Generalized Kramers-Wannier Duality for spin systems with non-commutative symmetry

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Abstract

The main goal of this lecture is to discuss some old and new results concerning Kramers-Wannier Duality for spin systems with non-abelian symmetry.

In 1941 H. Kramers and G. Wannier discovered a special symmetry which relates low-temperature and high-temperature phases in the planar Ising model. The corresponding transformation, the Kramers-Wannier transform, is a special nonlocal substitution in the partition function. The existence of such transformations is a general property of lattice spin systems. Generalization of KW transform to spin systems with non-abelian symmetry is essential for many problems in statistical physics and field theory. This problem is very difficult and can't be carried out by classical methods (like Fourier transform in commutative case). We present new results which solve this problem for finite non-abelian groups.

Introduction

In the classical paper of Kramers and Wannier [1] a special symmetry was discovered, which relates low-temperature and high-temperature phases in the planar Ising model. The corresponding transformation, the Kramers-Wannier (KW) transform, is a special nonlocal substitution of a variable in the partition function. This substitution transforms the partition function W defined by the initial "spin" variables taking values in Z_2 and determined on the vertices of the original lattice L to the partition function $\tilde{W}$ determined on the dual lattice L^* spin variables taking values in Z_2.

Furthermore, we will use the following transformation of Boltzman factor

$$\beta \longrightarrow \beta^* = arth \;\; e^{-2\beta}, \qquad \beta = (kT)^{-1} \tag{0..1}$$

to get the correct form of the dual partition function $\tilde{W}$.

The existence of such transformations is a general property of lattice spin systems that possess a discrete (and not only discrete) group of symmetry. The KW-transform allows the determination, for many physically important systems, of the point of phase transition in cases when the explicit analytical form of a partition function is unknown.

Generalizations of the KW transform in spin systems with different symmetry groups is essential for many problems in statistical physics and field theory. In fact, it is very important to carry out KW transforms for 4-dimensional gauge theories in which corresponding phases are free quarks and quarks confinement. In this case we need to construct KW transforms for non-abelian groups.

The KW-transform for systems with a commutative symmetry group K, particularly Z_n and Z (like the Ising Z_2-model), can be carried out by general methods. In this case the KW-transform is a Fourier transform from a spin system on the lattice L to the spin system on the dual lattice $\tilde{L}$ with spin variables taking values in the group $\hat{K}$, the group of characters of K. This result was obtained by a number of authors, see [2,3,4] and references cited in it. From the mathematical point of view this result is a generalization of the classical Poisson summation formula for the group Z.

In this lecture we present some results which solve this problem for non-commutative groups. Our lectures based on the paper [5]. For the sake of a volume limit we omit some examples but add the outline of our construction for the compact case. The efficacy of our approach was illustrated by examples of KW transforms for the icosahedron I_5 and dihedral groups D_n [5]. These examples are also interesting for physical applications, for example, to search out the line of phase transitions in quasicrystals with the icosahedral symmetry or discotic liquid crystals with the symmetry D_n.

The main result of our paper is the definition of the generalized KW-transform, based on the mapping of the group algebra $C(G)$ to the space of complex-valued functions on G. The construction of this transformation clarifies its real meaning and offers far-reaching generalization papers [2,6,7].

The layout of the lecture is as follows. In section 1 we recall, following the paper [2], the construction of the KW-transform for abelian groups. In section 2 we introduce some relevant algebra notions like the group algebra $C(G)$ and the space of regular functions $C[G]$. We also construct the canonical pairing of $C(G)$ with $C[G]$. In section 3 we describe orbits of the adjoint representation and the regular representation of the group G. In the section 4 we carry out the generalized KW-transform for finite groups and in the section 5 apply our general results to special cases of subgroups of the group SO(3). In the section 6 we study the compact case.

In the conclusion we discuss some applications of these results, in particular some connections with quantum groups.

