Stability of Certain QCD Sum Rules for Simple Model Spectra

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Abstract

The Analytic Continuation by Duality (ACD) method, a QCD sum rule, has been used in low-energy QCD phenomenology and for technicolor models. ACD is an ill-posed problem in the sense of Hadamard so the stability of the method is not assured. Nevertheless, the method has apparently been successfully applied to many low-energy effects. Hence it is useful to test the stability of the method for simple model spectra that mimic the actual QCD spectrum. We carry out such tests for (1) the original ACD, which employs polynomial approximations to 1/s integration kernel in L_p norms, and (2) a modified method, which uses Taylor series approximation. The original method that employs best-L_p approximation is unstable and the estimates tend to oscillate with increasing order of the approximation. The modified method is stable, but its convergence is very slow and it tends to provide only a lower bound for the estimates.

1. Introduction

Shifman, Vainshtein and Zakharov used the Operator Product Expansion (OPE) of the current-current correlators in their extension of the QCD to the subasymptotic region [1]. Theoretical predictions of many low-energy QCD effects were obtained, although some phenomenological input – namely, the vacuum condensates – was necessary.

The Analytic Continuation by Duality (ACD) is used to evaluate dispersive integrals in QCD using the first few terms of the OPE, but essentially no other phenomenological input. The ACD method was developed and applied to several QCD effects in the early 1980's [2]. It uses local quark-hadron duality and has been called a "strictly local" QCD sum rule [3]. The ACD approximates the kernel of a dispersive integral by a polynomial, which allows the integration to be performed in the complex plane.

The analytic continuation is an ill-posed problem [4]: small variations in the input parameters may lead to large variations in the results. Specifically, the ACD was shown to be an ill-posed problem [5], which relates to the

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fact that polynomial approximation is obtained from a minimization of the L_p-norm.

In the modified version of the ACD method proposed by Nasrallah [6], the $1/s$ kernel is approximated by truncated Taylor series around the upper cut-off R. The new method and its comparison to the original ACD are the main subjects of this contribution.

The reliability of the ACD-derived estimates has already been investigated for model spectra [7]. The original ACD was found to be unreliable due to several sources of error that are difficult to control. An instability was found for most models and divergence for some models. Here we investigate the stability of the modified ACD for two simple model spectra: vector-meson dominance and Breit-Wigner resonances.

The ACD method has been applied not only in the low-energy QCD context, but also to technicolor models [8]. The unknown spectrum of the Walking Technicolor is particularly sensitive to the reliability problems of the ACD. The relatively broad applicability of the ACD has lead us to abstain from reference to specific physical phenomena in this contribution, except in the justification of our models.

2. ACD Method

Suppose we need to find a matrix element of a process with the vacuum polarization $\Pi(s)$ in the non-perturbative regime. The problem may lead to the dispersive integral

$$\int_{s_0}^{\infty} \frac{1}{s}\, \mathrm{Im} F(s) ds,$$

where $F(s)$ is a simple function of $\Pi(s)$ and s_0 is the threshold of the process. The integration kernel may be another function instead of $1/s$, but the problems with this particular kernel are most common. The corresponding sum rule is called Das-Mathur-Okubo sum rule within QCD context whereas Peskin and Takeuchi used the term "zeroth order Weinberg sum rule" [9] in technicolor context.

The function $F(s)$ is analytic in the entire complex plane except on the branch cut along the real axis from s_0. Therefore, the function is analytic within the contour C shown in Fig. 1 and Cauchy's integral formula applies:

$$F(t) = \frac{1}{2\pi i} \oint_C \frac{F(s)}{s-t}\, ds. \tag{1}$$

The integral consists of two parts: the integral along the circle of radius R and the sum of the integrals along the segment. For the latter we use the discontinuity relation

$$F(s+i\varepsilon) - F(s-i\varepsilon) = 2\pi i\, \mathrm{Im} F(s+i\varepsilon),$$

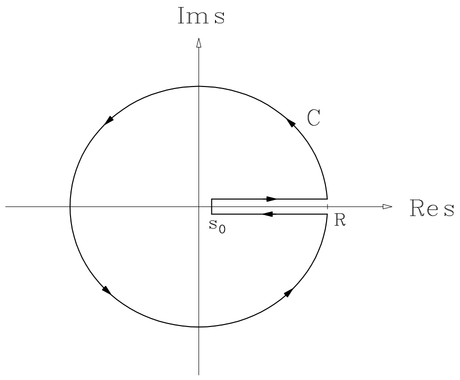

Figure 1: The contour of integration.

