done in seven inequivalent ways (the seven lines of the Fano's plane). The general $N = 8$ action should be expressed in terms of the octonionic structure constants. With respect to (26), an extra-term could in principle be present. It is given by $\int dt\beta(x)C_{ijkl}\psi_i\psi_j\psi_k\psi_l$, where $C_{ijkl}$ is the octonionic tensor of rank 4

$$C_{ijkl} = \frac{1}{6}\,\epsilon_{ijklmnp}C_{mnp} \tag{29}$$

($\epsilon_{ijklmnp}$ is the seven-indices totally antisymmetric tensor). Please notice that the rank-4 tensor is obviously vanishing when restricting to the quaternionic subspace. The term $\int dt\beta(x)C_{ijkl}\psi_i\psi_j\psi_k\psi_l$ breaks the $N = 8$ supersymmetries and cannot enter the invariant action. For what concerns the other terms, starting from the general action (with $i, j, k = 1, \ldots, 7$)

$$\begin{aligned} S \;=\; \int dt\{\alpha(x)[\dot{x}^2 - \psi\dot{\psi} - \psi_i\dot{\psi}_i + {g_i}^2] \\ +\; \alpha'(x)[\psi g_i\psi_i - \frac{1}{2}\,C_{ijk}g_i\psi_j\psi_k] - \frac{\alpha''(x)}{6}\,[C_{ijk}\psi\psi_i\psi_j\psi_k]\} \end{aligned} \tag{30}$$

it is easily proven that the invariance under the $Q_i$ generator ($i = 1, \ldots 7$) is broken by terms which, after integration by parts, contain at least a second derivative $\alpha''$. The full $N = 8$ invariance (the invariance under $Q_8$ is automatically guaranteed) requires imposing the constraint $\alpha''(x) = 0$. Therefore $\alpha$ is a linear function in $x$ (we recall that $\alpha$ is unconstrained for the corresponding $N = 4$ case). The most general $N = 8$ off-shell invariant action of the $(1, 8, 7)$ multiplet is given by

$$S = \int dt\{(ax + b)[\dot{x}^2 - \psi\dot{\psi} - \psi_i\dot{\psi}_i + {g_i}^2] + a[\psi g_i\psi_i - \frac{1}{2}\,C_{ijk}g_i\psi_j\psi_k]\}\,. \tag{31}$$

It is quite remarkable that the octonionic structure constants enter the $N = 8$ invariant actions as coupling constants. It is worth pointing out that the non-associativity of the octonions plays no role here. The supersymmetry transformations are ordinary (associative) transformations, the octonionic structure constants expressing the non-vanishing entries of ordinary matrices.

## 7. The fusion algebra of the irreps

The notion of fusion algebra of the supersymmetric vacua of the $N$-extended one dimensional supersymmetry was introduced in [14]. The fusion algebra encodes information concerning the decomposition into irreps of the tensor products of irreps. This information can be relevant in constructing multilinear invariants; we recall in fact that in any given multiplet the field(s) with highest dimension is mapped, under the supersymmetry transformations, into the time-derivative of a lower-dimensional field. Its integral can furnish an invariant term of the action.

The fusion algebras can also be nicely presented in terms of their associated graphs, see [18]. The tensoring of two zero-energy vacuum-state irreps (irreps associated with the zero energy eigenvalue of the hamiltonian operator $H$) can be symbolically written as

$$[i] \times [j] \;=\; N_{ij}{}^{k}[k]\,, \tag{32}$$

where $N_{ij}{}^{k}$ are non-negative integers specifying the decomposition of the tensored-products irreps into its irreducible constituents. The $N_{ij}{}^{k}$ integers satisfy a fusion algebra with the following properties :

1) Constraint on the total number of component fields,

$$\forall\; i,j \quad \sum_{k} N_{ij}{}^{k} \;=\; 2n\,, \tag{33}$$

where $n$ is the number of bosonic (fermionic) fields in the given irreps.

