# Deformed Gauge Theories*

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Abstract

In this lecture we discuss two possible ways of introducing gauge theories on noncommutative spaces. In the first approach the algebra of gauge transformations is unchanged, but the Leibniz rule changes (compared with gauge theories on commutative space). Consistency of equations of motion forces gauge field to be enveloping algebra-valued, which leads to new degrees of freedom. In the second approach we have to go to the enveloping algebra again if we want noncommutative gauge transformations to close in the algebra. However, no new degrees of freedom appear here because of the Seiberg-Witten map. This map enables one to express noncommutative gauge parameter and fields in terms of the corresponding commutative variables.

## 1. Introduction

The concept of symmetry is very important in physics. Classically, symmetries are described by Lie groups or Lie algebras and the physical space is the representation space of the symmetry algebra. Also, fundamental interactions (electromagnetic, strong,...) are described in terms of theories which are invariant under local transformations. The question arises if one can generalize this notions to noncommutative (deformed) spaces as well.

The concept of a noncommutative (NC) space first appeared[1] when people were discussing divergences in quantum field theory [3]. Our motivation for working with NC spaces comes from the fact that we do not know much about physics at very small scales (or at very high energies). Therefore, we can assume that in this situation coordinates do not commute anymore and we no longer have continuous space-time. Then we try to see what new physics this assumption might lead to. More about "nowadays" motivation readers can find in various review articles [4].

In this lecture we introduce concept of gauge theories on NC spaces. We work with the simples example of NC spaces, the canonically deformed

* The talk given by the author is based on common work with Paolo Aschieri, Christian Blohmann, Frank Meyer, Stefan Schraml, Peter Schupp and Julius Wess [1].

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1 For a detailed review of history of noncommutative spaces see [2].

space. In Section 2 some basic properties of this space are described. In Sections 3 and 4 we present two different ways to introduce gauge theory on the canonically deformed space and we try to compare them. In Section 3 twisted gauge theory is discussed, while Section 4 deals with Seiberg-Witten gauge theory.

## 2. Deformed spaces

The noncommutative space $\hat{\mathcal{A}}_{\hat{x}}$ is the associative algebra, freely generated by $\hat{x}^\mu$ coordinates and divided by the ideal generated by

$$[\hat{x}^\mu, \hat{x}^\nu] = i\Theta^{\mu\nu}(\hat{x}), \qquad \mu = 0, \dots n, \tag{1}$$

where $\Theta^{\mu\nu}(\hat{x})$ is an arbitrary polynomial of coordinates [5].

The elements of this space are all possible polynomials in the coordinates $\hat{x}^\mu$. Before proceeding further, we clarify the notation we use. Coordinates $\hat{x}^\mu$ generate the abstract algebra $\hat{\mathcal{A}}_{\hat{x}}$, while the operators $\hat{\partial}_\rho$, $\hat{L}_{\alpha\beta}$, ... are maps of the abstract algebra $\hat{\mathcal{A}}_{\hat{x}}$ into itself. Variables without the hat symbol, like $x^\mu$, $\partial_\rho$, ... are usual commutative variables. Sometimes we use $\mathcal{A}_x$ to denote the space of commuting coordinates. The defining relation of the deformed space (1) is very general and one usually considers some special examples of it. Among them there are three very important ones

$$\begin{aligned} \text{Canonically deformed} \quad [\hat{x}^\mu, \hat{x}^\nu] &= i\theta^{\mu\nu}, & (2)\\ \text{Lie algebra deformed} \quad [\hat{x}^\mu, \hat{x}^\nu] &= iC^{\mu\nu}_\lambda \hat{x}^\lambda, & (3)\\ q\text{-deformed} \qquad \hat{x}^\mu \hat{x}^\nu &= \frac{1}{q} R^{\mu\nu}{}_{\rho\sigma} \hat{x}^\rho \hat{x}^\sigma. & (4) \end{aligned}$$

In the case of canonically deformed spaces (which from now on we call $\theta$-deformed spaces) [6], $\theta^{\mu\nu} = -\theta^{\nu\mu}$ is an antisymmetric constant matrix of mass dimension $-2$. For Lie algebra deformed spaces [7] $C^{\mu\nu}_\lambda$ are Lie algebra structure constants of mass dimension $-1$. And finally, $R^{\mu\nu}{}_{\rho\sigma}$ is the dimensionless $R$-matrix of the quantum space [8]. This three examples fulfil the Poincaré-Birkoff-Witt (PBW) property.