where the limit $\varepsilon \to 0$ is understood. We obtain

$$F(t) = \frac{1}{2\pi i} \oint_{|s|=R} \frac{F(s)}{s-t}\, ds + \frac{1}{\pi} \int_{s_0}^{R} \frac{\mathrm{Im}F(s)}{s-t-i\varepsilon}\, ds. \tag{2}$$

In the limit $t \to 0$ Eq. 2 becomes

$$F(0) = \frac{1}{2\pi i} \oint_{|s|=R} \frac{F(s)}{s}\, ds + \frac{1}{\pi} \int_{s_0}^{R} \frac{\mathrm{Im}F(s)}{s}\, ds. \tag{3}$$

It can be shown that asymptotically $F(s) \sim 1/s$ so the integral over the circle $|s| = R$ vanishes as $R \to +\infty$. Therefore, in this limit we get the dispersion relation

$$\overline{F} \equiv F(0) = \frac{1}{\pi} \int_{s_0}^{+\infty} \frac{\mathrm{Im}F(s)}{s}\, ds. \tag{4}$$

In contrast to the approach that leads to the dispersion relation (4), the ACD method makes the *second* integral in (3) vanish. This is achieved by approximating the kernel $1/s$ by a polynomial. The polynomial

$$p_N(s) = \sum_{n=0}^{N} a_n(N) s^n$$

is analytic in the entire complex plane and, therefore, the product $p_N(s)F(s)$ is analytic inside the contour C. Applying Cauchy's theorem we get

$$0 = \frac{1}{2\pi i} \oint_C p_N(s)F(s)ds = \frac{1}{2\pi i} \oint_{|s|=R} p_N(s)F(s)ds + \frac{1}{\pi} \int_{s_0}^{R} p_N(s)\mathrm{Im}F(s)ds. \tag{5}$$

Subtract (5) from (3):

$$F(0) = \frac{1}{2\pi i}\oint_{|s|=R}\left[\frac{1}{s} - p_N(s)\right]F(s)ds + \frac{1}{\pi}\int_{s_0}^{R}\left[\frac{1}{s} - p_N(s)\right]\mathrm{Im}F(s)ds. \quad (6)$$

Eq. (6) is exact since no approximation has been introduced yet. If we find $p_N(s)$ that converges to $1/s$ on the interval $[s_0, R]$ then the second integral in (6) will tend to be small:

$$\overline{F} = \frac{1}{2\pi i}\oint_{|s|=R}\left[\frac{1}{s} - p_N(s)\right]F(s)ds + \delta F_{\mathrm{fit}}(N) = \overline{F}_N + \delta F_{\mathrm{fit}}(N), \quad (7)$$

where $\delta F_{\mathrm{fit}}(N)$ will be called "fit error". The fit error depends not only on N but also on 1) the interval $[s_0, R]$ and 2) the fit routine. The fit routines will be considered in subsecs. 2.1 and 2.2.

Let us assume that the large-s expansion

$$F(s) = \sum_{m=1}^{M}\frac{h_m(s)}{s^m} + O\left(\frac{1}{s^{M+1}}\right) \quad (8)$$

is valid around the circle $|s| = R$. The expansion (8) may be OPE, which is an asymptotic series, but it may also be the large-momentum expansion of an exactly known vacuum polarization, which may be convergent series. The latter case will be considered in Sec. 4. The substitution of (8) into (7) yields

$$F_N = F_{N,M} + \delta F_{\mathrm{trunc}}(N, M),$$

where

$$F_{N,M} = \frac{1}{2\pi i}\oint_{|s|=R}\left[\frac{1}{s} - p_N(s)\right]\sum_{m=1}^{M}\frac{h_m(s)}{s^m}\,ds,$$

$$\delta F_{\mathrm{trunc}}(N, M) = \frac{1}{2\pi i}\oint_{|s|=R}\left[\frac{1}{s} - p_N(s)\right]\left[F(s) - \sum_{m=1}^{M}\frac{h_m(s)}{s^m}\right]ds,$$

$\delta F_{\mathrm{trunc}}$ being the *truncation error* coming from the residual term in (8). In order to suppress the neglected terms by inverse powers of R, we need to take $M > N$.

