Garfinkle-Horovitz-Strominger Dilaton Black Hole Gravitational Lensing in the Strong Deflection Limit *

Galin N. Gyulchev† and Stoytcho S. Yazadjiev‡

Department of Theoretical Physics,
Faculty of Physics, Sofia University,
5 James Bourchier Boulevard, 1164 Sofia, BULGARIA

Abstract

In the present work we study numerically equatorial lensing by the charged, static, spherically symmetric Garfinkle-Horowitz-Strominger dilaton black hole in the strong deflection limit. In this approximation we compute the strong deflection limit coefficients and the bending angle of the light ray and it is shown that the last one decreases with the increases of the parameter Q/M. All of the lensing quantities are compared to particular case as Gibbons–Maeda–Garfinkle–Horowitz–Strominger and Reissner–Nördstrom black holes.

1. Introduction

One of the consequences of Einstein's General Theory of Relativity is that light rays are deflected by gravity. The phenomena resulting from the deflection of electromagnetic radiation in a gravitational field are referred to as *gravitational lensing* (GL) and an object causing a detectable deflection is known as a *gravitational lens*. The basic theory of GL was developed by Liebes [1], Refsdal [2] and Bourassa, Kantowski [3]. Detailed discussions on GL are the monographs by Schneider, Ehlers, Falco [4] and Zakharov [5] and the reviews by Blandford and Narayan [6], Refsdal and Surdej [7], Narayan and Bartelmann [8] and Wambsganss [9].

Nowadays, gravitational lensing is a violently developing area of research, and it has found applications ranging from the search of extrasolar planets and compact dark matter to the estimation of the values of the cosmological parameters [10]. In most of these cases it is possible to assume that the gravitational field is weak, hence the angle of deflection of the light in the field of a spherically symmetric body with mass M can be approximated by: $\hat{\alpha} \approx 4GM/uc^2$, where u is the impact parameter [4]. It is well known that

* Work supported by the Bulgarian National Science Fund under Grants MUF04/05 (MU 408) and VU-F-205/06 and by the Sofia University Research Fund N.60.

† e-mail address: gyulchev@phys.uni-sofia.bg

‡ e-mail address: yazad@phys.uni-sofia.bg

for a Schwarzschild black hole, the deflection angle diverges as $u \to 3\sqrt{3}M$, allowing photons to wind many times around the lens before reaching the observer. In this case, an infinite set of images at both sides of the black hole appears [11]. These images are produced by photons, making one or more complete loops around the black hole before reemerging in the observer's direction. These relativistic images could provide a profound test of general relativity in its strong field regime. In this context, the works of Bekenstein and Sanders [12] and Virbhadra, Narasimha and Chitre [13], may be of interest.

In Ref. [14] analytical formulae for the position and the magnification of the images were obtained, defining a strong field limit for the deflection angle. A full generalization of the strong field limit for any spherically symmetric spacetime was drawn in Ref. [15] and applied to several black hole metmetrics, allowing a non-degenerate discrimination among different collapsed objects. In work [16], the method was used to examine the characteristics of a Gibbons-Maeda-Garfinkle-Horowitz-Strominger (GMGHS) charged black hole of heterotic string theory. In works [17, 18] the cases of Kerr and Kerr-Sen black holes and the relevance of their spins in the strong deflection limit approximation were discussed for light rays travelling on quasi-equatorial trajectories. Waiting for an analytical treatment including non-equatorial trajectories, the general case is explored numerically in [19].

Presently, the general relativity is not the only viable theory of gravitation. As we have already mentioned, the strong gravitational lensing could provide profound examination of the space–time around different kinds black holes. Therefore, following the method of Bozza [15] in the present work we wish to study gravitational lensing in the strong deflection limit due to a static, spherically symmetric, charge, dilaton Garfinkle-Horowitz-Strominger black hole [20] in the heterotic string theory with the aim of investigating the influence of the dilaton field in the behaviour of the bending angle in the strong deflection limit.

