On Osserman manifolds*

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Abstract

Here we give a short survey on the theory of pseudo-Riemannian Osserman manifolds, which arises from Osserman conjecture.

1. Introduction

Let (M, g) be a $m-$dimensional pseudo-Riemannian manifold of signature (p, q). By $\varepsilon_X = g(X, X)$ we will denote the norm of the vector $X \in T_pM$, and depending on their norm we distinguish the following types of tangent vectors: spacelike ($\varepsilon_X > 0$), timelike ($\varepsilon_X < 0$), null ($\varepsilon_X = 0\,, X \neq 0$), definite ($\varepsilon_X \neq 0$) and unit ($|\varepsilon_X| = 1$). By S_pM, S_p^+M, and S_p^-M we will denote all unit nonnull, spacelike and timelike vectors in T_pM, respectively.
Let ∇ be the Levi-Civita connection and let R be the associated Riemannian curvature tensor, $R(X, Y) := [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$. The Jacobi operator $\mathcal{R}_X : Y \longrightarrow R(Y, X)X$ is a symmetric endomorphism of the tangent bundle TM. For nonnull X, $\mathcal{R}_X$ preserves the orthogonal space $\{X\}^\perp$, and we will use the notation $\bar{\mathcal{R}}_X$ for the restriction of $\mathcal{R}_X$ to this space.
If M is a Riemannian manifold which is locally a rank-one symmetric space or if M is flat, then the local isometry group acts transitively on the unit sphere bundle SM[1], and hence the eigenvalues of $\mathcal{R}_X$ are constant on SM. Based on the paper [34], Osserman in [33] wondered if the converse held, this question is usually known as Osserman conjecture.

If the eigenvalues of the Jacobi operator $\mathcal{R}_X$ are constant on SM, then M is locally a rank-one symmetric space or a flat space ?

This question was starting point of the field. In the years which has followed many authors have worked in this and related fields. For more details about this topic one can find in monographs [27], [23].

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[1] Spaces which satisfy this condition are called *two-point homogenous spaces*. Two-point homogeneous spaces are $\mathbb{R}^n, \mathbb{R}\mathrm{P}^n, \mathrm{S}^n, \mathrm{H}^n, \mathbb{C}\mathrm{P}^n, \mathbb{C}\mathrm{H}^n, \mathbb{H}\mathrm{P}^n$, $\mathbb{H}\mathrm{H}^n$, $\mathbb{C}ay\,\mathrm{P}^2$, and $\mathbb{C}ay\mathrm{H}^2$.

Our paper is organized as follows. Section 1 is devoted to the introduction in this topic and motivation. In Section 2 we give some basic notations and notions which we will use in the rest of the paper. Section 3 is devoted to the Riemannian case which is almost closed after papers of Chi and Nikolayevsky. The last Section 4 is devoted to the Osserman manifolds in pseudo-Riemannian settings. Except the Lorentzian case, which is solved by Blažić, Bokan and Gilkey (see [6]), there are a lot of open problems. Here we are discussing some aspects of these problems, giving known and new results and some new ideas and tools for further treatment of Osserman type problems.

2. Preliminaries

2.1. Notions and notations.

For natural generalizations of Osserman conjecture to the pseudo-Riemannian manifolds the following notions were defined. One says that (M, g) is *spacelike (resp. timelike) pointwise Osserman* if the characteristic polynomial of $\mathcal{R}_X$ is independent on $X \in S_p^+M$ (resp. S_p^-M). Manifolds such that the characteristic polynomial of $\mathcal{R}_X$ is constant on the bundle S^+M(resp. S^-M) of unit spacelike (resp. timelike) vectors are called *globally spacelike (resp. timelike)* Osserman. Also, the notions of pointwise (global) null Osserman condition are well-defined. But, in this article we will not consider such manifolds. In the higher signature setting, unlike in the Riemannian setting, the eigenvalue structure of a symmetric endomorphism does not determine the conjugacy class, but its Jordan normal form. We say that (M, g) is *spacelike pointwise Jordan-Osserman (resp. timelike pointwise Jordan-Osserman)* if the Jordan normal form of $\mathcal{R}_X$ is independent on $X \in S_p^+M$ (resp. on S_p^-M). Similarly, we define *globally spacelike (timelike) Jordan-Osserman manifolds.*