For known s-dependence of the large-s expansion coefficients we can calculate $F_{N,M}$, but the usual approach is to neglect the s-dependence of h_m's. The coefficients are, therefore, replaced by their value at $s = -R$:

$$h_m(s) \approx h_m(-R) \equiv \hat{h}_m.$$

This approximation causes an additional error:

$$F_{N,M} = F_{\mathrm{ACD}} + \delta F_{\mathrm{AC}}(N, M),$$

where

$$F_{\mathrm{ACD}} = \frac{1}{2\pi i} \oint_{|s|=R} \left[\frac{1}{s} - p_N(s) \right] \sum_{m=1}^{M} \frac{\hat{h}_m}{s^m}\, ds \tag{9}$$

is the ACD estimate of the function $F(t)$ at $t = 0$ and

$$\delta F_{\mathrm{AC}}(N, M) = \frac{1}{2\pi i} \oint_{|s|=R} \left[\frac{1}{s} - p_N(s) \right] \sum_{m=1}^{M} \frac{h_m(s) - \hat{h}_m}{s^m}\, ds$$

is called the *analytic continuation error*. This approximation may not be safe since it alters the analytic structure of the function $F(t)$. The only non-zero contributions to the integral (9) come from the simple poles so that Eq. (9) simplifies to

$$F_{\mathrm{ACD}} = - \sum_{n=0}^{\min(N,M-1)} \hat{h}_{n+1} a_n(N). \tag{10}$$

Unlike the dispersion relations, the ACD technique depends on several approximations. Along with the three errors defined above, there is another source of error: the *numerical uncertainty* in the coefficients $\hat{h}_m$. This source of error will not be considered here. If the spectral function is known exactly, the dispersion relation gives a reliable estimate of $\overline{F}$ so the comparison to F_{ACD} will determine if the ACD estimate is reliable.

2.1. Best-L_p Polynomial Approximations

The usual fit routines minimize the L_p norm

$$d_p = \left[\int_a^b |d(x)|^p w(x)\, dx \right]^{1/p},$$

where

$$d(x) = \frac{1}{x} - p_N(x)$$

and $w(x)$ is the weighting function. The most natural choices are $p = 1, 2, \infty$ and $w(x) = 1$, but there is no *a priori* reason to favour the (unweighted) least-square fit (which was apparently the case in nearly all of the ACD applications thus far). Hence we have developed the best-L_1,-L_2 and -L_∞ approximations to $1/x$ [7]; the coefficients $a_n(N)$ of the fit are found from fairly simple recursive formulae. The dependence of the ACD estimates on the fit routine was shown to be considerable in some cases.

2.2. Truncated Taylor Series Approximation

An approximation of the function $1/s$ by Taylor expansion around $s = R$ has been proposed [6]. The primary motivation is that such an approximation emphasizes the better-known part of the spectrum, although the strong dependence of the (best-L_p derived) ACD estimates on the lower cut-off also played a role.

For $0 < s \leq R$ the function $1/s$ can be expanded around $s = R$ in geometric series:

$$\frac{1}{s} = \frac{1}{R}\sum_{n=0}^{+\infty}\left(1 - \frac{s}{R}\right)^n . \tag{11}$$

The expansion (11) has to be truncated at finite N. The truncated series is easily summarizable:

$$\frac{1}{R}\sum_{n=0}^{N}\left(1 - \frac{s}{R}\right)^n = \frac{1}{s}\left[1 - \left(1 - \frac{s}{R}\right)^{N+1}\right]. \tag{12}$$

The function (12) is a polynomial of order N. With the help of the binomial formula we get

$$p_N(s) = \frac{1}{s}\left[1 - \left(1 - \frac{s}{R}\right)^{N+1}\right] = \sum_{n=0}^{N}\binom{N+1}{n+1}\frac{(-1)^n}{R^{n+1}}s^n$$

so the coefficients of the polynomial $p_N(s)$ are

$$a_n(N) = \frac{(-1)^n}{R^{n+1}}\binom{N+1}{n+1}. \tag{13}$$

We have obtained a simple explicit formula for the fit coefficients, which was not possible for the best-L_p approximations (except for very low values of N).

Since $0 < s \leq R$ the difference $d(s)$ is always non-negative:

$$d(s) = \frac{1}{s} - p_N(s) = \frac{1}{s}\left(1 - \frac{s}{R}\right)^{N+1} \geq 0. \tag{14}$$

The approximation has the effect of making $d(s)$ and its derivatives up to the order N vanish at $s = R$. However, the polynomial $p_N(s)$ is not actually an approximation to the function $1/s$ as it only bounds it from below. We will see in Sec. 4 that the function $\mathrm{Im}F(s)$ for a QCD-like spectrum is such that the fit error will tend to be positive for all N while this

The fact that the truncated Taylor series does not minimize any norm of the function $F(s)$ may be expected to have profound consequences even beyond the fit error itself.

