Galilean superalgebra and supersymmetric field theory in 2+1 space

A. V. Galkin
Taras Schevchenko Kiev National University,
Kiev, UKRAINE

1. Introduction

The Galilei group is the spacetime symmetry of non-relativistic systems as the Poincare group is the symmetry of relativistic systems. We can consider Galilei algebra in $2+1$ Minkowski space as subalgebra of the Poincare algebra in $3+1$ Minkowski space or as Inonu-Wigner contraction of Poincare algebra in $2+1$ Minkowski space [1].

The non-relativistic field theory is of vital importance in modern theoretical physics. First, most of non-relativistic theories are the non-relativistic limit of corresponding relativistic theories. Using non-relativistic theories we can study the generic field-theories concepts such as, for example, renormalization group in simpler way [3]. Second, Galilean field theories are second-quantized descriptions of quantum mechanics and they are rather suitable for description of many body problems in non-relativistic system, for example, Aharonov-Bohm scattering problem. Finally, Galilean field theories are also used in the relativistic superstring theory [2].

Supersymmetric generalization of the Galilean algebra were first proposed by Puzalowski in $3+1$ Minkowski space [4]. Puzalowski constructed representations of Galilean superalgebra with a single supercharge Q which can be understood as non-relativistic limit of $N=1$ Poincare superalgebra.

In the present paper we consider representations of Galilean superalgebra and Galilean parasuperalgebra in 2+1 Minkowski space in terms of grassmannian variables and physical models invariant under the Galilean superalgebra.

2. Galilei superalgebra $N=2$ in $2+1$ space

The Galilei superalgebra in $2+1$ space has seven operators of the Galilei algebra AG(1,2) P_0, P_i, K_i, J and M which satisfy the following commutation relations

$$[P_i, K_j] = i\delta_{ij}M, \quad [P_i, J] = -i\varepsilon_{ij}P_j, \quad [P_0, K_i] = iP_i, \quad [K_i, J] = -\varepsilon_{ij}K_j,$$

$$\varepsilon_{12} = -\varepsilon_{21} = 1, \qquad \varepsilon_{11} = \varepsilon_{22} = 0 \tag{1}$$

and two supercharges Q and R

$$
\begin{aligned}
&\{Q, Q^{\dagger}\} = M, \quad \{R, R^{\dagger}\} = 2P_0, \\
&[Q, J] = \frac{Q}{2}, \quad [R, J] = -\frac{R}{2}, \\
&[R, K^{-}] = iQ, \quad \{R, Q^{\dagger}\} = -P^{+},
\end{aligned}
\tag{2}
$$

where $K^{-} = K_1 - iK_2$ and $P^{+} = P_1 + iP_2$. The rest commutation relations between the generators of the Galilean superalgebra are equal to zero.

The Galilei superalgebra $N = 2$ in $2+1$ space is subalgebra of the Poincare superalgebra N=1 in 3+1 Minkowski space. It can be verified using (2.2) and the following relations

$$
\begin{aligned}
&M = P_0 + P_3, && P_0 = \tfrac{1}{2}\,(P_0 - P_3), \\
&K_i = J_{0i} + J_{3i}, && J = J_{12}, \\
&Q = \frac{1}{\sqrt{2}}\,Q_2, && R = \tfrac{1}{\sqrt{2}}\,Q_1\,.
\end{aligned}
\tag{3}
$$

The Galilei superalgebra in $2+1$ can be also found from $N = 2$ Poincare superalgebra with nontrivial central charge in $2+1$ by the contraction. One can choose new basis

$$
\begin{aligned}
&\tilde{P}_0 = -\frac{1}{\epsilon^2}\,M + \frac{1}{\epsilon}\,P_0, \\
&\tilde{P}_1 = P_1, \qquad \tilde{P}_2 = P_2, \\
&\tilde{J} = J_{12}, \qquad \tilde{G}_1 = \epsilon J_{01}, \qquad \tilde{G}_2 = \epsilon J_{02}, \\
&\tilde{Q}_1^1 = \frac{\sqrt{\epsilon}}{\sqrt{2}}\,(Q_1^1 + Q_2^{\dagger 2}), \qquad \tilde{Q}_1^2 = \frac{1}{\sqrt{2\epsilon}}\,(Q_1^1 - Q_2^{\dagger 2}), \\
&\tilde{Q}_2^1 = \frac{1}{\sqrt{2\epsilon}}\,(Q_1^2 + Q_2^{\dagger 1}), \qquad \tilde{Q}_2^2 = \frac{\sqrt{\epsilon}}{\sqrt{2}}\,(Q_1^2 - Q_2^{\dagger 1}), \\
&Z = -\frac{2}{\epsilon^2}\,M
\end{aligned}
\tag{4}
$$

(we suppose that M commutes with any generators of Poincare superalgebra). In the commutation and anticommutation relations of Poincare superalgebra in $2+1$ we put $\epsilon \longrightarrow 0$ and then denoting

$$
R = 2(\tilde{Q}_2^2 + \tilde{Q}_1^{\dagger 2}), \quad Q = \tilde{Q}_2^1 + \tilde{Q}_1^{\dagger 1}
$$

we come to (1) and (2).