2.2. Algebraic curvature tensor

An *algebraic curvature tensor* R in a pseudo-Euclidean space $V \cong \mathbb{R}^m$ of signature (p, q) is a $(3, 1)$ tensor having the same symmetries as the curvature tensor of a Riemannian manifold,

$$\begin{aligned} &R(X,Y) + R(Y,X) = 0, \\ &R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0, \\ &g(R(X,Y)Z,W) = g(R(Z,W)X,Y). \end{aligned} \tag{1}$$

One says R is an *Osserman algebraic curvature tensor* if the associated Jacobi operator has characteristic polynomial constant on the unit pseudospheres S_p^+M and S_p^-M. Similarly, R is a *spacelike (timelike) Jordan-Osserman algebraic curvature tensor* if the associated Jacobi operator has Jordan normal form constant on the unit pseudosphere S_p^+M (S_p^-M).

Remark 1. Using the theory of normal coordinates Gilkey (see [25]) showed that any algebraic curvature tensor is geometrically realizable. More precisely, he showed that for any algebraic curvature tensor R on V, there exists a pseudo-Riemannian manifold (M, g), a point $P \in M$, an isometry $\Psi : T_P M \longrightarrow V$ such that for all tangent vector fields $X, Y, Z, W \in T_P M$ holds $R(\Psi X, \Psi Y, \Psi Z, \Psi W) = {}^g R(X, Y, Z, W)$, where ${}^g R$ is the curvature tensor of (M, g).

Example 1. The simplest examples of Osserman algebraic curvature tensors are:

(i1) The curvature tensor R^c of a metric of constant sectional curvature is given up to scale by

$$R^c(X, Y)Z = R^c(X, Y, Z) = g(X, Z)Y - g(Y, Z)X\,. \tag{2}$$

(i2) The curvature tensor of the Fubini-Study metrics on $\mathbb{CP}(m/2)$ is given (up to scale) by $R = R^c + R^I$ where

$$R^I(X, Y, Z) = g(Y, IZ)IX - g(X, IZ)Y - 2\,g(X, IY)Z\,, \tag{3}$$

and where I is an almost complex structure making g Hermitian.

Example 2. An algebraic curvature tensor R in $\mathbb{R}^m$ has a *Cliff(ν)*-structure ($\nu \geq 0$), if there exist anticommuting skew-symmetric orthogonal operators $I_1, \ldots, I_\nu$, and the numbers $\lambda_0, \lambda_1, \ldots \lambda_\nu$, with $\lambda_i \neq \lambda_j$ ($i \neq j$), such that

$$R = \lambda_0 + \frac{1}{3}\sum_{i=1}^{\nu}(\lambda_i - \lambda_0)\,R^{I_i} \tag{4}$$

The fact that skew-symmetric operators I_i are orthogonal and anticommute is equivalent to each of the following sets of equations:

$$g(I_i X, I_j X) = \delta_{ij} g(X, X) \qquad \text{and} \qquad I_i I_j + I_j I_i = -2\delta_{ij}\,\mathrm{Id}$$

for all $i, j = 1, \ldots, \nu$ and all $X \in \mathbb{R}^m$. For any unit vector X, the Jacobi operator R_X has constant eigenvalues $\lambda_0, \lambda_1, \ldots, \lambda_\nu$. The eigenspace corresponding to the eigenvalue λ_i, $i \neq 0$, is $E_{\lambda_i}(X) = \mathrm{span}_{j:\lambda_j = \lambda_i}(I_i X)$, and the λ_0-eigenspace is $E_{\lambda_0}(X) = (\mathrm{span}(X, I_1 X, \ldots, I_\nu X))^\perp$, provided $\nu < m - 1$.

A Riemannian manifold M has a *Cliff(ν)*-structure if its curvature tensor at every point does.

Osserman algebraic curvature tensors with Clifford structure were introduced by Gilkey [25], and Gilkey, Swann and Vanhecke [26]. They showed that a *Cliff*(ν) algebraic curvature tensor (manifold) is Osserman (pointwise Osserman, respectively). Since the curvature tensor of the Cayley projective plane (or its hyperbolic dual) doesn't allow any Clifford structure, there exists at least one Osserman algebraic curvature tensor having no Clifford structure, [30].

One of the most natural approaches to the study of Osserman type problems was suggested in [26]. It has two steps:

(1) classifying the Osserman (Jordan-Osserman) algebraic curvature tensors,

(2) finding those Osserman (Jordan-Osserman) algebraic curvature tensors which can be realized as the curvature tensors of a pseudo-Riemannian manifold.