1. KW-duality for abelian systems

Let us recall the construction of KW-duality for commutative groups. We shall follow the paper [2]. Let us consider a planar square lattice L with unit edge. Let $x = \{x_\mu\} = \{x_1, x_2\}$ (where x_1 and x_2 are integers) represent a vertex, and $e^\alpha_\mu = \{e^1_\mu, e^2_\mu\} = \delta^\alpha_\mu$ basis vectors of L. We will often use the notation $x + \hat{\alpha} \equiv \{x_\mu + e^\alpha_\mu\}$. A double index x, α is convenient for denoting the edge in the lattice which connects the vertices x and $x + \hat{\alpha}$. In what follows we shall also need the dual lattice, $\tilde{L}$ whose vertices are at the centers of the faces of the original lattice L. We denote the coordinates of a vertex of $\tilde{L}$ by $\tilde{x}$:

$$\tilde{x} = \{x_\mu + 1/2e^1_\mu + 1/2e^2_\mu\}.$$

We define spin variables s_x on vertices of L, these take values in some manifold M, which we call the spin space. We confine ourselves to the case of a finite set M.

The simplest Hamiltonian of such a spin system involves only interactions of nearest neighbors

$$\mathcal{H} = \sum_{x,\alpha} H(s_x, s_{x+\hat{\alpha}}), \tag{1..1}$$

where the Hamiltonian $H(s, s')$ is a real function of a pair of points from M, with the properties

$$H(s, s') = H(s', s), \tag{1..2a}$$

$$H(s, s') \geqslant 0 \text{ for arbitrary } s, \quad s' \in M, \quad H(s, s) = 0. \tag{1..2b}$$

The Hamiltonian prescribes a structure similar on M to a metric structure (which in the general case is not metric, since we nowhere require that the triangle inequality hold), which we shall call the H structure.

Of particular interest are examples in which the manifold M is a homogeneous space, i.e., there exists a group G of transformations of M which preserves the H structure: $H(gs, gs') = H(s, s')$ for arbitrary $s, s' \in M$. In this case the spin system has global symmetry with group G.

Important special cases are systems on groups. For these the spin manifold coincides with a group G: $s_i = g_i \in G$, and the Hamiltonian is invariant under left and right translations:

$$H(hg, hg') = H(gh, g'h) = H(g, g') \text{ for arbitrary } h \in G. \tag{1..3}$$

The general H function of the system on the group can therefore be put in the form

$$H(g_1, g_2) = H(g_1 g_2^{-1}) = \sum_p h(p)\chi_p(g_1 g_2^{-1}), \tag{1..4}$$

where $\chi_p(g)$ are the characters of the p-th irreducible representations of the group G, and the constants $h(p)$ are chosen so that H has the properties (1..2) and are otherwise arbitrary.

The partition function of the general spin system with the Hamiltonian (1..1) is

$$Z = \sum_{s_x \in M} \prod_{x,\alpha} W(s_x, s_{x+\alpha}), \tag{1..5}$$

where

$$W(s, s') = exp\{-H(s, s')\}. \tag{1..6}$$

According to Eq. (1..2) the function W has the properties

$$W(s, s') = W(s', s), \quad 0 \leqslant W(s, s') \leqslant 1, \quad W(s, s) = 1. \tag{1..7}$$

For the system on a group we have also

$$W(g_1, g_2) = W(g_1 g_2^{-1}), \qquad W(g^{-1}) = W(g). \tag{1..8}$$

For a spin system on a group G the sum over states (1..5) can be put in the following equivalent form:

$$Z = \sum_{g_{x,\alpha} \in G} \prod_{x,\alpha} W(g_{x,\alpha}) \prod_{\hat{x}} \delta(Q_{\tilde{x}}, I), \tag{1..9}$$

where the summation variables $g_{x,\alpha}$ are defined on the edges of the lattice

$$Q_{\tilde{x}} = g_{x,1} \quad g_{x+\hat{1},2} g^{-1}_{x+\hat{2},1} g^{-1}_{x,2}, \tag{1..10}$$

and the δ-function is defined by the formula

$$\delta(g, I) = \begin{cases} 1, & \text{if } \ g = I, \\ 0 & \text{otherwise.} \end{cases}$$

In fact, the general solution of the connection equation $Q_{\tilde{x}} = I$ is

$$g_{x,\alpha} = g_x g^{-1}_{x+\hat{\alpha}}$$

and this brings us back to Eq. (1..5).