3. ACD Estimation as an Ill-Posed Problem

Shortly after the first applications of the ACD method [2], Caprini and Verzegnassi showed that the ACD approximation is an ill-posed problem in the sense of Hadamard [5]. This means that small uncertainties in the input parameters can lead to large uncertainties in the ACD-derived estimates.

We will show that the ACD estimation of the dispersive integral 4 is an ill-posed problem indeed. The ACD method approximates the dispersive integral

$$F(t) = \frac{1}{\pi} \int_{s_0}^{+\infty} \frac{\mathrm{Im}F(s)}{s-t}\, ds \tag{15}$$

with the function

$$\widetilde{F}(t) = \sum_{m=1}^{M} \frac{\hat{h}_m}{t^m}. \tag{16}$$

In order to ascertain how close the approximation of $F(t)$ by $\widetilde{F}(t)$ is we consider the functional distance

$$\varepsilon_0 = \inf_{F} \left[\frac{1}{2\pi} \int_0^{2\pi} \left| F(t) - \widetilde{F}(t) \right|^2 d\theta \right]^2, \tag{17}$$

where the infimum is taken over the functions satisfying the constraint (4) with $\overline{F}$ fixed, but arbitrary.

Substituting (15) and (16) into (17) we find

$$\varepsilon_0^2 = \inf_{F} \left[\frac{1}{\pi^2} \int_{s_0}^{+\infty} \int_{s_0}^{+\infty} K(s,t) \mathrm{Im}F(s) \mathrm{Im}F(t)\, ds\, dt - \frac{2}{\pi} \int_{s_0}^{+\infty} g(s) \mathrm{Im}F(s)\, ds + \sum_{m=1}^{M} \frac{\hat{h}_m^2}{R^{2m}} \right], \tag{18}$$

where

$$K(s,t) = \frac{1}{2\pi} \int_0^{2\pi} \frac{d\theta}{(s - Re^{-i\theta})(t - Re^{i\theta})}\ , \ g(s) = \frac{1}{2\pi} \int_0^{2\pi} \sum_{m=1}^{M} \frac{\hat{h}_m}{R^m e^{im\theta}} \frac{d\theta}{(s - Re^{-i\theta})}.$$

The integrals over θ can be evaluated by applying Cauchy's theorem for the unit circle in z-plane. The results are

$$K(s,t)=\begin{cases}\dfrac{1}{R^2-st}=\dfrac{1}{R^2}\displaystyle\sum_{n=0}^{+\infty}\left(\frac{st}{R^2}\right)^n & s,t<R,\\ -\dfrac{1}{R^2-st}=\dfrac{1}{st}\displaystyle\sum_{n=0}^{+\infty}\left(\frac{R^2}{st}\right)^n & s,t>R,\\ 0 & \text{otherwise},\end{cases}$$
$$g(s)=\begin{cases}-\displaystyle\sum_{m=1}^{M}\frac{s^{m-1}\hat{h}_m}{R^{2m}} & s<R,\\ 0 & s>R.\end{cases}\tag{19}$$

The problem has been reduced to minimization of the integral (18) subject to the constraint (4). By introducing the Lagrange multiplier λ the problem is further reduced to the minimization of the functional

$$\begin{aligned}\mathcal{L}(\mathrm{Im}F(t),\lambda)=&\frac{1}{\pi^2}\int_{s_0}^{+\infty}\int_{s_0}^{+\infty}K(s,t)\,\mathrm{Im}F(s)\,\mathrm{Im}F(t)ds\,dt\\ &-\frac{2}{\pi}\int_{s_0}^{+\infty}g(s)\,\mathrm{Im}F(s)ds+\lambda\left[\frac{2}{\pi}\int_{s_0}^{+\infty}\frac{1}{s}\,\mathrm{Im}F(s)-\overline{F}\right]ds\;.\end{aligned}\tag{20}$$

The function $1/s$ is approximated by the polynomial $p_N(s)$ on the branch cut so that $1/s$ in (20) is actually replaced by the function

$$f(s)=\begin{cases}\displaystyle\sum_{n=1}^{+\infty}a_n s^n, & s<R,\\ \dfrac{1}{s}, & s>R.\end{cases}$$