The standard tool for (2) is the second Bianchi identity. The difficult part is (1), but thanks to the remarkable construction of [25, 26], the right candidate for (1) is given in Example 2.

Let Φ be a linear map of a pseudo-Euclidean vector space V of signature (p,q) and dimension $m = p+q$. If $\Phi^* = \pm\,\Phi$ we define

$$R_\Phi(X,Y)Z = \begin{cases} g(\Psi Y,Z)\Phi X - g(\Psi Z,Z)\Phi Y & \text{if } \Psi = \Psi^*, \\ g(\Psi Y,Z)\Phi X - g(\Psi Z,Z)\Phi Y - 2g(\Psi X,Y)\Phi Z & \text{if } \Psi = -\Psi^*. \end{cases}$$

Let $\mathcal{C}(V)$ the space of algebraic curvature tensors on V and let

$$\mathcal{S}(V) = \operatorname{span}\{R_\Phi \mid \Phi^* = \Phi\} \subseteq \mathcal{C}(V),$$
$$\mathcal{A}(V) = \operatorname{span}\{R_\Phi \mid \Phi^* = -\Phi\} \subseteq \mathcal{C}(V).$$

Using representation theory Fiedler in [18] proved

$$\mathcal{C}(V) = \mathcal{S}(V) + \mathcal{A}(V).$$

Set of algebraic curvature tensors on V is completely described in

Theorem 2.1 ([27]) $\mathcal{C}(V) = \mathcal{S}(V) = \mathcal{A}(V)$.

2.3. k-stein manifolds

Following Carpenter, Gray and Wilmore [15], a pseudo-Riemannian manifold (M,g) is called k-stein at a point $p \in M$ if there exist constants C_t for every $1 \le t \le k$ such that

$$\operatorname{Tr}(K_X^t) = (\varepsilon_X)^t C_t \tag{5}$$

holds for each $X \in S_pM$. Pseudo-Riemannian manifold (M,g) is k-stein if it is k-stein in each point $p \in M$.

Now, we give some basic results on k-stein manifolds, for details see [27].

Theorem 2.2 *(a) Let (M,g) be a pseudo-Riemannian manifold of signature (p,q). Then the following statements are equivalent.*

(i1) If $p \ge 1$ then M is timelike Osserman at $p \in M$.

(i2) If $p \le \dim M - 1$ then M is spacelike Osserman at $p \in M$.

(i3) M is k-tein at $p \in M$ for every k.

(b) Pseudo-Riemannian manifold (M,g) of dimension $m \ge 3$ is k-stein at a point $p \in M$ if relation (5) holds for every $X \in T_pM$.

(c) Pseudo-Riemannian manifold (M,g) of signature (p,q) is Einstein manifold iff it is $1-$stein.

3. Riemannian case

Since the eigenspaces of Jacobi operators on Osserman manifolds form a distributions on unit sphere bundle SM, the following theorem become essential tool in Riemannian settings.

Theorem 3.1 (see [1, 4]) *For a integer $m = 2^r m_0$ with m_0 odd number, we define the Adams number $\nu(r)$ by $\nu(i) = 2^i - 1$, for $i = 0, 1, 2, 3$, and $\nu(i+4) = \nu(i) + 8$.*

(a) SM admits a $q-$dimensional distribution for $2q \leq m-1$ iff $q \leq \nu(r)$.

(b) $\mathbb{R}^m$ admits a Cliff(ν)-module structure iff $\nu \leq \nu(r)$.

Let us remark that above theorem gives topological restrictions on the number of distributions of small dimension on SM. Using this theorem, Chi at the end of eighties showed the following theorem.

Theorem 3.2 ([12, 13, 14]) *Let M be a m-dimensional Riemannian manifold. Osserman conjecture holds in the following cases*

(a) $m \equiv 1$ mod 2, *(b) $m \equiv 2$ mod 4,* *(c) $m = 4$,*

(d) $m = 4n, n = 2, 3, \dots$ and M is simple connected compact quaternionic Kähler manifold with vanishing second Betti number,

(e) M is Kähler manifold of negative or positive sectional curvature.