3. Representations of the Galilei superalgebra and parasuperalgebra $N = 2$ in $2 + 1$ and invariant physical models

Representations of the Galilei superalgebra in 2+1 in terms of grassmannian variables can be found using the representation of the Poincare superalgebra in $3 + 1$ in terms of grassmannian variables. The generators (1) and (2) have the form

$$P_0 = i\frac{\partial}{\partial t}, \quad P_i = i\frac{\partial}{\partial x_i}, \quad i = 1, 2, \tag{5}$$

$$\begin{aligned} K_1 &= tP_1 - x_1 M + i\theta_1\frac{\partial}{\partial\theta_2} + i\theta_1^\dagger\frac{\partial}{\partial\theta_2^\dagger}, \quad K_2 = tP_2 - x_2 M + \theta_1\frac{\partial}{\partial\theta_2} - \theta_2^\dagger\frac{\partial}{\partial\theta_1^\dagger}, \\ J &= x_1 P_2 - x_2 P_1 + \frac{1}{2}(\theta_1\frac{\partial}{\partial\theta_1} - \theta_2\frac{\partial}{\partial\theta_2} - \theta_1^\dagger\frac{\partial}{\partial\theta_1^\dagger} + \theta_2^\dagger\frac{\partial}{\partial\theta_2^\dagger}) \end{aligned} \tag{6}$$

and supercharges Q, R have the following form

$$\begin{aligned} Q &= \frac{\partial}{\partial\theta_1} - 2i\theta_1^\dagger P_0 + i\theta_2^\dagger(P_1 + iP_2), \\ Q^\dagger &= -\frac{\partial}{\partial\theta_1^\dagger} - 2i\theta_1 P_0 - i\theta_2(P_1 - iP_2), \\ R &= \frac{\partial}{\partial\theta_2} - i\theta_1^\dagger(P_1 - iP_2) - i\theta_2^\dagger M, \\ R^\dagger &= -\frac{\partial}{\partial\theta_2^\dagger} + i\theta_1(P_1 + iP_2) + i\theta_2 M\,. \end{aligned} \tag{7}$$

We can also define the covariant derivatives, which are used to build the physical models invariant under the Galilean superalgebra as

$$\begin{aligned} D_1 &= \frac{1}{\sqrt{2}}(\frac{\partial}{\partial\theta_1} - 2\theta_1^\dagger P_0 + \theta_2^\dagger(P_1 + iP_2)), \\ D_1^\dagger &= \frac{1}{\sqrt{2}}(-\frac{\partial}{\partial\theta_1^\dagger} + 2\theta_1 P_0 - \theta_2(P_1 - iP_2)), \\ D_2 &= \frac{1}{\sqrt{2}}(\frac{\partial}{\partial\theta_2} + \theta_1^\dagger(P_1 - iP_2) - \theta_2^\dagger M), \\ D_2^\dagger &= \frac{1}{\sqrt{2}}(\frac{\partial}{\partial\theta_2^\dagger} - \theta_1(P_1 + iP_2) + \theta_2 M)\,. \end{aligned} \tag{8}$$

Covariant derivatives (8) satisfy the following anticommutation relations

$$\begin{aligned} &\{D_1, R\} = \{D_1, Q\} = \{D_2, R\} = \{D_2, Q\} = 0, \\ &\{D_1^\dagger, R\} = \{D_1^\dagger, Q\} = \{D_2^\dagger, R\} = \{D_2^\dagger, Q\} = 0, \\ &\{D_1, R^\dagger\} = \{D_1, Q^\dagger\} = \{D_2, R^\dagger\} = \{D_2, Q^\dagger\} = 0, \\ &\{D_1^\dagger, R^\dagger\} = \{D_1^\dagger, Q^\dagger\} = \{D_2^\dagger, R^\dagger\} = \{D_2^\dagger, Q^\dagger\} = 0\,. \end{aligned} \tag{9}$$