Systems on commutative groups are a special case, in which the δ-function in Eq. (1..9) can be factorized in the following way:

$$\begin{aligned} \delta(Q_{\tilde{x}}, I) &= \sum_p \chi_p(Q_{\tilde{x}}) \\ &= \sum_p \chi_p(g_{x,1}) \chi_p(g_{x+\hat{1},2}) \chi_p^{-1}(g_{x+\hat{2},1}) \chi_p^{-1}(g_{x,2}). \end{aligned} \tag{1..11}$$

This sort of factorization is of decisive importance and allow for a unified presentation of the KW transform for all commutative groups.

We note that for a commutative group G all irreducible representations are one-dimensional and their characters χ_p form a commutative group $\hat{G}$ (the character group) with a group multiplication defined in accordance with the tensor product of representations. By definition

$$\chi_{p_1 p_2}(g) = \chi_{p_1}(g) \chi_{p_2}(g), \qquad\qquad \chi_{p^{-1}}(g) = \chi_p^{-1}(g),$$

and the unit element of $\tilde{G}$ corresponds to the identity representation of G. Accordingly, the summation in Eq. (1..11) can be regarded as a summation over the elements of the dual group $\hat{G}$.

Substituting the expansion (1..11) in Eq. (1..9), an obvious regrouping of factors yields

$$Z = \sum_{s_{x,\alpha}\in G} \prod_{x,\alpha} W(g_{x,\alpha}) \prod_{\tilde{x}} \sum_{p_{\tilde{x}}} \chi_{p_{\tilde{x}}}(g_{x,1}) \chi_{p_{\tilde{x}}}(g_{x+\hat{1},2}) \chi_{p_{\tilde{x}}^{-1}}(g_{x+\hat{2},1}) \chi_{p_{\tilde{x}}^{-1}}(g_{x,2})$$

$$= \sum_{p_{\tilde{x}}\in G} \prod_{\tilde{x},\alpha} \tilde{W}(p_{\tilde{x}} p_{\tilde{x}+\hat{\alpha}}^{-1}), \qquad (1..12)$$

$$\tilde{W}(p_{\tilde{x}} p_{\tilde{x}+\hat{\alpha}}^{-1}) = \sum_{g\in G} W(g) \chi_{p_{\tilde{x}}}(g) \chi_{p_{\tilde{x}+\alpha}^{-1}}(g) = \sum_{g\in G} W(g) \chi_{p_{\tilde{x}} p_{\tilde{x}+\hat{\alpha}}^{-1}}(g). \qquad (1..13)$$

The expression (1..12) defines a new, dual, spin system on the dual group $\hat{G}$ with a new Hamiltonian $\tilde{H}$, which is defined the formula

$$\exp\{-\tilde{H}(p)\} = \tilde{W}(p). \qquad (1..14)$$

The result can be formulated in the following way.

Proposition 1.1. *A spin system on a commutative group G with a Hamiltonian $H(g)(g \in G)$ is equivalent to a spin system on the character group $\hat{G}$ (and on the dual lattice) with the Hamiltonian $\tilde{H}(p)(p \in \hat{G})$ given by the Fourier transform*

$$\exp\{-\tilde{H}(p)\} = \sum_{g\in G} \exp\{-H(g)\} \chi_p(g). \qquad (1..15)$$

This is a Kramers-Wannier transform. In contradistinction to the "order variables" g_x the name "disorder variables" can be given to the dual spins $p_{\tilde{x}}$.

2. Algebraic constructions

For further details of exploiting algebraic constructions one can consult the books [8,9].

A) The group algebra $C(G)$ of G.

Let G be a finite group of order n with elements $\{g_1 = e, ..., g_n\}$.