The deformation quantization allows us to to describe the properties of a noncommutative space in a perturbative way, order by order in the deformation parameter. In the zeroth order the commutative space-time is obtained. The main idea of the deformation quantization is to represent a noncommutative space on the space of commuting coordinates. PBW property enables us to map an arbitrary element $\hat{f}(\hat{x})$ of $\hat{\mathcal{A}}_{\hat{x}}$ to the space of commuting coordinates. To extend this vector space isomorphism to an algebra morphism one has to map the multiplication in the abstract algebra $\hat{\mathcal{A}}_{\hat{x}}$ to the space of commuting coordinates

$$\hat{f} \cdot \hat{g}(\hat{x}) \mapsto f \star g(x) \in \mathcal{A}_x. \tag{5}$$

The new product in $\mathcal{A}_x$ is noncommutative and we call it star product ($\star$-product) of two functions. The algebra of noncommuting coordinates $\hat{\mathcal{A}}_{\hat{x}}$ is then isomorphic to the algebra of commuting variables with the $\star$-product as multiplication. In the following we consider one special example of noncommutative spaces, the $\theta$-deformed space.

### 2.1. $\theta$-deformed space

The abstract derivative $\hat{\partial}_\mu$ that is consistent with (2) fulfils

$$[\hat{\partial}_\rho, \hat{x}^\mu] = \delta^\mu_\rho. \tag{6}$$

One can check this consistency by applying the derivative to (2). That is, using (6) one calculates

$$\hat{\partial}_\rho([\hat{x}^\mu, \hat{x}^\nu] - i\theta^{\mu\nu}) = \ldots = ([\hat{x}^\mu, \hat{x}^\nu] - i\theta^{\mu\nu})\hat{\partial}_\rho = 0.$$

Since no new commutation relations on coordinates arise we conclude that $\hat{\partial}_\rho$ is a good derivative.

In the next step we represent this abstract space on the space of commuting coordinates. The $\star$-product is given by the Moyal-Weyl $\star$-product [9]

$$\begin{aligned} f \star g\,(x) \;&=\; \lim_{x\to y} e^{\frac{i}{2}\theta^{\rho\sigma}\frac{\partial}{\partial x^\rho}\frac{\partial}{\partial y^\sigma}} f(x)g(y) \\ &= \sum_{n=1}^{\infty}\Big(\frac{i}{2}\Big)^n \frac{1}{n!}\,\theta^{\rho_1\sigma_1}\ldots\theta^{\rho_n\sigma_n}\Big(\partial_{\rho_1}\ldots\partial_{\rho_n} f(x)\Big)\Big(\partial_{\sigma_1}\ldots\partial_{\sigma_n} g(x)\Big). \end{aligned} \tag{7}$$

It is convenient to introduce here the inverse of the $\star$-product. Namely, the usual product of two functions $f$ and $g$ can be written in terms of the higher order differential operator $X^\star_f$ which is $\star$-multiplied with the function $g$

$$f\cdot g\,(x) \;=\; f\star g - \frac{i}{2}\,\theta^{\rho\sigma}(\partial_\rho f)\star(\partial_\sigma g) + \ldots . = X^\star_f \star g. \tag{8}$$

The differential operator $X_f$ is given by

$$X^\star_f \;=\; \sum_{n=0}^{\infty}\Big(-\frac{i}{2}\Big)^n\frac{1}{n!}\,\theta^{\rho_1\sigma_1}\ldots\theta^{\rho_n\sigma_n}\Big(\partial_{\rho_1}\ldots\partial_{\rho_n} f(x)\Big)\star\partial_{\sigma_1}\ldots\partial_{\sigma_n}. \tag{9}$$

In order to represent the derivative $\hat{\partial}_\rho$ we use the following scheme

$$\begin{array}{rcl} \hat{f}(\hat{x}) & \longmapsto & f(x) \\ \hat{\partial}\downarrow & & \downarrow \partial^\star\star \\ (\hat{\partial}\hat{f})(\hat{x}) & \longmapsto & (\partial^\star\star f)(x). \end{array} \tag{10}$$