4. Galilei superalgebra in 3+1 dimensions.

The minimal nontrivial generalization of $N = 2$ Galilei superalgebra in 2+1 is $N = 4$ superalgebra in $3 + 1$ dimensions with commutation relations

$$
\begin{gathered}
\{\bar{Q}_1, Q_1\} = 2P_0, \ \ \{\bar{Q}_1, Q_2\} = 0, \ \ \{\bar{Q}_1, Q_3\} = P_3, \ \ \{\bar{Q}_1, Q_4\} = P_1 - iP_2, \\
\{\bar{Q}_2, Q_1\} = 0, \ \ \{\bar{Q}_2, Q_2\} = 2P_0, \ \ \{\bar{Q}_2, Q_3\} = P_1 + iP_2, \ \ \{\bar{Q}_2, Q_4\} = -P_3, \\
\{\bar{Q}_3, Q_1\} = P_3, \ \ \{\bar{Q}_3, Q_2\} = P_1 - iP_2, \ \ \{\bar{Q}_3, Q_3\} = 2M, \ \ \{\bar{Q}_3, Q_4\} = 0, \\
\{\bar{Q}_4, Q_1\} = P_1 + iP_2, \ \ \{\bar{Q}_4, Q_2\} = -P_3, \ \ \{\bar{Q}_4, Q_3\} = 0, \ \ \{\bar{Q}_4, Q_4\} = 2M, \\
\begin{array}{lll}
[Q_1, J_1] = -\frac{1}{2}\, Q_2, & [Q_1, J_2] = -\frac{i}{2}\, Q_2, & [Q_1, J_3] = -\frac{1}{2}\, Q_1, \\
[Q_2, J_1] = -\frac{1}{2}\, Q_1, & [Q_2, J_2] = \frac{i}{2}\, Q_2, & [Q_2, J_3] = \frac{1}{2}\, Q_2, \\
[Q_3, J_1] = -\frac{1}{2}\, Q_4, & [Q_3, J_2] = -\frac{i}{2}\, Q_4, & [Q_3, J_3] = -\frac{1}{2}\, Q_3, \\
[Q_4, J_1] = -\frac{1}{2}\, Q_3, & [Q_4, J_2] = \frac{i}{2}\, Q_3, & [Q_4, J_3] = \frac{1}{2}\, Q_4, \\
[Q_1, G_1] = iQ_4, & [Q_1, G_2] = -Q_4, & [Q_1, G_3] = iQ_3, \\
[Q_2, G_1] = iQ_3, & [Q_2, G_2] = -Q_3, & [Q_2, G_3] = -iQ_4.
\end{array}
\end{gathered}
\tag{10}
$$

In terms of grassmannian variables the supercharges Q_i $(i = 1, 2, 3, 4)$ can be written as

$$
\begin{aligned}
Q_1 &= -i\frac{\partial}{\partial\theta_1} + \bar{\theta}_1\frac{\partial}{\partial t} + \bar{\theta}_2\frac{\partial}{\partial x} + i\bar{\theta}_2\frac{\partial}{\partial y} + \bar{\theta}_1\frac{\partial}{\partial z}\,, \\
Q_2 &= -i\frac{\partial}{\partial\theta_2} + \bar{\theta}_2\frac{\partial}{\partial t} + \bar{\theta}_1\frac{\partial}{\partial x} + i\bar{\theta}_1\frac{\partial}{\partial y} + \bar{\theta}_2\frac{\partial}{\partial z}\,, \\
Q_3 &= -i\frac{\partial}{\partial\theta_3} - i\bar{\theta}_3 M + \bar{\theta}_2\frac{\partial}{\partial x} - i\bar{\theta}_2\frac{\partial}{\partial y} + \bar{\theta}_1\frac{\partial}{\partial z}\,, \\
Q_4 &= -i\frac{\partial}{\partial\theta_4} - i\bar{\theta}_4 M + \bar{\theta}_1\frac{\partial}{\partial x} + i\bar{\theta}_1\frac{\partial}{\partial y} - \bar{\theta}_2\frac{\partial}{\partial z}\,, \\
\bar{Q}_1 &= i\frac{\partial}{\partial\bar{\theta}_1} - \theta_1\frac{\partial}{\partial t} + \theta_2\frac{\partial}{\partial x} + i\theta_2\frac{\partial}{\partial y} + \theta_1\frac{\partial}{\partial z}\,, \\
\bar{Q}_2 &= i\frac{\partial}{\partial\bar{\theta}_2} - \theta_2\frac{\partial}{\partial t} + \theta_1\frac{\partial}{\partial x} + i\theta_1\frac{\partial}{\partial y} + \theta_2\frac{\partial}{\partial z}\,, \\
\bar{Q}_3 &= i\frac{\partial}{\partial\bar{\theta}_3} + \theta_3 M - \theta_2\frac{\partial}{\partial x} - i\theta_2\frac{\partial}{\partial y} - \theta_1\frac{\partial}{\partial z}\,, \\
\bar{Q}_4 &= i\frac{\partial}{\partial\bar{\theta}_4} + i\theta_4 M - \theta_1\frac{\partial}{\partial x} + i\theta_2\frac{\partial}{\partial y} + \theta_2\frac{\partial}{\partial z}\,.
\end{aligned}
\tag{11}
$$