Definition 1. *The group algebra $C(G)$ of G is n-dimensional algebra over the complex field $\mathbb{C}$ with basis $\{g_1 = e, ...g_n\}$. A general element $u = c(g) \in C(G)$ is*

$$u = \sum \alpha_i g_i. \qquad (2..1)$$

The product of two elements (convolution) $u, v \in C(G)$ is defined as

$$uv = \Big(\sum_{i=1}^{n} \alpha_i g_i\Big)\Big(\sum_{i=1}^{n} \beta_j g_j\Big) = \sum_{1}^{k} (\gamma_k g_k), \quad \gamma_k = \sum_{g_i g_j = g_k} \alpha_i \beta_j. \tag{2..2}$$

B) The ring of functions $C[G]$ on G.

Definition 2. *$C[G]$ is a linear space of all complex-valued functions on G and the product is defined pointwise:*

$$\Big(f_1 \cdot f_2\Big)(g) = f_1(g) f_2(g). \tag{2..3}$$

C) The canonical pairing $< \cdot, \cdot >$ of spaces $C(G)$ and $C[G]$ is a map

$$< \cdot, \cdot >: C(G) \otimes C[G] \longrightarrow \mathbb{C},$$

defined as follows: for $u \in C(G)$ and $f \in C[G]$

$$u \otimes f \quad \longrightarrow \ < u, f >= \sum \alpha_i f(g_i). \tag{2..4}$$

We choose as a basis in $C[G]$ functions such that $< g_i, g^j >= \delta_i^j$ here δ_i^j is the Kronecker symbol.
This pairing enables us to identify $C(G)$ and $C[G]$ as vector spaces.

3. Canonical actions of the Group G

We now define two canonical representations, the adjoint representation on $C(G)$ and the regular representation on $C[G]$.

A) $T(g) : C(G)$

The adjoint representation is defined on the basis consisting of elements of G by

$$g : \ g_i \ \longrightarrow \ g g_i g^{-1}. \tag{3..1}$$

The adjoint representation ad G decomposes in the direct sum of irreducible representations and split $C(G)$ in the sum of subspaces invariant under the adjoint action.

Each irreducible subspace H_i relates with the orbit of ad G (3..1). The number of H_i is equal to m, the number of elements in the space $C(G)/[C(G), C(G)]$, here $[C(G), C(G)]$ denotes the commutant of $C(G)$.

B) $\tilde{T}(g) : C[(G]$

Let us define the canonical representation $\tilde{T}$ in the space $C[G]$ as the (right) regular representation as:

$$T(g) : f \Longrightarrow T(g) : f(g_k) = f(g_k g), \quad g \in G, \quad f(g) \in C[G]. \tag{3..2}$$

It is well known that, in the decomposition of the regular representation into irreducible ones all irreducible representations appear with multiplicity equal to the dimension of the representation.

$$\tilde{T} = \sum d_k V_k \,,$$

where V_k is the irreducible representation of degree k and d_k is the degree (dimension) of V_k (multiplicity of irreducible representation).

Proposition 3.1. *The number m of irreducible representations $\tilde{T}$ is equal to the number of orbits of T.*

C) The canonical scalar product in the space $C[G]$ is

$$< f_1, f_2 >= \frac{1}{n} \sum_{k=1}^{n} f_1(g_k) \bar{f}_2(g_k), \quad f_1, f_2 \in C[G]. \qquad (3..3)$$

The characters $\chi_i(g)$ of the irreducible representation of G form the set of orthogonal functions with respect to the scalar product (3..4).

Now we construct the basis in the space $C[G]$. Let us choose the character $\chi_k(g)$ and act on $\chi_k(g)$ by the group G with the help of the right regular representation:

$$R_{g_l} \chi_k(g), \quad l = 1, ..., n. \qquad (3..4)$$

We obtain the space V_k with dim $V_k = |\chi_k(g)|^2$. As a result we get the factorization of $C[G]$:

$$C[G] = \sum_{k \in \mathfrak{M}_G} V_k, \quad \mathfrak{M}_G = \{k = 1, .., m_G\},$$

where m_G is the number of irreducible representations of G.