Variation of the functional (20) leads to the integral equation

$$\int_{s_0}^{+\infty}K(s,t)\,\mathrm{Im}F(t)\,dt-g(s)+\frac{\lambda}{2}f(s)=0,\tag{21}$$

which, upon the substitution of $K(s,t)$ and $g(s)$ from (19), gives the equations:

$$\frac{1}{\pi}\frac{1}{R^2}\sum_{n=1}^{+\infty}\frac{s^n}{R^{2n}}\int_{s_0}^{R}t^n\mathrm{Im}F(t)dt+\sum_{n=1}^{M-1}\frac{s^n\hat{h}_{n+1}}{R^{2(n+1)}}+\frac{\lambda}{2}\sum_{n=1}^{+\infty}a_n s^n=0,\quad s<R,$$
$$\frac{1}{\pi}\sum_{n=1}^{+\infty}\frac{R^{2n}}{s^{n+1}}\int_{R}^{+\infty}\frac{\mathrm{Im}F(t)}{t^{n+1}}dt+\frac{\lambda}{2}\frac{1}{s}=0,\qquad s>R.$$

The first equation gives an infinite set of conditions

$$\begin{aligned}\frac{1}{\pi}\frac{1}{R^{2n+2}}\int_{s_0}^{R} t^n \mathrm{Im}F(t)dt + \frac{\hat{h}_{n+1}}{R^{2n+2}} + \frac{\lambda}{2}\,a_n = 0, \quad & n \le M-1 \\ \frac{1}{\pi}\frac{1}{R^{2n+2}}\int_{s_0}^{R} t^n \mathrm{Im}F(t)dt + \frac{\lambda}{2}\,a_n = 0, \quad & n > M-1\end{aligned} \tag{22}$$

while the second equation leads to another infinite set of conditions:

$$\frac{1}{\pi}\int_{R}^{+\infty}\frac{\mathrm{Im}F(t)}{t}\,dt + \frac{\lambda}{2} = 0, \tag{23}$$

$$\int_{R}^{+\infty}\frac{\mathrm{Im}F(t)}{t^{n+1}}\,dt = 0, \quad n \ge 1. \tag{24}$$

A function that satisfies all the conditions (22)–(24) may not exist. Hence we fix a finite number of moments of $\mathrm{Im}F(t)$ and then take the limit $N \to +\infty$ [5].

Let us find the Lagrange multiplier λ. From (22) we have

$$\frac{1}{\pi}\int_{s_0}^{R} t^n \mathrm{Im}F(t)dt = \begin{cases} -\hat{h}_{n+1} - \dfrac{\lambda}{2}\,a_n R^{2n+2}, & n \le M-1, \\ -\dfrac{\lambda}{2}\,a_n R^{2n+2}, & n > M-1. \end{cases} \tag{25}$$

On the other hand, the constraint (4) can be written as

$$\frac{1}{\pi}\int_{s_0}^{R} t^n \mathrm{Im}F(t)dt + \frac{1}{\pi}\int_{R}^{+\infty}\frac{\mathrm{Im}F(t)}{t}\,dt = \overline{F},$$

which, upon substitution of (25) and (23), gives the Lagrange multiplier:

$$\frac{\lambda}{2} = -\frac{\overline{F} - F_{\mathrm{ACD}}}{\sum_{n=0}^{N} a_n^2 R^{2n+2} + 1}, \tag{26}$$

where F_{ACD} is given by (10).

Substituting the integral in (21) into (18) we obtain

$$\varepsilon_0^2 = -\frac{1}{\pi}\int_{s_0}^{+\infty}\left[g(s) + \frac{\lambda}{2}f(s)\right]\mathrm{Im}F(s)\,ds + \sum_{n=0}^{M-1}\frac{\hat{h}_{n+1}^2}{R^{2n+2}}. \tag{27}$$

Using the explicit expression (19) for $g(s)$ and relations (25) and (23) and after some cancellations the Eq. (27) becomes

$$\varepsilon_0^2 = \frac{\lambda^2}{4}\left(\sum_{n=0}^{N} a_n^2 R^{2n+2} + 1\right),$$

which, along with (26), yields

$$\varepsilon_0^2 = \frac{\left(\overline{F} - F_{\mathrm{ACD}}\right)^2}{\sum_{n=0}^{N} a_n^2 R^{2n+2} + 1}. \tag{28}$$