Then the representation of $\hat{\partial}_\rho$ on the space of commuting coordinates is given by the usual partial derivative

$$\hat{\partial}_\rho \mapsto \partial^\star_\rho = \partial_\rho \tag{11}$$

and it has the undeformed Leibniz rule

$$\begin{aligned}(\partial^\star_\rho \star (f \star g)) = \partial_\rho (f \star g) &= (\partial^\star_\rho \star f) \star g + f \star (\partial^\star_\rho \star g) \\ &= (\partial_\rho f) \star g + f \star (\partial_\rho g).\end{aligned} \tag{12}$$

Finally, to be able to write down actions we have to introduce an integral on this space. A possible definition is

$$\int f = \int \mathrm{d}^4 x\ f. \tag{13}$$

This integral is cyclic

$$\int \mathrm{d}^4 x\ f \star g = \int \mathrm{d}^4 x\ g \star f = \int \mathrm{d}^4 x\ f \cdot g. \tag{14}$$

From (14) it follows

$$\int \mathrm{d}^{n+1} x\ (f_1 \star f_2 \star \ldots \star f_k) = \int \mathrm{d}^{n+1} x\ (f_k \star f_1 \star \ldots \star f_{k-1}), \tag{15}$$

that is cyclic permutations under the integral are allowed. This is important if we want to define the variational principle. Namely, we can always bring the function to be varied to one side of the product under the integral and then vary it

$$\frac{\delta}{\delta g(x)} \int \mathrm{d}^{n+1} x\ f \star g \star h = \frac{\delta}{\delta g(x)} \int \mathrm{d}^{n+1} x\ g(h \star f) = h \star f. \tag{16}$$

## 3. Twisted gauge theory

Before we start with NC gauge theories [2] we review shortly the gauge theory on commutative space, which we call the undeformed gauge theory. A NC gauge theory we also call deformed gauge theory.

Gauge theories are based on a gauge group. This is a compact Lie group with generators $T^a$

$$[T^a, T^b] = i f^{abc} T^c. \tag{17}$$

Infinitesimal transformation of the matter field $\psi$ is given by

$$\delta_\alpha \psi(x) = i\alpha(x)\psi(x), \tag{18}$$

[2] Note that NC gauge theory stands for gauge theory on NC spaces and has nothing to do with the choice of the gauge group.

where $\alpha(x) = \alpha^a(x)T^a$ is a Lie algebra-valued gauge parameter. Transformations (18) close in the algebra

$$\delta_\alpha\delta_\beta - \delta_\beta\delta_\alpha = \delta_{-i[\alpha,\beta]}. \tag{19}$$

The Leibniz rule is given by

$$\delta_\alpha(\phi\cdot\psi) = (\delta_\alpha\phi)\cdot\psi + \phi\cdot(\delta_\alpha\psi). \tag{20}$$

### 3.1. Gauge transformations

The deformed gauge transformations [1], [10] are defined as follows:

$$\delta^\star_\alpha\psi = iX^\star_\alpha \star\psi = iX^\star_{\alpha^a}\star T^a\psi = i\alpha\cdot\psi, \tag{21}$$

where $X^\star$ operator was introduced in (9). From the fact that $X^\star_f\star X^\star_g = X^\star_{f\cdot g}$ we conclude that this transformations close in the algebra (19)

$$[\delta^\star_\alpha,\delta^\star_\beta]\psi = -i\delta^\star_{[\alpha,\beta]}\psi. \tag{22}$$

Now we have to learn how products of fields transform. We start with defining the transformation law of $\star$-product of two fields like in the commutative case

$$\begin{aligned}\delta^\star_\alpha(\phi\star\psi) &= iX^\star_{\alpha^a}\star\{T^a_\phi\phi\star\psi + \phi\star T^a_\psi\psi\}\\ &= i\alpha^a\cdot\{T^a_\phi\phi\star\psi + \phi\star T^a_\psi\psi\}.\end{aligned} \tag{23}$$

Generators $T^a_\phi$ and $T^a_\psi$ are generators of the gauge group in the appropriate representation and in the last line the definition of $X^\star_f$ given in (9) was used.