Orthonormalizing the set of functions (3..5) we obtain the basis in the space V_k. Since V_k are pairwise orthogonal, applying this procedure to all characters χ_k we obtain the desired basis in $C[G]$.

Definition 3. *We shall call the dual space $\hat{G}$ to G the basis in $C[G]$ which we construct in the section C.*

Motivations for such definition ensue from the case of a commutative group K. The characters of K are one-dimensional and the action of G on characters is simply the multiplication on the scalar, the eigenvalue of the operator R_g. The derived basis is the same as the set of elements of the group $\hat{K}$.

4. The KW-transform for finite groups

Let us consider the adjoint representation ad G of G, on the space $C(G)$, induced by

$$g : g_k \;\longrightarrow\; g g_k g^{-1}.$$

Let us denote by g_k^G the orbit relative to the adjoint action for $g_k \in G$, and by $\delta_k \in C[G]$ it characteristic function:

$$\delta_k(g_s) = \begin{cases} 1, & \text{if } \ g_s \in g_k^G, \\ 0, & \text{otherwise.} \end{cases}$$

Let m_G be the number of conjugacy classes relative to the adjoint action of G. Let us choose representations of the classes

$$g_1, ..., g_{k_j}.$$

Lemma 4.1. *A linear map $W : C(G) \longrightarrow \mathbb{C}$ satisfies the condition*

$$W(g_k) = W(g_l g_k g_l^{-1}) \tag{4..1}$$

for every $g_l \in G$, off

$$W = \sum_{j=1}^{m} \gamma_j \delta_{k_j} \in C[G] = Hom(C(G), \mathbb{C}), \qquad i.e.$$

$$W(g_s) = \sum \gamma_j \delta_{k_j}(g_s). \tag{4..2}$$

We obtain a general form of the adjoint invariant linear mapping, if we choose as $\gamma = (\gamma_1, ..., \gamma_m)$, the vector of free parameters.

Now we shall find the form of a general linear mapping:

$$\hat{W} : C[G] \longrightarrow \mathbb{C}$$

determined by the characters $\chi^i(G)$.

The set of characters $\chi^1, ..., \chi^m$ of the irreducible representation of G form the orthonormalized basis (relative to the scalar product (3..4)) in $C[G]$. Here and further χ^1 is the character of the trivial one-dimensional representation.

We get $\hat{W} = \sum \hat{\gamma}_j \chi^j$ as

$$\hat{W}(\psi) = \sum_{j=1}^{m} \hat{\gamma}_j < \chi^j, \psi > \tag{4..3}$$

since characters of representations by Lemma 4.1 are ad-invariant functions, we introduce the matrix $\Gamma = \gamma_j^l$ using the expansion

$$\chi^l = \sum_{j=1}^{m} \gamma_j^l \delta_{k_j}. \tag{4..4}$$

Let us denote by $g^0, ..., g^{m-1}$ the orthonormalized basis in the algebra $C[G]$, dual to the basis $g_0, .., g_{m-1}$ in the group algebra $C(G)$, i.e. $< g^i, g_j >= \delta^i_j$.

Let D be the duality map:

$$D : C(G) \longrightarrow C[G], \qquad D(g_k) = g^k. \tag{4..5}$$

Theorem 4.1. *If we pose* $\gamma_j = \sum\limits_{l=1}^{m} \gamma^l_j \hat{\gamma}_l,\ (j = 1, ..., m)$ *then by the canonical duality D the linear map*

$$W : C(G) \longrightarrow \mathbb{C}, \qquad W(g) = \sum_{j=1}^{m} \gamma_j \delta_{k_j}(g)$$

pass to the linear map

$$\hat{W} : C[G] \longrightarrow \mathbb{C}, \qquad \hat{W}(\psi) = \sum_{j=1}^{m} \hat{\gamma}_j < \chi^j, \psi >$$

and maps W and $\hat{W}$ themselves will be determined by the same function, more precisely

$$W(g_s) = \sum_{j=1}^{m} \gamma_j \delta_{k_j}(g_s) = n \sum_{j=1}^{m} \hat{\gamma}_j \chi^j(g_s) = n\hat{W}(g^s). \tag{4..6}$$