2. The Garfinkle-Horowitz-Strominger dilaton black hole

We consider a family solutions of Garfinkle-Horowitz-Strominger obtained in low energy string theory, which describe static, spherically-symmetric black holes. The line element of this solution is

$$ds^2 = -\chi^2 dt^2 + \frac{dr^2}{\chi^2} + R^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{1}$$

where the functions χ^2 and R^2 are

$$\chi^2 = \left(1 - \frac{r_+}{r}\right)\left(1 - \frac{r_-}{r}\right)^{\frac{1-\alpha^2}{1+\alpha^2}}, \tag{2}$$

$$R^2 = r^2\left(1 - \frac{r_-}{r}\right)^{\frac{2\alpha^2}{1+\alpha^2}}, \tag{3}$$

and α is an arbitrary parameter. Here r_+ and r_- are independent parameters, which are connected with the mass M and the charge Q of the black hole:

$$2M = r_+ + \left(\frac{1-\alpha^2}{1+\alpha^2}\right) r_-,$$
$$Q^2 = \frac{r_+ r_-}{1+\alpha^2}. \tag{4}$$

If we introduce parameter $\gamma = \frac{1-\alpha^2}{1+\alpha^2}$, we can rewrite the metric (1) in the form

$$ds^2 = -\left(1-\frac{r_-}{r}\right)^{\gamma}\left(1-\frac{r_+}{r}\right)dt^2 + \left(1-\frac{r_-}{r}\right)^{-\gamma}\left(1-\frac{r_+}{r}\right)^{-1}dr^2 + r^2\left(1-\frac{r_-}{r}\right)^{1-\gamma}d\Omega^2. \tag{5}$$

Then the relations (4), which take the form

$$2M = r_+ + \gamma r_-, \tag{6}$$
$$2Q^2 = (1+\gamma) r_- r_+, \tag{7}$$

lead to black hole horizons

$$r_+ = M\left\{1+\sqrt{1-\frac{2\gamma}{(1+\gamma)}\left(\frac{Q}{M}\right)^2}\right\}, \tag{8}$$

$$r_- = \frac{M}{\gamma}\left\{1-\sqrt{1-\frac{2\gamma}{(1+\gamma)}\left(\frac{Q}{M}\right)^2}\right\}. \tag{9}$$

The transition between the black hole and naked singularity occurs at

$$\left(\frac{Q}{M}\right)^2 = \frac{2}{1+\gamma}. \tag{10}$$

Hereafter we will restrict our consideration up to extremal black hole. In the particular case $\gamma = 0$, the solution (1) coincides with static, spherically symmetric GMGHS charged solution investigated in SDL regime from Bhadra in [16], while in the particular case $\gamma = 1$ reconstructs the Reissner–Nördstrom solution.

The outer horizon r_+ and the inner horizon r_- are plotted in the Fig.1 as a function of the metric parameter γ for different value of the ratio Q/M. The black hole horizons coincide in situation $\gamma = 1$ for charge $Q = M$, when the extremal black hole is occurs. In particular case of GHSHS black hole there is no inner horizon.

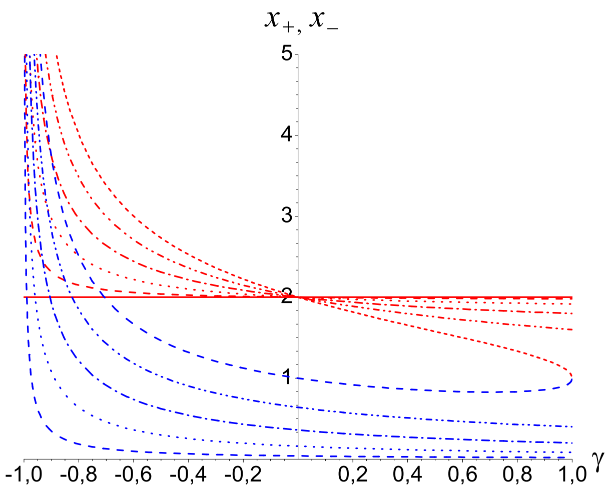

Figure 1: *The outer and the inner black hole horizons $x_+ = r_+/M$ (red line) and $x_- = r_-/M$ (blue line) as a function of metric parameter γ for $Q/M = 0$ (solid line) to $Q/M = 1$ (short, dashed line) by $Q/M = 0.2$*

3. Lensing in the equatorial plane

3.1. Deflection angle in the equatorial plane

Let us start with the conditions $\theta = \pi/2$ and set the light ray on the equatorial plane. In this case, if we substitute $x = r/M$ to a new radial coordinate nd measure all distances in units $M = 1$ we obtain the reduced metric in the form

$$ds^2 = -A(x)dt^2 + B(x)dx^2 + C(x)d\phi^2. \tag{11}$$

The metric coefficients are:

$$A(x) = \left(1 - \frac{x_-}{x}\right)^{\gamma} \left(1 - \frac{x_+}{x}\right), \tag{12}$$

$$B(x) = \left(1 - \frac{x_-}{x}\right)^{-\gamma} \left(1 - \frac{x_+}{x}\right)^{-1}, \tag{13}$$

$$C(x) = x^2 \left(1 - \frac{x_-}{x}\right)^{1-\gamma}. \tag{14}$$

The relation between the impact parameter (the perpendicular distance from the lens to the tangent to the null geodesic of the source) and the distance of closest approach of the light ray x_0 can be obtained from the conservation of the angular momentum of the scattering process, and it is given by

$$u(x_0) = \sqrt{\frac{C(x_0)}{A(x_0)}} = x_0^{1+\gamma} \sqrt{\frac{(x_0 - x_-)^{1-2\gamma}}{(x_0 - x_+)}}. \tag{15}$$