The limit $N \to +\infty$ is understood in (28). The series in the denominator of (28) is divergent:

$$\sum_{n=0}^{N} a_n^2 R^{2n+2} \xrightarrow[N\to+\infty]{} \infty$$

since the function is not analytic within the circle $|s| = R$. Therefore,

$$\lim_{N\to+\infty} \varepsilon_0^2 = 0.$$

The result $\varepsilon_0 \to 0$ means that it is impossible to predict the value of $\overline{F}$ from the condition that $F(t)$ is approximated on the circle $|s| = R$ by $\widetilde{F}(t)$. This is why we have to use the finite-order polynomial approximation to $1/s$. In this case it is possible to restrict the value of $\overline{F}$ to the interval

$$F_{\mathrm{ACD}} - \delta \leq \overline{F} \leq F_{\mathrm{ACD}} + \delta,$$

where

$$\delta = \varepsilon \sqrt{\sum_{n=0}^{N} a_n^2 R^{2n+2} + 1}.$$

Clearly, since the sum is infinite as $N \to +\infty$, the uncertainty of the ACD estimates goes to infinity as $N \to +\infty$. Notice that minimization of the functional distance (17) is not equivalent to the minimization of an L_p-norm, but our qualitative conclusion remains in effect.

The usual procedure in applying the ACD method is to use low values of N for which OPE is reasonably accurate. The problem is that a_n's are very sensitive to N so the ACD-derived estimates are expected to oscillate strongly; change from N to $N+1$ changes F_{ACD} considerably. The problem seems to have been overcome with the large error bars on F_{ACD}, at least in the early applications of the ACD [2], but the estimation of this 'total' error is obscure.

The truncated Taylor series approximation does not minimize a functional distance so the behavior of F_{ACD} is expected to be different.

4. ACD Estimates for Model Spectra

In order to investigate the stability of the ACD-derived results we introduce two simple model functions $F(s)$ for which the dispersive integral (4) can be evaluated exactly. The three types of error in the ACD results will be calculated.

In both models there is a single resonance of mass m_V in the vector channel and a single resonance of mass m_A in the axial channel. The parameters of the resonances are constrained by the condition that the spectra saturate the first and second Weinberg sum rules.

4.1. Vector-Meson Dominance Model

The first model contains two infinitely sharp (δ-function) resonances:

$$\mathrm{Im}F(s) = -\pi[f_V^2\delta(s-m_V^2) - f_A^2\delta(s-m_A^2)],$$

$$F(s) = \frac{f_V^2}{s-m_V^2+i\epsilon} - \frac{f_A^2}{s-m_A^2+i\epsilon}.$$

Clearly,

$$\overline{F} = F(0) = \frac{f_A^2}{m_A^2} - \frac{f_V^2}{m_V^2} \tag{29}$$

provided $s_0 < m_V, m_A < R$. The first and second Weinberg sum rules impose the conditions

$$f_V^2 - f_A^2 = f^2, \qquad f_V^2 m_V^2 - f_A^2 m_A^2 = 0$$

so that from (29) one obtains

$$\overline{F} = (1+r)\frac{f^2}{m_V^2}, \tag{30}$$

where $r = m_V^2/m_A^2$. The QCD value is $r \approx 0.4$.
The large-momentum expansion for this model is

$$F(s) = \sum_{n=0}^{\infty} \frac{f_V^2 m_V^{2n} - f_A^2 m_A^{2n}}{s^{n+1}} = \frac{f^2}{s} + \sum_{n=2}^{\infty} \frac{f^2 m_V^{2n}}{s^{n+1}} X_{2n}(r),$$

where

$$X_n(r) \equiv \frac{1 - r^{1-\frac{n}{2}}}{1-r}. \tag{31}$$

The coefficients of the expansion do not depend on s so that for $M > N$ the only error for this model is the fit error:

$$\delta F_{\rm fit} = \frac{f^2}{(1-r)m_V^2}\left[\left(1 - \frac{m_V^2}{R}\right)^{N+1} - r^2\left(1 - \frac{m_V^2}{rR}\right)^{N+1}\right] > 0.$$

The fit error falls off monotonously with N so $F_{\rm ACD}$ approaches $\overline{F}$ from below. Thus $F_{\rm ACD}$ is only a lower bound for $\overline{F}$. In order to get a close bound we need to decrease R, but obviously not into the region of the resonances.