The proofs of (a), (b) are direct consequences of Theorem 3.1 (a). In the case (a) such manifolds allow only trivial distributions on SM, and the Jacobi operator $\mathcal{R}_X$ has at most two eigenvalues: 0 of multiplicity 1 and λ_0 of multiplicity $m-1$ (for the eigenvalue structure of $\mathcal{R}_X$ we will use the following notations $[(0,1), (\lambda_0, m-1)]$). This eigenvalue structure implies that curvature tensor of M is given up to scale by (2), i.e. M has constant sectional curvature. In the case (b) the eigenvalue structure of $\mathcal{R}_X$ is $[(0,1), (\lambda_1, 1), (\lambda_0, m-2)]$ and this eigenvalue structure implies that curvature of M is given up to scale by (3), i.e. the curvature tensor of M is curvature tensor of the Fubini-Study metrics on $\mathbb{CP}(m/2)$ and consequently M has either constant sectional curvature or is covered by a standard projective space or its noncompact dual. In the proof of (b) and (c) Chi showed the following property for extremal eigenvalues of Jacobi operator: if X and Y are unit vectors then

$$\mathcal{R}_X Y = \lambda Y \quad \text{if and only if} \quad \mathcal{R}_Y X = \lambda X. \tag{6}$$

This property of algebraic curvature tensors is known as Rakić duality, and it is proved in [36] for any eigenvalue of pointwise Riemannian manifold.

In 1999, Dotti and Druetta [16] showed that Osserman conjecture holds for non flat homogeneous manifold of nonpositive curvature.

Using the remarkable construction given in [25, 26], duality principle and some other results, Nikolayevsky proved in almost all cases that Osserman condition implies existence of Clifford structure on M, i.e., he proved Osserman conjecture in almost all cases (2003 – 2007). More precisely, he proved the following theorem.

Theorem 3.3 *([30, 31, 32]) Let M^m be a Riemannian globally Osserman manifold of dimension $m \geq 2$, or a pointwise Osserman manifold of dimension $m \neq 2, 4$. Then M^m is flat or locally rank-one symmetric except, possibly, in the following case: $m = 16$ and the Jacobi operator has an eigenvalue of multiplicity 7 or 8.*

Remark 2. Previous theorem shows that there is no too big difference between globally and pointwise Osserman conditions, except in dimension 2, where any Riemannian manifold is pointwise Osserman, and in dimension 4, where exist pointwise Osserman manifolds that are not symmetric (*generalized complex space forms*, see [26, Corollary 2.7]).

Nikolayevsky in [32] announce the following theorem.

Theorem 3.4 *A pointwise Osserman manifold of dimension 16 whose Jacobi operator has two eigenvalues, of multiplicity 7 and 8, respectively, is locally isometric to the Cayley projective plane or to its hyperbolic dual.*

4. Non-Riemannian case

4.1. Algebraic curvature tensors

Here we will start with well-known facts about relation between the introduced notions in the Section 2.

Theorem 4.1 *(a) [22] Let (M, g) be a pseudo-Riemannian manifold and $p \in M$. Then (M, g) is timelike Osserman at p iff (M, g) is spacelike Osserman at p.*

(b) [28, 27] Let R be a spacelike Jordan-Osserman algebraic curvature tensor on the vector space T_pM of signature (p, q), where $p < q$. Then $\mathcal{R}_X$ is diagonalizable for any $X \in S_p^+M$.

(c) [27] Let J be an arbitrary linear map of a vector space V of dimension m. There exists $l = l(m)$ and an algebraic curvature tensor R on $\mathbb{R}^{(2^l,2^l)}$ so that $\mathcal{R}_X$ is conjugate to $\pm J\oplus$ if $X \in S^{\pm}V$.

Remark 3. It is clear from Theorem 4.1 (a) that the notions pointwise spacelike Osserman and pointwise timelike Osserman are equivalent, and if (M, g) is either of them, then (M, g) is said to be *Osserman.* The same is true for globally spacelike and timelike Osserman manifolds. Since Jordan normal form of a linear map determines its eigenvalue structure, the pointwise spacelike or timelike Jordan-Osserman manifolds are Osserman manifolds, too. The converse is not true, in [27] one can find examples of spacelike (or timelike) Osserman algebraic curvature tensors which are not timelike (or spacelike) Jordan-Osserman algebraic curvature tensor. Moreover, there exist examples of spacelike Jordan-Osserman algebraic curvature tensors which are not timelike Jordan-Osserman algebraic curvature tensors and vice versa. It means that notions of timelike and spacelike pointwise (and consequently globally) Jordan-Osserman manifolds aren't equivalent.

