Graviton propagator from spin networks *

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Abstract

We describe Rovelli's approach to construct the graviton propagator in Loop Quantum Gravity and present an alternative approach which solely relies on the spin network wave function.

1. Introduction

The problem of quantization of General Relativity (GR) can be formulated in the narrowest sense as a problem of finding a finite quantum perturbation theory of a spin-two field in the flat spacetime. String theory has a solution for this problem [1]; however, for the purposes of exploring the strongly curved spacetime regions one needs a non-perturbative formulation of quantum gravity. One formulation of this kind is Loop Quantum Gravity (LQG) [2], which posseses the following attractive features:

a) it is manifestly independent from the background metric,

b) it predicts a quantization of spatial geometry in the form of spin network states.

The feature (a) is desirable because in the regions of large metric fluctuations the notion of a classical background metric is useless, while the feature (b) embodies our intuitive notions about the discreteness of quantum geometry. However, the weak point of LQG is a demonstration that a finite perturbation theory of a spin-two field appears in the semi-classical limit. In this talk we present arguments in favor of the existence of such a limit in LQG.

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2. LQG, spin networks and spin foams

LQG is based on the canonical quantization of GR in terms of the Ashtekar variables. The spacetime four-manifold M is restricted to $\mathbf{R}\times\Sigma$, where Σ is a three-manifold. The metric g on M can be decomposed as

$$g_{\mu\nu}dx^\mu dx^\nu = (-N^2 + N^m N_m)\,dt^2 + 2N_m\,dx^m dt + \gamma_{mn}\,dx^m dx^n\,, \qquad (1)$$

where $x^\mu = (t, x^m) = (t,\vec{x})$ are coordinates on M adapted to $M = \mathbf{R}\times\Sigma$, γ is a metric on Σ, N is the laps while N^m is the shift vector. The Hamiltonian analysis of the Einstein-Hilbert action implies that the laps and the shift vector are Lagrange multipliers while the dynamics resides in the spatial metric γ and its canonically conjugate momentum π. Instead of γ one can use the triads e_m^a due to the relation $\gamma_{mn} = e_m^a e_{na}$, so that one can perform the following series of canonical transformations

$$(\pi^{mn},\gamma_{mn}) \to (p_a^m, e_m^a) \to (p_m^a, e_a^m) \to (A_m^a, E_a^m)\,, \qquad (2)$$

where

$$A_m^a = p_m^a + \beta\omega_m^a(e)\,, \quad E_a^m = (\det e)\,e_a^m\,, \qquad (3)$$

are the Ashtekar variables. The parameter β in front of the spin connection ω_i^a can be real or imaginary so that the Ashtekar-Sen (AS) connection A_i^a can be real or complex. The constraints of GR take the following form

$$\begin{aligned} G_a &= \partial_m E_a^m + \epsilon_{abc}A_m^b E^{mc}\,, \qquad & D_m &= F_{mn}^a E_a^n\,,\\ H &= \epsilon_{abc}F_{mn}^a E^{mb}E^{nc} + (\beta^2 \pm 1)H'\,, \end{aligned} \qquad (4)$$

where F is the connection curvature two-form and H' is the nonpolynomial piece of the Hamiltonian constraint. The $\pm$ sign refers to the signature of the spacetime metric ($+$ for the Minkowski and $-$ for the Euclidian case). In the Minkowski case the constraints become polynomial for $\beta = i$, so that the AS connection is complex, while in the Euclidean case this happens for $\beta = 1$ and hence the AS connection is real.

The theory is quantized by choosing the connection representation, i.e. the functionals $\Psi[A]$ of the $SU(2)$ valued connection A which satisfy the constraints. The constraints G and D imply that Ψ is a gauge and diffeomorphism invariant functional, while H generates the dynamics, i.e. it is the analog of the Schroedinger equation. One can show that a basis in the physical Hilbert space is given by the spin network functionals [3]

$$W_s[A] = Tr\left(\bigotimes_{l\in E_\gamma} U_l^{(j_l)}(A) \bigotimes_{v\in V_\gamma} \iota_v\right)\,, \qquad (5)$$