In general, only a few terms of the OPE are known [1, 2] so in actual applications $M \le 5$, while N is under our control. Hence we have, besides the logical choice $N = M - 1$, investigated the cases with $N \ge M$ when the truncation error is present. The results are shown in Table 1. For best-L_p approximations the fit error converges to zero only after some oscillations, which are more pronounced for greater p [7]. For Taylor series approximation $F_{\rm ACD}$ reaches a local maximum as a function of N with M fixed (not shown in the Table); the maximum value is not necessarily close to the exact value (30).

The dependence of the estimates on the upper cutoff is shown in Table 2. As expected, the modified ACD depends on R stronger than the original method. For $M \le 5$, R needs to be very small to get meaningful estimates.

4.2. Breit-Wigner Model

In a more realistic model we suppose a Breit-Wigner type imaginary part on the real axis:

$$\mathrm{Im}F(s) = -f_V^2 \frac{\sqrt{s}\,\Gamma_V}{(s-m_V^2)^2 + s\Gamma_V^2} + f_A^2 \frac{\sqrt{s}\,\Gamma_A}{(s-m_A^2)^2 + s\Gamma_A^2},$$

which is obtained from the spectral function

$$F(s) = \frac{f_V^2}{s - m_V^2 + i\sqrt{s}\,\Gamma_V} - \frac{f_A^2}{s - m_A^2 + i\sqrt{s}\,\Gamma_A}. \tag{32}$$

The first Weinberg sum rule gives the same constraint as for the vector-meson dominance model:

$$f_V^2 - f_A^2 = f^2,$$

but the second Weinberg sum rule diverges. However, this problem can be fixed by imposing the rule that the OPE term of dimension 4 vanishes [7], which gives conditions

$$f_V^2 m_V^2 - f_A^2 m_A^2 = 0, \qquad f_V^2 \Gamma_V^2 - f_A^2 \Gamma_A^2 = 0.$$

With the additional conditions

$$\frac{\Gamma_V^2}{\Gamma_A^2} = r, \qquad \xi \equiv \frac{\Gamma_V}{m_V} = \frac{\Gamma_A}{m_A}$$

the estimate of $\overline{F}$ remains (30). For QCD $\xi \approx 0.2$. Interestingly, $\overline{F}$ does not depend on the width of the resonances even though the spectral function does. The large-momentum expansion for this model is

$$\frac{f^2}{s - m^2 + i\sqrt{s}\,\Gamma} = \frac{f^2}{s} + \frac{f^2}{s} \sum_{n=0}^{\infty} \frac{(-1)^n m^{2n}}{s^n} \left[U_{2n}\Big(\frac{\xi}{2}\Big) + i\frac{\sqrt{s}}{m} U_{2n-1}\Big(\frac{\xi}{2}\Big) \right],$$

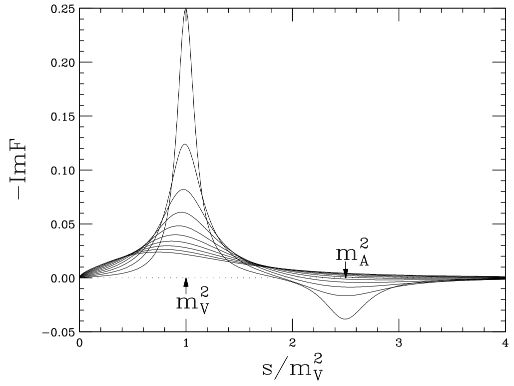

Figure 2: The Spectral Functions for the Breit–Wigner Model. The ratio ξ is varied from $\xi = 1.0$ to $\xi = 0.1$ at 0.1 intervals with smaller values of ξ corresponding to more prominent peaks. The arrows indicate the positions of the lowest-lying resonances in the vector and axial channels. [7]

where

$$U_n(\cos\theta) = \frac{\sin(n+1)\theta}{\sin\theta} \tag{33}$$

are the Chebyshev polynomials of the second kind. Taking into account the above conditions we find

$$\begin{aligned} F(s) =& \frac{f^2}{s}\left[1 - i\frac{m_V}{\sqrt{s}}X_1(r)U_1\Big(\frac{\xi}{2}\Big)\right] + \\ &+ \frac{f^2}{s}\sum_{n=2}^{\infty}\frac{(-1)^n m_V^{2n}}{s^n}\left[X_{2n}(r)U_{2n}\Big(\frac{\xi}{2}\Big) + i\frac{\sqrt{s}}{m_V}X_{2n-1}(r)U_{2n-1}\Big(\frac{\xi}{2}\Big)\right], \end{aligned}$$

where $X_n(r)$ is defined in (31).