Remark 4. The (b) statement of the previous theorem shows that Jacobi operator $\mathcal{R}_X$ $(X \in S^{\pm}M)$ of a spacelike (timelike) Jordan-Osserman manifold of signature (p, q), $p < q$ $(p > q)$ must be diagonalizable, i.e. in the case of non-neutral metrics notions of pointwise spacelike or timelike Jordan-Osserman manifolds coincide with the notion of Osserman manifold (see Remark 3.). In the case of Osserman manifolds in neutral signature (p, p) those notions are not the same and the Jordan normal form of corresponding Jacobi operator should be arbitrarily complicated as Theorem 4.1 (c) shows.

Motivated by the results of previous theorem as well as the usefulness of the duality principle in the Riemannian settings, recently in [2] the notion of duality principle is extended to the pseudo-Riemannian manifolds. More precisely, we introduce the following definition:

Let R be an Osserman algebraic curvature tensor. For $\lambda \in \mathbb{R}$ we say that it satisfies the duality principle if for all mutually orthogonal unit vectors X, Y holds

$$\mathcal{R}_X(Y) = \varepsilon_X \lambda Y \quad \Longrightarrow \quad \mathcal{R}_Y(X) = \varepsilon_Y \lambda X\,. \tag{7}$$

If the duality principle holds for all $\lambda \in \mathbb{R}$ then we say that duality principle holds for the algebraic curvature tensor R (or for the pseudo-Riemannian Osserman manifold (M, g) whose curvature tensor is R).

For pointwise spacelike (or timelike) Jordan-Osserman manifold with diagonalizable Jacobi operator (so called *diagonalizable Osserman manifolds*), in [2] the following theorem was proved

Theorem 4.2. *Let (M, g) be a diagonalizable Osserman manifold.*

(a) If the duality principle holds for $\lambda \in \mathbb{R}$ then implication (7) holds for all $X, Y \in T_pM$ with $\varepsilon_X \neq 0$.

(b) If all eigenvalues of $\mathcal{R}_X$ $(X \in S^{\pm}M)$ are different then the duality principle holds in M.

(c) If for every $X \in S_pM$ doesn't exist null eigenvector of $\mathcal{R}_X$, then the duality principle holds in M. Specially, if M is Riemannian manifold the duality principle holds.

4.2. Lorentzian case

Let M be a Lorentzian manifold (of signature $(1, m-1)$) the answer to the Osserman conjecture is the most simple by some miracle on the pointwise level. The first results on timelike (spacelike) Osserman manifolds were obtained by Garcia-Rio, Kupeli and Vazquez-Abal. They showed that Osserman conjecture holds for timelike Osserman manifolds and if $\dim M = 3, 4$, [20, 21, 23]. Let us mention here that equivalence of spacelike and timelike pointwise Osserman condition was established later in [22]. The complete positive answer is given by N. Blažić, N. Bokan and P. B. Gilkey in [6]. More precisely, they proved the following theorem.

Theorem 4.3 ([6]) *Let M be a spacelike or timelike pointwise Osserman manifold then M is flat or locally rank-one space.*

The proof follows from the fact that spacelike (or timelike) pointwise Osserman condition implies that curvature tensor of M is given by (2). It means that sectional curvature $K(\sigma)$ of an arbitrary 2-plane $\sigma \subset T_pM$ depends of $p \in M$, and then by Schur type theorem (which holds in Lorentzian setting) implies that M is the space of constant sectional curvature.

4.3. 4-dimensional neutral Jordan-Osserman manifolds

Let M be a pointwise Jordan-Osserman manifold of signature (p, q) where $r = \min\{p, q\} \geq 2$. In this settings the most followed approach is that one proposed in [26]. The problem with this approach lies in the fact that the space of symmetric operators in pseudo-Euclidean space of signature (p, q) is very complicated if $r > 2$. In the case of non-neutral pointwise spacelike (timelike) Osserman manifolds, Theorem 4.1 (b) reduces problem to the diagonalizable case. But in the case of neutral signature (p, p) the situation is very complicated at the algebraic level, as a consequence of Theorem 4.1 (c). The only case which is considered in details is the case of 4-dimensional neutral spacelike (timelike) Osserman manifold. It turns out that geometry of such manifolds is very rich (more then in other non-neutral cases) and it is studied by many authors, see [23], [27], [7], [3], [10], [17], etc.