where s is a spin network consisting of a closed graph Γ, an assignment of $SU(2)$ irreps j_l to the edges l of Γ and an assignment of the corresponding intertwiner operators ι_v to the vertices of Γ, while the $U_l(A) = P\,\exp \int_l A$ are the corresponding edge holonomies. This implies that

$$\Psi[A] = \sum_s I(s) W_s[A] \tag{6}$$

and $I(s)$ are the invariants of the spin network s embedded in the 3-manifold Σ which are associated to the functional Ψ. The relation (6) can be used to define an abstract spin network basis $|s\rangle$, see [4], so that

$$|\Psi\rangle = \sum_s I(s)|s\rangle\,. \tag{7}$$

In the spin network basis one can naturally introduce the matrix elements $Z_M(s,s')$ of the evolution operator associated to $M = \Sigma \times I$ manifold such that $Z_M(s,s')$ is given by the spin-foam transition amplitude

$$Z_M(s,s') = \sum_L \prod_f A_f(L) \prod_l A_l(L) \prod_v A_v(L)\,. \tag{8}$$

The L represents a labeling of the faces f of the dual two-complex $\mathcal{C}^*$ associated to M with the irreducible representations of the spacetime symmetry group, while A_f, A_l and A_v are the corresponding face, edge and vertex amplitudes, respectively, see [3]. The set of labels L can also include the interwiner labels for the edges of $\mathcal{C}^*$, so that one can think of the labelled complex $\mathcal{C}^*$ as a world-sheet of a spin network, hence the name spin foam. In the case of LQG theory the symmetry group is $SO(3,1)$, or $SO(4)$ in the Euclidean case, and it is not known which class of representations have to be used nor the exact form of the simplicial amplitudes. The most popular model is the Barret-Crane (BC) model [5, 6, 7], where

$$A_f = \dim R_f\,, \quad A_l = \theta(R_{f_1},\cdots,R_{f_4})\,, \quad A_v = \sigma_4(R_{f_1},\cdots,R_{f_{10}})\,, \tag{9}$$

and R is a simple representation of $SO(3,1)$ or $SO(4)$ while θ and σ_4 are certain evaluations of the θ_4 and the four-simplex spin networks, respectively.

An important feature of the BC amplitude (9) is that it can be represented as a Feynman diagram of a group field theory given by the action

$$\begin{aligned} S_{BC}(\phi) \;=\; & \frac{1}{2}\int_{G^4}\int_{G^4} d^4g_1 d^4g_2\, \phi(\vec{g}_1)\, K(\vec{g}_1,\vec{g}_2)\, \phi(\vec{g}_2) \\ & + \frac{\lambda}{4!}\int_{G^4}\cdots\int_{G^4} d^4g_1\cdots d^4g_5\, V(\vec{g}_1,\cdots,\vec{g}_5)\, \phi(\vec{g}_1)\cdots\phi(\vec{g}_5)\,, \end{aligned} \tag{10}$$

where G is the spacetime symmetry group, $\phi(\vec{g}) = \phi(g_1,\cdots,g_4)$ and K and V are functions depending on the model, see [8, 9].

3. Graviton propagator

When the spacetime curvature is small compared to l_P^{-2}, where l_P is the Planck length, one can write the spacetime metric as

$$g_{\mu\nu} = \eta_{\mu\nu} + l_P h_{\mu\nu}, \tag{11}$$

where h is a spin-two field. The corresponding propagator is defined as

$$G_{\mu\nu,\rho\sigma}(x,y) = \langle 0|T(h_{\mu\nu}(x)h_{\rho\sigma}(y))|0\rangle\,, \tag{12}$$

where $|0\rangle$ is the flat spacetime vacuum. In the limit when $\hbar \to 0$ and the Newton constant $G_N \to 0$ one has

$$G(x,y) \approx \frac{\text{const.}}{|x-y|^2}\,, \tag{13}$$

which is also the asymptotics for large distances, i.e. $|x-y| >> l_P$.
In the context of LQG, the analog of (12) is given by

$$G_{mn,rs}(0,\vec{x};t,\vec{y}) = \langle\Psi_0|h_{mn}(0,\vec{x})h_{rs}(t,\vec{y}))|\Psi_0\rangle\,, \tag{14}$$