Proof. For any g_s we have

$$\begin{aligned} W(g_s) &= \sum_{j=1}^{m} \gamma_j \delta_{k_j}(g_s) = \sum_{j=1}^{m}\sum_{l=1}^{m} \gamma^l_j \hat{\gamma}_l \delta_{k_j}(g_s) = \sum_{l=1}^{m} \hat{\gamma}_l (\sum_{j=1}^{m} \gamma^l_j \delta_{k_j})(g_s) \\ &= \sum_{l=1}^{m} \hat{\gamma}_l \chi^l(g_s) = n \sum_{l=1}^{m} \hat{\gamma}_l < \chi^l, g^s >= n\hat{W}(g^s). \end{aligned}$$

□

Definition 4. We shall call the transform

$$W(g_s) = \sum \gamma_j \delta_{k_j}(g_s) \quad \longrightarrow \quad \hat{W}(g^s) = \frac{1}{n} \sum_l \gamma_l \chi^l(g^s),$$

$$\text{where:} \qquad \gamma_j = \sum_{l=1}^{m} \gamma^l_j \hat{\gamma}_l \tag{4..7}$$

the Kramers-Wannier transform for finite groups.

In the next section we consider several examples which confirm the coincidence of our approach with former one in the known cases and enables us to find explicit K-W transforms in some earlier unknown cases.See also for other examples [5].

5. Examples

A) Commutative case $G = \mathbb{Z}_n$.

Let us consider first the special case $G = \mathbb{Z}_3 = \{1, g, g^2\}$. In this case $\delta_j = \delta(g - g^{j-1})$, $j = 1, 2, 3$. Then

$$\begin{aligned} \chi^1 &= \delta_1 + \delta_2 + \delta_3 \,, \\ \chi^2 &= \delta_1 + z\delta_2 + z^2\delta_3 \,, \\ \chi^3 &= \delta_1 + z^2\delta_2 + z\delta_3 \,, \quad \text{as } z^4 = z, \end{aligned} \tag{5..1}$$

where $z = \exp(2\pi i/3)$, and $\chi^k(g^j) = z^{(k-1)j}$, $(k = 1, 2, 3)$ are the characters of one-dimensional representations. Hence

$$\Gamma = (\gamma_j^l) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & z & z^2 \\ 1 & z^2 & z \end{pmatrix} \tag{5..2}$$

and we get $\hat{\gamma} = \Gamma^{-1}\gamma$.

If we choose $\gamma_1 = 1$, $\gamma_2 = \gamma_3 = \gamma$, we obtain

$$\hat{\gamma}_1 = \frac{1 + 2\gamma}{3} \,, \qquad \hat{\gamma}_2 = \hat{\gamma}_3 = \frac{1 - \gamma}{3} \,, \tag{5..3}$$

and hence:
$$\hat{\gamma}_2/\hat{\gamma}_1 = \frac{1 - \gamma}{1 + 2\gamma} \,. \tag{5..4}$$

For the general case of the group $\mathbb{Z}_n$ we have to replace the formula (5..1) for characters $\chi^1, ..., \chi^n$ to

$$\begin{array}{ccl} \chi^1 & = & \delta_1 + \delta_2 + ... + \delta_n \,, \\ \chi^2 & = & \delta_1 + z\delta_2 + ... + z^{n-1}\delta_n \,, \\ \cdots & \cdots & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \chi_l^n & = & \delta_1 + z^{(n-1)}\delta_2 + ... + z\delta_n \,, \end{array} \tag{5..5}$$

and for $\Gamma = (\gamma_j^l)$ we get

$$\Gamma = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & z & \cdots & z^{n-1} \\ \cdots & \cdots & \cdots & \cdots \\ 1 & z^{n-1} & \cdots & z \end{pmatrix} . \tag{5..6}$$

In the special case of choosing parameters γ_j: $\gamma_1 = 1, \gamma_2, ..., \gamma_n = \gamma$ we obtain

$$\frac{\hat{\gamma}_j}{\hat{\gamma}_1} = \frac{1 - \gamma}{1 + (n-1)\,\gamma} \,. \tag{5..7}$$

These formulas coincide with the similar one in the paper [2].