If we now take the usual Leibniz rule (20) and just insert $\star$s everywhere we obtain

$$\begin{aligned}\delta^\star_\alpha(\phi\star\psi) &= (\delta^\star_\alpha\phi)\star\psi + \phi\star(\delta^\star_\alpha\psi)\\ &= i(\alpha^aT^a_\phi\phi)\star\psi + i\phi\star(\alpha^aT^a_\psi\psi).\end{aligned} \tag{24}$$

Clearly, RHS of (23) and (24) are not equal because the $\star$-product (7) is not commutative. Therefore we formulate our deformed Leibniz rule as follows

$$\delta^\star_\alpha(\phi\star\psi) = (\delta^\star_\alpha\phi)\star\psi + \phi\star(\delta^\star_\alpha\psi) + \text{ additional terms.} \tag{25}$$

This additional terms are found by comparing RHSs of the demanded transformation law (23) and (24). Expanding the $\star$-products up to first order in the deformation parameter $\theta$ and arranging terms leads to

$$\begin{aligned}\text{a.t.} &= \frac{1}{2}\,\theta^{\rho\sigma}\alpha^a\Big((T^a_\phi\partial_\rho\phi)\cdot(\partial_\sigma\psi) + \partial_\rho\phi\cdot(T^a_\psi\partial_\sigma\psi)\Big) + \mathcal{O}(\theta^2)\\ &= -\frac{i}{2}\,\theta^{\rho\sigma}((\delta^\star_{\partial_\rho\alpha}\phi)\star(\partial_\sigma\psi) + (\partial_\rho\phi)\star(\delta^\star_{\partial_\sigma\alpha}\psi)) + \mathcal{O}(\theta^2),\end{aligned}$$

which gives the following Leibniz rule

$$\delta^\star_\alpha(\phi \star \psi) = (\delta^\star_\alpha\phi) \star \psi + \phi \star (\delta^\star_\alpha\psi) - \frac{i}{2}\,\theta^{\rho\sigma}((\delta^\star_{\partial_\rho\alpha}\phi) \star (\partial_\sigma\psi) + (\partial_\rho\phi) \star (\delta^\star_{\partial_\sigma\alpha}\psi)) + \mathcal{O}(\theta^2). \tag{26}$$

One can continue like this to second and higher orders. The full result for the deformed Leibniz rule[3] is finally

$$\delta^\star_\alpha(\phi \star \psi) = \sum_{n=0}^{\infty} \frac{1}{n!}\Big(-\frac{i}{2}\Big)^n \theta^{\rho_1\sigma_1}\ldots\theta^{\rho_n\sigma_n}\big((\delta_{\partial_{\rho_1}\ldots\partial_{\rho_n}\alpha}\phi) \star (\partial_{\sigma_1}\ldots\partial_{\sigma_n}\psi) + (\partial_{\rho_1}\ldots\partial_{\rho_n}\phi) \star (\delta_{\partial_{\sigma_1}\ldots\partial_{\sigma_n}\alpha}\psi)\Big). \tag{27}$$

Note that the zeroth term in the sum is just

$$(\delta^\star_\alpha\phi) \star \psi + \phi \star (\delta^\star_\alpha\psi). \tag{28}$$

### 3.2. Field strength tensor

Having found the Leibniz rule for $\delta^\star_\alpha$ transformations, we proceed like in the commutative case. First, covariant derivative is introduced as

$$D_\mu\psi = \partial_\mu\psi - iA_\mu \star \psi, \tag{29}$$

$$\delta^\star_\alpha(D_\mu\psi) = iX^\star_{\alpha^a} \star T^a(D_\mu\psi) = i\alpha(D_\mu\psi), \tag{30}$$

where $A_\mu$ is NC gauge field. Using (27) gives

$$\delta^\star_\alpha A_\mu = \partial_\mu\alpha + iX^\star_{\alpha^a} \star [T^a, A_\mu] = \partial_\mu\alpha + i[\alpha, A_\mu]. \tag{31}$$

From this transformation law it follows that $A_\mu$ can be taken to be Lie algebra valued.