2) The symmetry property

$$N_{ij}{}^{k} \;=\; N_{ji}{}^{k}\,. \tag{34}$$

3) The associativity condition,

$$[i] \times ([j] \times [k]) \;=\; ([i] \times [j]) \times [k] \tag{35}$$

which implies the commutativity of the $(N_i)^k_j \equiv N^k_{ij}$ fusion matrices.

In a graphical presentation of the fusion algebra the irreps correspond to points. $N^k_{ij}$ oriented lines (with arrows) connect the $[j]$ and the $[k]$ irrep if the decomposition $[i] \times [j] \;=\; N_{ij}{}^{k}[k]$ holds. The arrows are dropped from the lines if the $[j]$ and $[k]$ irreps can be interchanged. The $[i]$ irrep should correspond to a generator of the fusion algebra. This means that the whole set of $N_l = N_{lj}{}^{k}$ fusion matrices is produced as sum of powers of the $N_i = N_{ij}{}^{k}$ fusion matrix.

It is particularly instructive to present explicitly the $N = 2$ case. It admits four irreps (if we discriminate their statistics, bosonic or fermionic), given by

$$[1] \equiv (2,2)_{Bos};\;\; [2] \equiv (1,2,1)_{Bos};\;\; [3] \equiv (2,2)_{Fer};\;\; [4] \equiv (1,2,1)_{Fer} \tag{36}$$

The corresponding $N = 2$ fusion algebra is realized in terms of four $4 \times 4$, mutually commuting, matrices given by

$$\begin{aligned}
&N_1 = \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 2 & 0 & 2 \\ 1 & 0 & 1 & 2 \\ 0 & 2 & 0 & 2 \end{pmatrix} \equiv X; \qquad N_2 = N_4 = \begin{pmatrix} 0 & 2 & 0 & 2 \\ 0 & 2 & 0 & 2 \\ 0 & 2 & 0 & 2 \\ 0 & 2 & 0 & 2 \end{pmatrix} \equiv Y;\\
&N_3 = \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 2 & 0 & 2 \\ 1 & 2 & 1 & 0 \\ 0 & 2 & 0 & 2 \end{pmatrix} \equiv Z.
\end{aligned} \tag{37}$$

The fusion algebra admits three distinct elements, $X, Y, Z$ and one generator (we can choose either $X$ or $Z$), due to the relations

$$Y = \frac{1}{8}\left(X^3 - 2X\right), \qquad Z = -\frac{1}{4}\left(X^3 - 6X^2 + 4X\right). \tag{38}$$

The vector space spanned by $X, Y, Z$ is closed under multiplication

$$\begin{aligned} X^2 = Z^2 = ZX &= X + 2Y + Z, \\ XY = Y^2 = YZ &= 4Y. \end{aligned} \tag{39}$$

This fusion algebra corresponds to the "smiling face" graph of [18].

## 8. Conclusions

The supersymmetric quantum mechanics is a fascinating subject with several open problems. The potentially most interesting one concerns perhaps the construction of off-shell invariant actions whose lagrangians have the correct dimensions of a kinetic term, for large values of $N$ (let's say $N > 8$). These types of actions could provide some hints towards an off-shell formulation of higher-dimensional supergravity and $M$-theory. We recall that, up to now, no one-dimensional sigma model with non-trivial action (namely, possessing a non-constant background metric) was found for $N > 8$. Even for $N \leq 8$ the program of classifying the whole set of off-shell invariant actions for each given irreducible multiplet has not been completed yet. A large class of $N = 8$ off-shell actions was produced in [6]. However, the action of the $N = 8$ $(1, 8, 7)$ model here discussed was not contained in that list.

To complete this program could be particularly valuable in the light of the recent results (discussed in Section **4**) concerning the different connectivities of some $N = 5$ and $N = 6$ irreducible multiplets. Indeed, multiplets presenting fields with different mass-dimensions have an obvious physical meaning. The number of 0-dimensional bosonic fields corresponds to the dimensionality of the target manifold of the one-dimensional sigma models. It would be interesting to understand the possible physical implications of the multiplets with same content of fields of given dimension which, nevertheless, differ in connectivity. The $N = 5$ and $N = 6$ cases which present these features have not been studied in the literature yet. The manifestly supersymmetric "linear approach" of constructing off-shell invariants, that we outlined in Section **6**, looks promising in addressing this problem.