B) The group S_3.

This is the first non-trivial example of non-abelian groups which was studied in [2]. Following our general approach we split the group S_3 in 3 classes of conjugacy elements or 3 orbits:

$$S_3 = \{\Omega_1 = \{e\}, \qquad \Omega_2 = \{a, a^2\}, \qquad \Omega_3 = \{b, ab, a^2b\}\}\,.$$

The characteristic functions are:

$$\delta_1 = \delta(\Omega_1) = \delta(g-e), \qquad \delta_2 = \delta(\Omega_2), \qquad \delta_3 = \delta(\Omega_3).$$

Following our general procedure (see 4.4) and using

$$\begin{aligned} \chi^1 &= \delta_1 + \delta_2 + \delta_3 \\ \chi^2 &= \delta_1 + \delta_2 - \delta_3 \\ \chi^3 &= 2\delta_1 - \delta_2 \end{aligned} \qquad \text{we get the matrix} \quad \Gamma = (\gamma_j^l) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 2 & -1 & 0 \end{pmatrix}$$

and hence $\hat{\gamma} = \Gamma^{-1}\gamma$

$$\begin{aligned} \hat{\gamma}_1 &= \frac{1}{6}\,(\gamma_1 + 2\gamma_2 + 3\gamma_3), \\ \hat{\gamma}_2 &= \frac{1}{6}\,(\gamma_1 + 2\gamma_2 - 3\gamma_3), \qquad \hat{\gamma}_3 = \frac{1}{3}(\gamma_1 - \gamma_2), \end{aligned} \tag{5..8}$$

with the following relation: $\hat{\gamma}_1 + \hat{\gamma}_2 + 2\hat{\gamma}_3 = \gamma_1$.

If we choose the free parameters $\gamma_1, \gamma_2, \gamma_3$ as $1, \gamma_2, \gamma_3$ we obtain two independent parameters $\hat{\eta}_1, \hat{\eta}_2$

$$\hat{\eta}_1 = \frac{\hat{\gamma}_2}{\hat{\gamma}_1} = \frac{1 + 2\gamma_2 - 3\gamma_3}{1 + 2\gamma_2 + 3\gamma_3}\,, \qquad \hat{\eta}_2 = \frac{\hat{\gamma}_3}{\hat{\gamma}_1} = \frac{2(1-\gamma_2)}{1 + 2\gamma_2 + 3\gamma_3}\,, \tag{5..9}$$

which coincide with the formula (5..7) in the paper [2].

Remark 1. Let us mention the missing of factor 2 in the nominator of $\hat{\eta}_2$ in (5..7) in the paper [2].

6. The KW transform for compact groups

In this section we give an outline of construction of KW transform for compact "gauge" groups. The corresponding construction can be carried out parallel to the finite case. However, it is substantially more complicated. As formerly, we restrict to the case of a square lattice $L \subset R^2$. [1]. All necessary materials regarding the theory of representations of compact groups can be find in [10, 11] .

[1]The detailed proof and some generalizations to high dimensional spaces and different classes of lattices will be published in the joint paper with V.Buchstaber .

A) Let G be a compact connected group. G is isomorphic to $A \times G_1$, where A is a compact abelian group (torus T^n) and G_1 is a semisimple compact group. In the case of an abelian group A the KW transform can be carried out by the general method of section 2. Therefore, in what follows we restrict to the case of a semisimple compact group G .

B) Group algebra $C(G)$. The natural analog of a group algebra for finite group will be some functional space endowed with the product operation as a convolution. It is possible to choose as such space $L^1(G, dg)$, the space of summable functions, or $L^2(G, dg)$, or a subspace $H(G, dg)$ of continuous functions on G.It is more convenient to consider a completion of these spaces by norm:

$$||f|| = \sup_T ||T(f)||,$$

where T runs over all unitary representations of the group G. The algebra $C^*(G)$ is called the C^*-algebra of G.