where $|\Psi_0\rangle$ is a LQG state representing the flat spacetime. If t is not to large, it is reasonable to assume that

$$h(t,\vec{x}) = e^{-iHt}h(0,\vec{x})e^{iHt}\,, \tag{15}$$

where H is the LQG Hamiltonian. Also $H|\Psi_0\rangle \approx 0$, so that

$$G_{mn,rs}(0,\vec{x};t,\vec{y}) = \langle\Psi_0|h_{mn}(\vec{x})U(t)h_{rs}(\vec{y}))|\Psi_0\rangle\,, \tag{16}$$

where $h_{mn}(\vec{x}) \equiv h_{mn}(0,\vec{x})$ and $U(t) = e^{-iHt}$. The problem with (16) is that it is difficult to find H explicitly, which is the problem of time in canonical quantum gravity. By inserting the identity operator $\sum_s |s\rangle\langle s|$ in front and behind of the U in (16) one arrives at

$$G_{mn,rs}(0,\vec{x};t,\vec{y}) = \sum_{s,s'}\langle\Psi_0|h_{mn}(\vec{x})|s\rangle\langle s|U(t)|s'\rangle\langle s'|h_{rs}(\vec{y}))|\Psi_0\rangle\,. \tag{17}$$

On the other hand one can take that [10]

$$h_{aa}(\vec{x})|s\rangle = \frac{1}{l_P}[j_x(j_x+1) - j_0(j_0+1)]|s\rangle\,, \tag{18}$$

where j_x is the spin of the spin network edge which emanates from the point x in the direction a, $h_{ab} = E_a^m E_b^n \delta_{mn}$ while j_0 is a spin parameter which appears in the flat spacetime vacuum state, see [13]. The case $a \neq b$ is more complicated, see [11], but the essential features stay the same. One can then define a propagator $\tilde{G}_{ab,cd}(x,y)$ such that the diagonal components are given by

$$\begin{aligned}\tilde{G}_{aa,bb}(x,y) &= \frac{1}{l_P^2}\sum_{s,s'}\bar{\Psi}_0(s)[j_x(j_x+1)-j_0(j_0+1)]Z_M(s,s') \\ &\quad \times [j_y(j_y+1)-j_0(j_0+1)]\Psi_0(s')\,, \end{aligned} \tag{19}$$

where $M=\Sigma\times I$ and Z_M is a spin foam amplitude for the transition $s\to s'$ while

$$G_{mn,rs}(x,y)=E_m^a(x)E_n^b(x)E_r^c(y)E_s^d(y)\tilde{G}_{ab,cd}(x,y)\,.$$

4. Rovelli's propagator

In Rovelli's approach [10, 11] the idea is to consider $\Psi_0(s)\Psi_0(s')$ as $\Psi_0(s\bigcup s')$ and to identify $Z_M(s,s')$ with the Hartle-Hawking wavefunction for $s\bigcup s'$ in the BC model. Hence one starts from a slightly different definition of the propagator and the diagonal components are then given by

$$\tilde{G}_R(x,y)=\frac{1}{l_P^2}\sum_{s}\bar{\Psi}_0(s)[j_x(j_x+1)-j_0(j_0+1)][j_y(j_y+1)-j_0(j_0+1)]W(s)\,, \tag{20}$$

where

$$W(s)=\int\mathcal{D}\phi\,O_s(\phi)\,e^{-S_{BC}(\phi)}\,. \tag{21}$$

$O_s(\phi)$ is the group field theory variable representing the the spin network state s in the BC model, see [12]. In the lowest-order perturbative expansion of $W(s)$ contribute only the spin networks with five vertices, and since in the BC formalism only the four-valent spin nets are allowed, one obtains that only the 4-simplex spin networks p are contributing. Hence

$$\begin{aligned}\tilde{G}_R(x,y) &= \frac{1}{l_P^2}\sum_{j_l}\bar{\Psi}_0(p)[j_x(j_x+1)-j_0(j_0+1)] \\ &\quad \times [j_y(j_y+1)-j_0(j_0+1)]W_1(p)+O(\lambda^2)\,. \end{aligned} \tag{22}$$