It should be mentioned that the classification of the irreducible multiplets of the one-dimensional $N$-extended supersymmetry finds application not only in the construction of the off-shell invariant actions of the one-dimensional supersymmetric quantum mechanics, but also in the two-dimensional supersymmetric quantum mechanical models (because we can decompose the problem in terms of the light-cone coordinates).

We conclude by pointing out that the supersymmetric quantum mechanics presents several interesting open questions which have still to be clarified.

One of the most puzzling concerns the similarities shared by both linear and non-linear representations of the $N$-extended one-dimensional supersymmetry algebra.

## Acknowledgments

I am grateful to the organizers for the kind invitation. The majority of the results here reported have been obtained jointly with Z. Kuznetsova and M. Rojas.

## References

[1] E. Witten, *Nucl. Phys.* **B 188** (1981) 513.

[2] G. Parisi and N. Sourlas, *Nucl. Phys.* **B 206** (1982) 321.

[3] K. B. Efetov, *Adv. Phys.* **32** (1983) 53.

[4] S.J. Gates Jr., W.D. Linch and J. Phillips, hep-th/0211034; S.J. Gates Jr., W.D. Linch III, J. Phillips and L. Rana, Grav. Cosmol. **8** (2002) 96.

[5] E. Ivanov, S. Krivonos and O. Lechtenfeld, JHEP 0303 (2003) 014; S. Bellucci, E. Ivanov, S. Krivonos and O. Lechtenfeld, Nucl. Phys. **B 684** (2004) 321.

[6] S. Bellucci, E. Ivanov, S. Krivonos and O. Lechtenfeld, *Nucl. Phys.* **B 699** (2004) 226 (also hep-th/0406015).

[7] M. Faux and S.J. Gates Jr., Phys. Rev. **D 71** (2005) 065002.

[8] S. Bellucci, A. Beylin, S. Krivonos and A. Shcherbakov, *Phys. Lett.* **B 633** (2006)382 (also hep-th/0511054); S. Bellucci, A. Beylin, S. Krivonos, A. Nersessian and E. Orazi, Phys. Lett. **B 616** (2005) 228; S. Bellucci and S. Krivonos, *Phys. Rev.* **D 74** (2006) 125024.

[9] F. Delduc and E. Ivanov, *Nucl. Phys.* **B 753** (2006) 211 (also hep-th/0605211); hep-th/0611247.

[10] C. Burdik, S. Krivonos and A. Shcherbakov, hep-th/0610009; S. Bellucci, S. Krivonos and A. Shcherbakov, *Phys. Lett.* **B 645** (2007) 299.

[11] C.F. Doran, M.G. Faux, S.J. Gates Jr., T. Hubsch, K.M. Iga and G.D. Landweber, hep-th/0611060.

[12] C.F. Doran, M.G. Faux, S.J. Gates Jr., T. Hubsch, K.M. Iga and G.D. Landweber, math-ph/0603012.

[13] A. Pashnev and F. Toppan, J. Math. Phys. **42** (2001) 5257 (also hep-th/0010135).

[14] Z. Kuznetsova, M. Rojas and F. Toppan, JHEP **0603** (2006) 098 (also hep-th/0511274).

[15] Z. Kuznetsova and F. Toppan, preprint hep-th/0701225.

[16] F. Toppan, hep-th/0612276.

[17] S. Bellucci and S. Krivonos, hep-th/0602199.

[18] F. Toppan, POS (IC2006) 033 (also hep-th/0610180).

[19] M.F. Atiyah, R. Bott and A. Shapiro, *Topology* (Suppl. 1) **3** (1964) 3.

[20] I.R. Porteous, "Clifford Algebras and the Classical Groups", Cambridge Un. Press, 1995.

[21] S. Okubo, *J. Math. Phys.* **32** (1991) 1657; *ibid.* **32** (1991) 1669.

[22] H.L. Carrion, M. Rojas and F. Toppan, JHEP 0304 (2003) 040.

[23] J. Baez, . *Bull. Amer. Math. Soc.* **39** (2002) 145.