However, introducing the field strength tensor $F_{\mu\nu}$ as

$$F_{\mu\nu} \star \psi = i[D_\mu \overset{\star}{,} D_\nu]\psi \tag{32}$$

gives

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu \overset{\star}{,} A_\nu],$$

$$\delta^\star_\alpha F_{\mu\nu} = iX^\star_{\alpha^a} \star [T^a, F_{\mu\nu}] = i[\alpha, F_{\mu\nu}]. \tag{33}$$

If we assume that $A_\mu = A^a_\mu T^a$, that is $A_\mu$ is Lie algebra-valued, for $F_{\mu\nu}$ we obtain

[3] Using the Hopf algebra language, this deformed Leibniz rule comes from the deformed coproduct of $\delta^\star_\alpha$ transformations. Deformed coproduct is obtained by twisting the usual coproduct for the $\delta_\alpha$ transformations [1], [10] and this is why we call this transformations twisted gauge transformations.

$$\begin{aligned} F_{\mu\nu} &= F^a_{1\mu\nu} T^a + F^{ab}_{2\mu\nu} \frac{1}{2}\{T^a, T^b\} = F_{1\mu\nu} + F_{2\mu\nu}, \\ F^a_{1\mu\nu} &= \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + \frac{1}{2} f^{abc}\{A^b_\mu \stackrel{\star}{,} A^c_\nu\}, \\ F^{ab}_{2\mu\nu} &= -i[A^a_\mu \stackrel{\star}{,} A^b_\nu], \end{aligned} \tag{34}$$

so $F_{\mu\nu}$ is enveloping algebra-valued. The good news is that each part transforms covariantly

$$\begin{aligned} \delta^\star_\alpha F_{1\mu\nu} &= iX^\star_{\alpha^a} \star [T^a, F_{1\mu\nu}] = i[\alpha, F_{1\mu\nu}], \\ \delta^\star_\alpha F_{2\mu\nu} &= iX^\star_{\alpha^a} \star [T^a, F_{2\mu\nu}] = i[\alpha, F_{2\mu\nu}]. \end{aligned} \tag{35}$$

If we want to stay in the Lie algebra we could take just $F^a_{\mu\nu}$ part of the full $F_{\mu\nu}$ and formulate the action with it only. Note that $F^a_{\mu\nu}$ also has the good classical limit, that is when $\theta \to 0$ it reduces to commutative $F_{\mu\nu}$. One can include matter fields as well and formulate "gauge + matter" action, calculate equations of motion (EOM), analyze their solutions and so on.

Unfortunately, some problems arise in this procedure. To see clearly what is causing them we now analyze only the gauge part of the action.

### 3.3. Equations of motion

Let us first write the action for the gauge field as

$$S_{\text{gauge}} = -\frac{1}{4}\int \mathrm{d}^4x \; F^a_{1\mu\nu} \star F_1^{\mu\nu\, a}, \tag{36}$$

that is taking only the Lie algebra-valued part of the full $F_{\mu\nu}$. Using the variational principle (16) gives

$$\frac{\delta S_{\text{gauge}}}{\delta A^a_\rho} = \partial_\mu F_1^{\mu\rho\, a} + \frac{1}{2} f^{abc}\{A^b_\mu \stackrel{\star}{,} F_1^{\mu\rho\, c}\} = 0. \tag{37}$$

From the antisymmetry of $F^a_{\mu\nu}$ it follows that $\partial_\rho\partial_\mu F^{\mu\rho^a} = 0$ and this leads to

$$\partial_\rho\Big(\frac{1}{2} f^{abc}\{A^b_\mu \stackrel{\star}{,} F_1^{\mu\rho\, c}\}\Big) = \partial_\rho J^{\rho\, a} = 0. \tag{38}$$

This is nice, it looks like we obtain the conserved current $J^{\rho\, a}$. However, when we check its conservation explicitly using EOM (37) we obtain that it is not conserved! This signalizes that we are doing something wrong. It comes out that the problem arose when we took only the Lie algebra-valued part of $F_{\mu\nu}$ and not the full $F_{\mu\nu}$.