The covariant derivatives take the following from

$$
\begin{aligned}
D_1 &= -i\frac{\partial}{\partial\theta_1} + i\bar{\theta}_1\frac{\partial}{\partial t} + i\bar{\theta}_2\frac{\partial}{\partial x} - \bar{\theta}_2\frac{\partial}{\partial y} + i\bar{\theta}_1\frac{\partial}{\partial z}\,,\\
D_2 &= -i\frac{\partial}{\partial\theta_2} + i\bar{\theta}_2\frac{\partial}{\partial t} + i\bar{\theta}_1\frac{\partial}{\partial x} - \bar{\theta}_1\frac{\partial}{\partial y} + i\bar{\theta}_2\frac{\partial}{\partial z}\,,\\
D_3 &= -i\frac{\partial}{\partial\theta_3} + \bar{\theta}_3 M + i\bar{\theta}_2\frac{\partial}{\partial x} + \bar{\theta}_2\frac{\partial}{\partial y} + i\bar{\theta}_1\frac{\partial}{\partial z}\,,\\
D_4 &= -i\frac{\partial}{\partial\theta_4} + \bar{\theta}_4 M + i\bar{\theta}_1\frac{\partial}{\partial x} - \bar{\theta}_1\frac{\partial}{\partial y} - i\bar{\theta}_2\frac{\partial}{\partial z}\,,\\
\bar{D}_1 &= i\frac{\partial}{\partial\theta_1} - i\theta_1\frac{\partial}{\partial t} + i\theta_2\frac{\partial}{\partial x} - \theta_2\frac{\partial}{\partial y} + i\theta_1\frac{\partial}{\partial z}\,,\\
\bar{D}_2 &= i\frac{\partial}{\partial\theta_2} - i\theta_2\frac{\partial}{\partial t} + i\theta_1\frac{\partial}{\partial x} - \theta_1\frac{\partial}{\partial y} + i\theta_2\frac{\partial}{\partial z}\,,\\
\bar{D}_3 &= i\frac{\partial}{\partial\theta_3} + i\theta_3 M - i\theta_2\frac{\partial}{\partial x} + \theta_2\frac{\partial}{\partial y} - i\theta_1\frac{\partial}{\partial z}\,,\\
\bar{D}_4 &= i\frac{\partial}{\partial\theta_4} - \theta_4 M - i\theta_1\frac{\partial}{\partial x} - \theta_2\frac{\partial}{\partial y} + i\theta_2\frac{\partial}{\partial z}\,.
\end{aligned}
\tag{12}
$$

Below we consider the physical models invariant under the Galilei superalgebra.

5. Physical models invariant under the Galilean superalgebra.

To build the physical models in $2+1$ we introduce scalar superfield Φ, which has 16 component, in the form

$$
\begin{aligned}
\Phi \;=\; & A(x) + \theta_\alpha\psi_\alpha + \theta^\dagger_\alpha\phi_\alpha + \theta_1\theta_2 B(x) + \theta_\alpha\theta^\dagger_\beta\Lambda_{\alpha\beta} + \theta^\dagger_1\theta^\dagger_2 C(x)\\
& + \theta_1\theta_2\theta^\dagger_\alpha\beta_\alpha + \theta_\alpha\theta^\dagger_1\theta^\dagger_2\varepsilon_\alpha + \theta_1\theta_2\theta^\dagger_1\theta^\dagger_2 D(x)\,,
\end{aligned}
\tag{13}
$$

where $x = (x_0, x_1, x_2)$.

Using covariant derivatives (8) we can build operator L which commutes with any generators of the Galilei superalgebra and act in the space of scalar superfields

$$L = D_1 D_2 - a D^\dagger_1 D^\dagger_2 + b D_2 D^\dagger_2 + cM.$$

So we have following equation invariant under the Galilean superalgebra in $2+1$

$$L\Phi = g_1\Phi\Phi\Phi^\dagger, \tag{14}$$

where a, b, c and g_1 are constants.

Using covariant derivatives (12), we can build the operator L for the case of the Galilei superalgebra in $3+1$ dimensions

$$L = (D_1D_4 - D_2D_4) + a(\bar{D}_1\bar{D}_4 - \bar{D}_2\bar{D}_4) + bD_3D_4 + c\bar{D}_3\bar{D}_4 + fM\,, \tag{15}$$

where a, b, c and f are constant. The invariant equation (14) does not change the form in this case. The scalar superfield in $3+1$ dimensions $\Phi = \Phi(t, x, y, z, \theta_1, \theta_2, \theta_3, \theta_4, \bar{\theta}_1, \bar{\theta}_2, \bar{\theta}_3, \bar{\theta}_4)$ has 64 components.

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