A key ingredient in Rovelli's approach is that

$$\Psi_0(s)=\exp\left(-\frac{1}{2j_0}\sum_{l,l'}\alpha_{ll'}(j_l-j_0)(j_{l'}-j_0)+i\sum_l j_l\theta_l\right)\,, \tag{23}$$

where θ_l are the dihedral angles associated with the triangulation of Σ whose dual one-simplex is the spin network graph Γ and α is a constant matrix. Then the semi-classical limit corresponds to $j_0\to\infty$, so that only the spin network states where the edges carry large spins are important. In the limit of large spins one has

$$W_1(p) \approx \cos\left(\sum_l j_l\theta_l + k\pi/4\right)\,, \tag{24}$$

where k is an integer.

By combining (22), (23) and that $E_a^m|\Psi_0\rangle \approx j_0\delta_a^m|\Psi_0\rangle$ one obtains

$$\begin{aligned} G_R^{(1)}(x;y) &= \frac{1}{j_0^4 N l_P^2}\sum_{j_p}\bar\Psi_0(p)[j_x(j_x+1)-j_0(j_0+1)] \\ &\quad\times [j_y(j_y+1)-j_0(j_0+1)]W_1(p)\,, \end{aligned} \tag{25}$$

where $N=\sum_{j_l}\bar\Psi_0(p)\Psi_0(p)$. In the limit $j_0\to\infty$ it follows from (24) that

$$G_R^{(1)}(x,y)\approx l_P^{-2}j_0^{-4}\frac{\sum_{j_l}(j_x^2-j_0^2)(j_y^2-j_0^2)\exp\left(-\frac{1}{j_0}\sum_{l,l'}\alpha_{ll'}(j_l-j_0)(j_{l'}-j_0)\right)}{\sum_{j_l}\exp\left(-\frac{1}{j_0}\sum_{l,l'}\alpha_{ll'}(j_l-j_0)(j_{l'}-j_0)\right)}\,. \tag{26}$$

The sums over j_l can be converted into Gaussian integrals which gives

$$G_R^{(1)}(x,y)\approx \mathrm{const.}\frac{\alpha^{-1}_{j_xj_y}}{j_0 l_P^2}\,, \tag{27}$$

as a leading term.

Since the flat spacetime state is described by a spin network whose spins are close to j_0, then the area A carried by an edge of p will be approximately $l_P^2\sqrt{j_0(j_0+1)}$. Since j_0 is large, we will have $A\approx l_P^2 j_0$. Since the points x and y are the ends of some edge of p, we can say that they are somewhere in the ball of radius $\sqrt{A}$, which means

$$|x-y|^2\approx A\approx l_P^2 j_0\,.$$

Hence the formula (27) implies that

$$G_R^{(1)}(x,y)\approx\frac{\mathrm{const.}}{|x-y|^2}$$

which is the correct classical asymptotics given by (13).

5. Alternative spin network propagator

Rovelli's construction of the propagator gives the correct classical behavior but it has certain shortcomings. For example, one has to know in advance that the phase-factors in $\Psi_0(s)$ have to be dihedral angles in order to obtain a correct propagator. However, it is not clear whether (23) is a solution of the Hamiltonian constraint or whether a solution exists whose large j_0 asymptotics is given by (23). Another shortcoming is that the Rovelli construction is a hybrid one, i.e. it mixes the LQG theory with the BC spinfoam Euclidean theory. This type of construction is not completely understood. For example, in LQG one can have any closed graph as a spin network graph, while in the BC theory only the four-valent closed graphs are allowed, see [12].

One way to avoid these problems would be to take the formula (19) for $t = 0$, which then corresponds to $Z_M(s, s') = \delta(s, s')$. Hence the diagonal components of the propagator will be given by

$$\tilde{G}(x; y) = \frac{1}{l_P^2} \sum_s \bar{\Psi}_0(s)[j_x(j_x+1) - j_0(j_0+1)][j_y(j_y+1) - j_0(j_0+1)]\Psi_0(s) \, . \tag{28}$$

In this case we do not need to know the spin-foam amplitude Z_M, and only the knowledge of $\Psi_0(s)$ is required. In refs [13, 14] it was described how to construct $\Psi_0(s)$ in the Euclidean gravity case as a spin-foam state sum for the quantum $SU(2)$ group at a root of unity. However, the problem with this construction is that at present it is difficult to calculate $\Psi_0(s)$ due to the lack of efficient techniques of dealing with sums of products of the corresponding quantum spin network evaluations.