So, in this subsection we deal with $4-$dimensional neutral Jordan-Osserman manifolds (M, g). In T_pM there exists an orthonormal basis (E_1, E_2, E_3, E_4) of T_pM where E_1, and E_2 are timelike and E_3, and E_4 are spacelike vectors, such that all non-vanishing components of its algebraic curvature tensor with respect to this basis are (see [3], [9], [7]):

(Ia) $\mathcal{R}'_X$ is diagonalizable,

$$R_{1221} = R_{3443} = \alpha\,, \quad R_{1331} = R_{2442} = -\beta\,, \quad R_{1441} = R_{3223} = -\gamma\,,$$

$$R_{1234} = \frac{-2\,\alpha + \beta + \gamma}{3}\,,\ R_{1423} = \frac{\alpha + \beta - 2\,\gamma}{3}\,,\ R_{1342} = \frac{\alpha - 2\,\beta + \gamma}{3}\,.$$

(Ib) $\mathcal{R}'_X$ has a complex root,

$$R_{1221} = R_{3443} = \alpha\,, \quad R_{1331} = R_{2442} = -\alpha\,, \quad R_{1441} = R_{3223} = -\gamma\,,$$

$$R_{2113} = R_{2443} = -\beta\,, \quad R_{1224} = R_{1334} = \beta\,,$$

$$R_{1234} = \frac{-\alpha + \gamma}{3}\,, \quad R_{1423} = \frac{2\,\alpha - 2\,\gamma}{3}\,,\ R_{1342} = \frac{-\alpha + \gamma}{3}\,.$$

(II) the characteristic polynomial of $\mathcal{R}'_X$ has a double root α,

$$R_{1221} = R_{3443} = -\alpha + \frac{1}{2}\,,\ R_{1331} = R_{2442} = \alpha + \frac{1}{2}\,,$$

$$R_{1441} = R_{3223} = -\beta, \quad -R_{2113} = -R_{2443} = R_{1224} = R_{1334} = \frac{1}{2}\,,$$

$$R_{1234} = \frac{\alpha - \frac{3}{2} + \beta}{3}\,, \quad R_{1423} = \frac{-2\,\alpha - 2\,\beta}{3}\,, \quad R_{1342} = \frac{\alpha + \frac{3}{2} + \beta}{3}\,.$$

(III) the characteristic polynomial of $\mathcal{R}'_X$ has a triple root α,

$$-R_{2112} = R_{1331} = R_{1441} = R_{2332} = R_{2442} = -R_{3443} = -\alpha$$

$$R_{2114} = R_{2334} = R_{3224} = R_{1442} = -\frac{\sqrt{2}}{2},$$

$$R_{3114} = R_{1223} = R_{1443} = R_{1332} = \frac{\sqrt{2}}{2}.$$

Theorem 4.4 ([7]) *Let M be a 4-dimensional pseudo-Riemannian Osserman manifold of signature $(2,2)$.*

(a) If M is of type (Ia) then universal covering space $\tilde{M}$ of M is one of the following manifolds

(i1) $\tilde{M}$ is a manifold of constant sectional curvature.

(i2) $\tilde{M}$ is a Kähler manifold of constant holomorphic sectional curvature.

(i3) $\tilde{M}$ is a para-Kähler manifold of constant para-holomorphic sectional curvature.

(b) M cannot be of type (Ib).

(c) if M is of types (II) (with $\beta = 4\,\alpha$) and (III) its curvature is locally given by above formulas.

In this signature (in [7]), using Wu's construction (see [39]), it was proved the existence of globally Osserman manifolds which are locally symmetric spaces of rank two, and consequently Osserman conjecture doesn't hold (see Example 3. below). Later it was shown (see[23]) that the Jacobi operators of a locally symmetric four-dimensional Osserman metric are either diagonalizable or two-step nilpotent. Manifolds with non-diagonalizable Jacobi operator have very rich geometry, for example in this class one can find also some recurrent space, harmonic space and etc.

Example 3. Let $M = \mathbb{R}^4$, (u_1, u_2, u_3, u_4) be the Descartes coordinates.
(i1) ([35])

$$\begin{aligned} g \;=\; & \frac{1}{6}\,(u_2^2 du_1 \otimes du_1 + u_1^2 du_2 \otimes du_2 - u_1 u_2 [du_1 \otimes du_2 + du_2 \otimes du_1]) \\ & - \frac{1}{2}\,([du_1 \otimes du_4 + du_4 \otimes du_1 + du_2 \otimes du_3 + du_3 \otimes du_2]). \end{aligned}$$

Then $(\mathbb{R}^4, g)$ is the timelike Osserman rank two symmetric space of type (II), the characteristic polynomial of $\mathcal{R}_X$ for unit timelike X is λ^4 and its characteristic polynomial is λ^2.