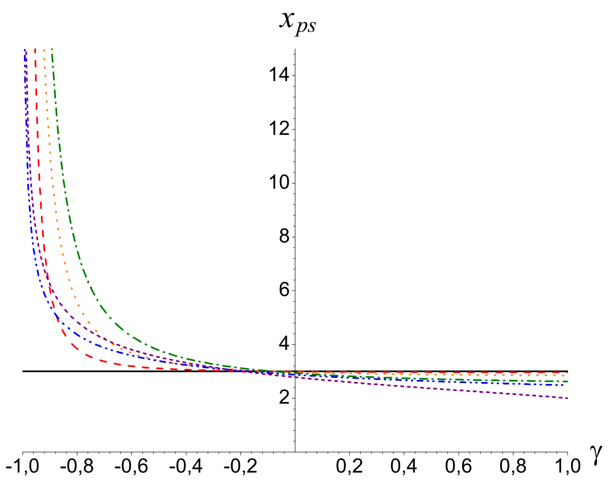

Figure 2: *The radius of the photon sphere as a function of metric parameter γ for charge $Q/M = 0$ (solid line) to charge $Q/M = 1$ (short, dashed line) by charge $Q/M = 0.2$*

A starting point of the strong field limit expansion is the photon sphere, which has been defined by Virbhadra and Ellis [21] (and has subsequently been investigated by Claudel, Virbhadra and Ellis [22]) as a timelike hypersurface $\{x = x_{ps}\}$ on which the light bending angle becomes unboundedly large when the closest distance of approach x_0 tends to x_{ps}. The photon sphere equation for static and spherically symmetric metric is

$$A(x_0)C'(x)|_{x=x_0} = A'(x)|_{x=x_0}C(x_0), \tag{16}$$

for which we require to admit at least one positive solution. For the GSH metric the photon sphere equation takes the form

$$2x_0^2 - (3x_+ + (2\gamma+1)x_-)x_0 + 2x_+x_-(1+\gamma) = 0. \tag{17}$$

The real root external to the outer horizon of this equation defines the radius of the photon sphere $x_{ps} = r_{ps}/M$ for the GHS black hole:

$$\begin{aligned} x_{ps} \;=\; & \frac{1}{4\gamma}\left[(\gamma-1)\sqrt{1-\frac{2\gamma}{(1+\gamma)}\left(\frac{Q}{M}\right)^2}+1+5\gamma\right] \\ & + \sqrt{\frac{1}{16\gamma^2}\left[(1-\gamma)\sqrt{1-\frac{2\gamma}{(1+\gamma)}\left(\frac{Q}{M}\right)^2}-1-5\gamma\right]^2-2\left(\frac{Q}{M}\right)^2}. \end{aligned} \tag{18}$$

The radius of the photon sphere is computed numerically and is plotted in Fig. 2 for different values of the ratio Q/M.

The bending angle of a light ray in a static spherically symmetric spacetime, described by the line element (11) is given by

$$\tilde{\alpha}(x_0) = \phi_f(x_0) - \pi, \tag{19}$$

$$\phi_f(x_0) = 2\int_{x_0}^{\infty} \left(\frac{B(x)}{C(x)}\right)^{1/2} \left[\frac{C(x)}{C(x_0)}\frac{A(x_0)}{A(x)} - 1\right]^{-1/2} dx, \tag{20}$$

where $\phi_f(x_0)$ is the total azimuthal angle. With the decreases of the distance of closest approach x_0 the deflection angle increases, and for a certain value of x_0 the deflection angle becomes 2π, so that the light makes a complete loop around the black hole. Let x_0 decreases further, then the light ray will wind several times around the lens before reaching the observer and finally when x_0 becomes equal to the radius of the photon sphere x_{ps} the deflection angle will become unboundedly large and the photon will be captured by the lens object.