The spectral function $\mathrm{Im}F(s)$ is shown in Fig. 2 for several values of r. It is clear from the figure and Eq. (14) that the fit error for Taylor series approximation will be positive.

The branch cut of the spectral function (32) begins at $s = 0$, which is below s_0. Hence the contribution of the interval $[0, s_0]$ will be taken into account separately. For $s_0 \ll m_V^2$ we have

$$-\mathrm{Im}F(s) \;\approx\; \left(\frac{f_V^2}{m_V^3} - \frac{f_A^2}{m_A^3}\right)\xi\sqrt{s} \;=\; \frac{\xi f^2 X_{-3}(r)\sqrt{s}}{m_V^3}$$

so that the “infrared” interval $[0, s_0]$ contributes to $\overline{F}$ the amount

$$F_{\mathrm{IR}}(s_0) \;=\; -\frac{1}{\pi}\int_0^{s_0}\frac{\mathrm{Im}F(s)}{s}ds \;\approx\; \frac{2\xi f^2 X_{-3}(r)\sqrt{s_0}}{\pi m_V^3}.$$

Table 1: F_{ACD} in units of its exact value (30). The cutoffs are $s_0 = 0.2m_V^2$ and $R = 5m_V^2$. The dimensionless parameters are $r = 0.4$, $\xi = 0.2$. For Breit-Wigner model $F_{\text{IR}} \approx 0.085f^2/m_V^2$ was subtracted from (30).

		vector-meson		Breit-Wigner	
M	N	best-L_2 fit	Taylor series	best-L_2 fit	Taylor series
2	1	1.198	0.286	1.160	0.283
	2	1.955	0.429	1.776	0.417
	3	2.709	0.571	2.269	0.546
3	2	1.564	0.414	1.615	0.411
	3	0.607	0.514	1.406	0.522
	4	−3.020	0.571	−0.036	0.611
4	3	1.375	0.524	1.589	0.525
	4	2.214	0.621	1.214	0.623
	5	9.002	0.721	1.201	0.706
5	4	1.018	0.616	1.201	0.622
	5	−0.875	0.688	1.285	0.706
	6	−13.495	0.733	3.610	0.778

Therefore, the ACD estimate is now corrected:

$$F'_{\text{ACD}} = F_{\text{IR}}(s_0) + F_{\text{ACD}}.$$

The results are shown in Table 1. The behavior of the estimates is qualitatively similar to the vector-meson dominance model. The much greater difference between the models for the original (best-L_2 fit) method appears to be a further evidence of its strong instability.

For Taylor series approximation there is a maximal value of F_{ACD} achieved at $N > M$ whereas for best-L_p fits increasing N beyond $M - 1$ leads to rapid divergence of the results. For the Breit-Wigner model with the same parameters as in Table 1, F_{ACD} reaches a local maximum at $N = 6$ for $M = 3$ and at $N = 11$ for $M = 4$.

5. Conclusions

The modified ACD method, which approximates $1/s$ kernel by a truncated Taylor series, was evaluated against the original form of the ACD. The advantage of the modified method is primarily its stability. However, for realistic values of M the results are far from the expected values even for the simple model spectra. The modified ACD tends to yield a *lower bound* for $\overline{F}$, although this may depend on the actual spectrum.

Another problem is the very strong dependence of the results on the upper cutoff R; the estimates may approach the expected value only for very low

Table 2: The dependence of $F_{\rm ACD}$ on the upper cutoff R. The other parameters as well as the units are the same as in Table 1 while $M = N + 1$.

		vector-meson		Breit-Wigner	
M	$R(m_V^2)$	best-L_2 fit	Taylor series	best-L_2 fit	Taylor series
3	4	1.567	0.508	1.699	0.500
	5	1.564	0.414	1.615	0.411
	7	1.424	0.301	1.440	0.303
	10	1.208	0.212	1.224	0.216
4	4	1.160	0.627	1.431	0.627
	5	1.375	0.524	1.589	0.525
	7	1.525	0.390	1.639	0.393
	10	1.477	0.279	1.537	0.284
5	4	0.877	0.719	0.937	0.729
	5	1.018	0.616	1.201	0.622
	7	1.313	0.470	1.505	0.476
	10	1.494	0.342	1.622	0.349

R (with $M \leq 5$). If the location of the resonances is unknown, such as for the Walking Technicolor, this problem may render estimates completely unreliable.

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