C) $C[G]$. $C[G]$ is a linear functional space (e.g. $L^1(G, dg), L^2(G, dg), ...$) and the product is defined pointwise :

$$(f_1 \cdot f_2)(g) = f_1(g)\, f_2(g)\,.$$

There is a well known theorem of I. Gelfand and D. Raikov asserting that for any locally compact group there exist irreducible unitary representations and the system of such representations is complete.

To construct an analog of a basis in $C(G)$ for a compact case we need some generalization of Schur-Frobenius theorem (see C in Sec. 4). In our case we use the theorem of Peter-Weyl.

Theorem (Peter-Weyl). *The set of linear combinations of matrix elements of irreducible representation is dense in the space $H(G, dg), L^2(G, dg)$.*

The orthogonal relations for matrix elements of a unitary representations can be proved in the same way as for finite groups.

D) The basis in $C(G)$. To construct a relevant basis in the space $C(G)$ we use the construction of irreducible representations by the orbit method. Let us recall that a coadjoint orbit of a group G is an orbit in the space g^* dual to the Lie algebra g of G. If we have an adjoint representation T of G we can determine the coadjoint representation T^* of G which acts in the space g^*. We call such a representation as a coadjoint representation.

Proposition. *For a compact semisimple group G a coadjoint representation of G is equivalent to the adjoint representation.*

This is evident, since exists Cartan-Killing Ad invariant form on g. For any compact group G there exists only finite number of co(adjoint) orbits Ω_i.The stabilizers of elements $x \in g$ form a finite number k of conjugate classes of subgroups of G. Let G_i $(1 \leq i \leq k)$ be a representative of these classes. Then any adjoint orbit is isomorphic to the coset space $\Omega_i = G/G_i$. So we can choose a basis $\delta_j(\Omega_i)$ in $C(G)$ $\delta_j(\Omega_i)$. To complete our proof we use the following statement of Gelfand and Naimark [10]. We omit some technical conditions.

Theorem (Gelfand and Naimark). *There exists a one-to-one correspondence between representations of a group algebra $C(G)$ and unitary representation of the group G. So as in the finite case we determine the KW transform for a compact group as*

$$\mathcal{D} : C(G) \to C[G]\,. \tag{6..1}$$

Conclusion

Our approach to the KW-transform has important applications. We briefly discuss some of them, intending to return to these problems in the forthcoming publications.

A) KW-transforms and Quantum groups.

We refer reader to [12,13] for all notations and following references in the theory of Hopf algebras and Quantum groups.

Let us consider the algebra $C[G]$. If we endow $C[G]$ by the operation of coproduct $\triangle : C[G] \longrightarrow C[G] \otimes C[G]$ induced by the multiplication in the group G, the algebra $C[G]$ becomes Hopf algebra. Using natural dual to $C[G]$ the algebra $C(G)$, we are able to construct another Hopf algebra, (quantum) double $D(G) = C[G] \otimes C(G)$ [12]. Since transformations W and $\hat{W}$ acts as $W : C(G) \longrightarrow \mathbb{C}$ and $\hat{W} : C[G] \longrightarrow \mathbb{C}$, i.e. $W \in C[G] = Hom(C(G), \mathbb{C})$ and $\hat{W} \in C[G] = Hom(C(G), \mathbb{C})$ that is $W \otimes \hat{W} \in D(G)$. The KW-transform yields to explicit solutions of Yang-Baxter equations related with the quantum group D(G).

This observation leads to very explicit formulas in the structure theory of quantum groups and quantum spin systems.

And last but not least.

B) In our lecture we consider spin systems with a global non-abelian symmetry. It is natural to ask about generalizing proposed technique to systems with a local (gauge) symmetry. The study of such systems including Ising and Potts chiral models, abelian and non-abelian gauge fields is very important for Quantum Field Theory and the Theory of Phase Transitions.

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