3.2. Strong deflection limit approach

We can find the behaviour of the deflection angle very close to the photon sphere following the evaluation technique for the integral (19) developed by Bozza [15]. The divergent integral is first splitted into two parts one of which $\phi_f^D(x_0)$ contains the divergence and the other $\phi_f^R(x_0)$ is regular. Both pieces are expanded around $x_0 = x_{ps}$ and with sufficiently large accuracy are approximated with the leading terms. At first, we express the integrand of (19) as a function of two new variables y and z which are defined by

$$y = A(x), \tag{21}$$

$$z = \frac{y - y_0}{1 - y_0}, \tag{22}$$

where $y_0 = A_0$.
The whole azimuthal angle then takes the form

$$\phi_f(x_0) = \int_0^1 R(z, x_0) f(z, x_0) dz, \tag{23}$$

where the functions are defined as follows

$$R(z, x_0) = \frac{2(1 - y_0)\sqrt{ByC_0}}{CA'}, \tag{24}$$

$$f(z, x_0) = \frac{1}{\sqrt{y_0 - [(1 - y_0)z + y_0]C_0/C}}\,. \tag{25}$$

All functions in (24) and (25) without the subscript $''0''$ are evaluated at $x = A^{-1}[(1 - y_0)z + y_0]$. The function $R(z, x_0)$ is regular for all values of

z and x_0 but $f(z, x_0)$ diverges when $z \to 0$, *i. e.* as one approaches the photon sphere. The integral (23) is then separated in two parts

$$\phi_f(x_0) = \phi_f^D(x_0) + \phi_f^R(x_0), \tag{26}$$

where

$$\phi_f^D(x_0) = \int_0^1 R(0, x_{ps}) f_0(z, x_0) \tag{27}$$

contains the divergence and

$$\phi_f^R(x_0) = \int_0^1 g(z, x_0) dz \tag{28}$$

is a regular integral in z and x_0. To find the order of divergence of the integrand, we expand the argument of the square root of $f(z, x_0)$ to second order in z and get the function $f_0(z, x_0)$:

$$f_0(z, x_0) = \frac{1}{\sqrt{\alpha z + \beta z^2}}\,, \tag{29}$$

where:

$$\alpha = \frac{(1-y_0)}{A_0' C_0}\,(A_0 C_0' - A_0' C_0), \tag{30}$$

$$\beta = \frac{(1-y_0)^2}{2C_0^2 A_0'^3}\,[\,2C_0 C_0' A_0'^2 + (C_0 C_0'' - 2C_0'^2) y_0 A_0' - C_0 C_0' y_0 A''\,]. \tag{31}$$

The function $g(z, x_0)$ is the difference between the original integrand and the divergent integrand

$$g(z, x_0) = R(z, x_0) f(z, x_0) - R(0, x_{ps}) f_0(z, x_0). \tag{32}$$

When x_0 becomes equal to x_{ps} the equation of photon sphere holds. Then, the coefficient α vanishes and the leading term of the divergence in f_0 is z^{-1}. Therefore the integral diverges logarithmically. The coefficient β takes the form

$$\beta_{ps} = \frac{(1-A_{ps})^2}{2C_{ps}{A'_{ps}}^2}\,(A_{ps} C''_{ps} - A''_{ps} C_{ps}). \tag{33}$$

Close to the divergence, Bozza [15] obtained the analytical expression of the deflection angle expanding the two parts of the original integral (19) around $x_0 = x_{ps}$ and approximating the leading terms. The result is

$$\alpha(\theta) = -\bar{a} \ln\left(\frac{\theta D_{OL}}{u_{ps}} - 1\right) + \bar{b} + O(u - u_{ps}), \tag{34}$$

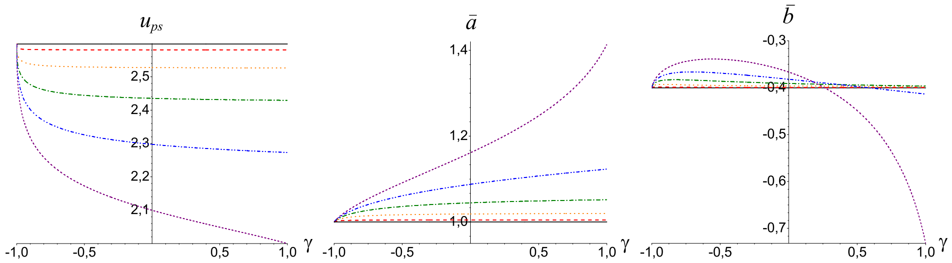

Figure 3: *Strong deflection limit coefficients as a function of metric parameter γ for charge $Q/M = 0$ (solid line) to charge $Q/M = 1$ (short, dashed line) by charge $Q/M = 0.2$*

where the coefficients $\bar{a}$, $\bar{b}$ and u_{ps} depend on the metric function evaluated numerically at x_{ps}. $\theta = u/D_{OL}$ is the angular separation between the lens and the image. The strong deflection limit coefficients of the expansion (34) are