Let us then assume that $\Psi_0(s)$ has the large j_0 asymptotics given by (23) but know we take the θ's to be arbitrary. In that case

$$G(x, y) \approx l_P^{-2} j_0^{-4} \frac{\sum_\Gamma \sum_{j_l} (j_x^2 - j_0^2)(j_y^2 - j_0^2) \exp\left(-\frac{1}{j_0} \sum_{l,l'} \alpha_{ll'} (j_l - j_0)(j_{l'} - j_0)\right)}{\sum_\Gamma \sum_{j_l} \exp\left(-\frac{1}{j_0} \sum_{l,l'} \alpha_{ll'} (j_l - j_0)(j_{l'} - j_0)\right)} \, , \tag{29}$$

which after the Gaussian integrations gives

$$G(x, y) \approx \text{const.} \frac{1}{j_0 l_P^2} \frac{\sum_\Gamma (\alpha^{-1}(\Gamma))_{j_x, j_y} \det^{-1/2} \alpha(\Gamma)}{\sum_\Gamma \det^{-1/2} \alpha(\Gamma)} \, . \tag{30}$$

In the case when $\alpha_{ll'} = \alpha \, \delta_{ll'}$ we obtain

$$G(x, y) \approx \text{const.} \frac{\delta_{j_x, j_y}}{j_0 l_P^2} \, , \tag{31}$$

in the leading order. The leading-order term is non-zero when $j_x = j_y$ and this condition means that the points x and y are on the same edge. Hence the distance $|x - y|$ is approximately $j_0 l_P$ and therefore the result

(31) implies the classical asymptotics (13). Note that if x and y are not on the same edge, then $G(x,y) \approx 0$. This means that we have the correct classical asymptotics for large distances, but for very large distances, i.e. $|x-y| >> j_0 l_P$, the propagator is zero. This is a non-perturbative effect and appears because we have included in our calculation all the spin network states.

6. Conclusions

Clearly the properties of the propagator are encoded in the state $\Psi_0(s)$. It is encouraging that one can find examples of $\Psi_0(s)$ which give the correct classical asymptotics for the propagator and that these examples have the Gaussian form. However, it still remains to be verified that such solutions of the Hamiltonian constraint exist. A class of solutions for $\Psi_0(s)$ in the Euclidian case has been constructed in [13, 14] and we expect to find out in the near future whether these solutions have the Gaussian form. A harder problem is extending these results to the Minkowski case.

References

[1] Green, Schwarz and Witten, Superstring Theory, vol 1,2 (Cambridge University Press, 1987).

[2] C. Rovelli, Quantum Gravity (Cambridge University Press, 2004).

[3] J. Baez, *Lect. Notes Phys.* **543**, 25 (2000).

[4] C. Rovelli and L. Smolin, *Phys. Rev. D* **52**, 5743 (1995).

[5] J. W. Barrett and L. Crane, *J. Math. Phys.* **39**, 3296 (1998).

[6] J. W. Barrett and L. Crane, *Class. Quant. Grav.* **17**, 3101 (2000).

[7] L. Crane, A. Perez and C. Rovelli, *Phys. Rev. Lett.* **87**, 181301 (2001).

[8] R. de Pietri, L. Freidel, K. Krasnov and C. Rovelli, *Nucl. Phys. B* **574**, 785 (2000).

[9] A. Perez and C. Rovelli, *Nucl. Phys. B* **599**, 255 (2001).

[10] C. Rovelli, *Phys. Rev. Lett.* **97**, 151301 (2006).

[11] E. Bianchi, L. Modesto, C. Rovelli and S. Speziale, *Class. Quant. Grav.* **23**, 6989 (2006).

[12] A. Miković, *Class. Quant. Grav.* **18**, 2827 (2001).

[13] A. Miković, *Class. Quant. Grav.* **21**, 3909 (2004).

[14] A. Miković, *Class. Quant. Grav.* **23**, 5459 (2006).