(i2) ([23])

$$\begin{aligned} g_{(f_1,f_2)} \;=\; & u_3 f(u_1,u_2) du_1 \otimes du_1 + u_4 f_2(u_1,u_2) du_2 \otimes du_2 + a[du_1 \otimes du_2 \\ & + du_2 \otimes du_1] + b[du_1 \otimes du_3 + du_3 \otimes du_1 + du_2 \otimes du_4 + du_4 \otimes du_2], \end{aligned}$$

where $\partial f_1/\partial u_2 + \partial f_2/\partial/\partial u_1 = 0$. Then, the characteristic polynomial of the Jacobi operator of $(M, g_{(f_1,f_2)})$ is $p_\lambda(\mathcal{R}_X) = \lambda^4$, i.e., it is independent of the nonnull vector X, but with different minimal polynomials $m_\lambda(\mathcal{R}_X) = \lambda^2$ or $m_\lambda(\mathcal{R}_X) = \lambda^3$. Also, there are examples when the minimal polynomials change degree from point to point. Functions f_1 and f_2 can be additionally chosen so that $(M, g_{(f_1,f_2)})$ is not locally symmetric.

Let us note that all previous examples are examples of Ricci flat manifolds. Recently in [17], it was find examples of Jordan-Osserman $(2,2)$ manifolds which are not Ricci-flat.

Example 4. Let $M = \mathbb{R}^4$ with usual coordinates (u_1, u_2, u_3, u_4). For any arbitrary real valued function $f(u_4)$, define a metric by

$$\begin{aligned} g \;=\; & du_1 \otimes du_3 + du_3 \otimes du_1 + du_2 \otimes du_4 + du_4 \otimes du_2 \\ & + \left(4\,k\,u_1^2 - \frac{1}{4k}\,f(u_4)^2\right) du_3 \otimes du_3 + 4\,k\,u_2^2\, du_4 \otimes du_4 \\ & + \left(4\,k\,u_1 u_2 + u_2 f(x_4) - \frac{1}{4k}\,f'(u_4)\right)(du_3 \otimes du_4 + du_4 \otimes du_3), \end{aligned}$$

where k is a nonzero constant. In [17] was shown that (M, g) is Osserman of type (II) with $\alpha = k$ and $\beta = 4\,k$, on the open set where

$$24\,k\,f(u_4)f'(u_4)\,u_2 - 12\,k\,f''(u_4)\,u_1 + 3\,f(u_4)f''(u_4) + 4\,f'(u_4)^2 \neq 0, \qquad (8)$$

and it is diagonalizable Osserman (type (Ia)) with eigenvalue structure $[(0,1), (k,2), (4k,1)]$ if equality in (8) holds. This means that in the case of manifold $(\mathbb{R}^4, g)$ its Jacobi operator changes its Jordan normal form and thus it is Osserman but not Jordan-Osserman.

As we mention before, geometry of $(2,2)$ Osserman manifold is very rich and here we will emphasize a few interesting characterizations. We start with a characterization of pointwise $(2,2)$ Osserman manifolds which is generalization of the same fact in Riemannian case.

Theorem 4.4 ([26, 3]) *Let M be a four dimensional Riemannian or neutral manifold then the following statements are equivalent:*

(a) M is pointwise Osserman manifold.

(b) There is a choice of orientation such that M is Einstein and self-dual or anti-self dual.

In the next theorem we give a characterization of four dimensional Kähler $(2,2)$ neutral Osserman manifolds.

Theorem 4.5 ([24]) *A four-dimensional Kähler metric of signature $(2,2)$ is pointwise Osserman if and only if it is an indefinite complex space form or a Ricci flat Kähler surface. Moreover, the Jacobi operators of a Jordan-Osserman four-dimensional Kähler metric which is not of constant holomorphic sectional curvature are nilpotent of degree two or three.*

We will finish this overview of $(2,2)$ Osserman manifolds with new very interesting fact.

Theorem 4.6 ([2]) *The duality principle holds for every 4-dimensional Osserman manifold.*

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