$$u_{ps} = x_{ps}^{1+\gamma}\sqrt{\frac{(x_{ps}-x_-)^{1-2\gamma}}{(x_{ps}-x_+)}}, \tag{35}$$

$$\begin{aligned}\bar{a} &= \frac{R(0,x_{ps})}{2\sqrt{\beta_{ps}}} \\ &= \frac{x_{ps}(x_{ps}-x_-)^{1/2}(x_{ps}-x_+)[-2x_{ps}+(\gamma+1)x_-]}{[x_{ps}(x_+ +\gamma x_-)-(\gamma+1)x_-x_+]}[x_{ps}^3 - x_-x_{ps}^2 \\ &\quad -(x_-^2\gamma+(\gamma+1)x_-x_+)x_{ps}+(\gamma+1)x_-^2x_{ps}]^{-1/2},\end{aligned} \tag{36}$$

$$\begin{aligned}\bar{b} &= -\pi + b_R + \bar{a}\ln\left\{\frac{2\beta_{ps}}{y_{ps}}\right\}, \\ \frac{2\beta_{ps}}{y_{ps}} &= \frac{2[(x_{ps}-x_-)^{\gamma}(x_{ps}-x_+)-x_{ps}^{\gamma+1}]^2}{(x_{ps}-x_-)^{2\gamma}(x_{ps}-x_+)^3[2x_{ps}-(\gamma+1)x_-]^2}[x_{ps}^3 - x_-x_{ps}^2 \\ &\quad -(x_-^2\gamma+(\gamma+1)x_-x_+)x_{ps}+(\gamma+1)x_-^2x_+],\end{aligned} \tag{37}$$

where b_R is the regular integral $\phi_f^R(x_0)$ evaluated at the point x_{ps} as follows

$$b_R = \int_0^1 g(z,x_{ps})dz. \tag{38}$$

For the same metrics the coefficient b_R cannot be obtained analytically. In such cases it must be evaluated numerically.

Fig. 3 shows the strong deflection limit coefficients as functions of metric parameter γ.

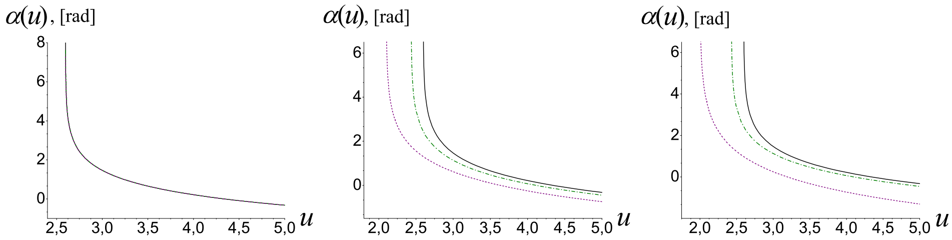

Figure 4: *Deflection angle $\tilde{\alpha}(u)$ as a function of impact parameter u for $\gamma = -1$ (left), $\gamma = 0$ (middle), $\gamma = 1$ (right) and for charge $Q/M = 0$ (solid line), $Q/M = 0.6$ (dashed, dotted line) and $Q/M = 1$ (short, dashed line)*

In Fig. 4 we plot the logarithmic behaviour of the bending angle $\tilde{\alpha}$ against the impact parameter u (scaled in terms of mass M) for several values of γ and black hole charge Q according to the current estimates for the black hole mass [23] point $M_{BH} = 3.6 \times 10^6 M_{\odot}$. In the left figure we plot curves for $\gamma = -1$ and charges $Q/M = 0, 0.6, 1$. It is obvious that these share equality features for the different charges. In the middle and the right figures we plot curves for $\gamma = 0$ (GMGHS black hole) and $\gamma = 1$ (Reissner–Nördstrom black hole) for charges $Q/M = 0, 0.6, 1$. All values of Q/M leads to quite similar behaviour of the deflection angle. For all plotted values of γ the bending angle strictly increases with a decrease in the impact parameter and becomes unboundedly large as the impact parameter approaches the impact parameter for their respective photon spheres (i. e $u \to u_{ps}$).

4. Conclusion

In this work we discuss the features of light propagation on the equatorial plane of the static and spherically symmetric black hole. Modelling the massive compact object at the center of the galaxy as a Garfinkle-Horovitz-Strominger dilaton black hole, we estimate the numerical values of the strong deflection limit coefficients and of the deflection angle of the light ray. When comparing this results to the corresponding quantities due to the GMGHS and Reissner–Nördstrom black holes, it is found that there is a dilaton field effect present in the behaviour of the bending angle of the light ray.

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