If we now try with the full $F_{\mu\nu}$, we obtain

$$S_{\text{gauge}} = -\frac{1}{4}\int \mathrm{d}^4x \; \mathrm{Tr}(F_{\mu\nu} \star F^{\mu\nu}) \tag{39}$$

and for the equations of motion

$$\frac{\delta S_{\text{gauge}}}{\delta(A_\rho)_{\text{AB}}} = (\partial_\mu F^{\mu\rho})_{\text{AB}} - i([A_\mu \stackrel{\star}{,} F^{\mu\rho}])_{\text{AB}} = 0, \tag{40}$$

where $A$, $B$ are matrix indices. Again, using the antisymmetry of $F_{\mu\nu}$, we obtain

$$\partial_\rho\Big(i[A_\mu \stackrel{\star}{,} F^{\mu\rho}]\Big) = \partial_\rho J^\rho = 0. \tag{41}$$

This result we check using EOM (40) and Jacobi identity. In

$$\partial_\rho[A_\mu \stackrel{\star}{,} F^{\mu\rho}] = [\partial_\rho A_\mu \stackrel{\star}{,} F^{\mu\rho}] + [A_\mu \stackrel{\star}{,} \partial_\rho F^{\mu\rho}] \tag{42}$$

in the first term we replace $\partial_\rho A_\mu$ by $\frac{1}{2}(\partial_\rho A_\mu - \partial_\mu A_\rho)$ because $F_{\mu\rho}$ is antisymmetric in $\mu$ and $\rho$. Then we express this term by $F_{\mu\rho}$ according to (33):

$$\frac{1}{2}\left(\partial_\rho A_\mu - \partial_\mu A_\rho\right) = \frac{i}{2}\, F_{\rho\mu} + \frac{i}{2}\, [A_\rho \stackrel{\star}{,} A_\mu]. \tag{43}$$

The $[F^{\mu\rho} \star F_{\mu\rho}]$ commutator vanishes and we are left with $\frac{i}{2}\,[[A_\rho \stackrel{\star}{,} A_\mu] \stackrel{\star}{,} F^{\mu\rho}]$ for the first term in (42). For the second term in (42) we use EOM (40). Finally, all terms that are left add up to zero if we use the Jacobi identity, so $J^\rho$ is conserved on EOM.

To conclude, the consistency of EOM forces us to go to the enveloping algebra and take $F_{\mu\nu}$ in the form (33). Then there is no reason to assume that $A_\mu$ itself is Lie algebra-valued[4]. Instead, we start with the enveloping algebra-valued gauge field

$$A_\mu = A_\mu^a T^a + A_\mu^{ab} \frac{1}{2}\,\{T^a, T^b\} + \dots . \tag{44}$$

Looking at (44) we see that in this way we are introducing extra degrees of freedom via fields $A_\mu^{ab}, \dots$. There will be infinitely many new fields, leading to problems again. One possible solution is to generate large masses for these fields via some kind of Higgs mechanism and in that way to make them unobservable for todays experiments. Nevertheless, this still remains an open question and needs to be analyzed in future.

Nevertheless, we obtained one important result and that is (41). Almost for free we got conserved current for the deformed symmetry. Note that in the case of deformed symmetry one cannot apply Nöther theorem (at least it is not clear how to do it).

[4] Even if it would be Lie algebra-valued in the beginning, it would become enveloping algebra-valued trough its evolution (40).

## 4. Seiberg-Witten gauge theory

In this section we describe the other approach to gauge theory on deformed spaces. The noncommutative gauge transformations are now defined as [11], [12], [13]

$$\delta_\alpha \psi = i\Lambda_\alpha \star \psi(x), \tag{45}$$

where $\Lambda_\alpha$ is the noncommutative gauge parameter and $\psi$ is the noncommutative matter field. Note that $\Lambda_\alpha$ is a function and not a differential operator like in (21). Before proceeding to the standard construction of a covariant derivative one should check if this transformations close in the algebra (19). Explicit calculation gives

$$\begin{aligned}(\delta_\alpha\delta_\beta\ -\ \delta_\beta\delta_\alpha)\psi(x) &= (\Lambda_\alpha \star \Lambda_\beta - \Lambda_\beta \star \Lambda_\alpha) \star \psi \\ &= \frac{1}{2}\Big([\Lambda^a_\alpha \stackrel{\star}{,} \Lambda^b_\beta]\{T^a, T^b\} + \{\Lambda^a_\alpha \stackrel{\star}{,} \Lambda^b_\beta\}[T^a, T^b]\Big) \star \psi. \end{aligned} \tag{46}$$

If we take $\Lambda_\alpha = \Lambda^a_\alpha T^a$, that is a Lie algebra-valued gauge parameter, algebra will not close because of the first term in the last line of (46) (anticommutator of two generators is no longer in the Lie algebra of generators).

There are two ways of solving this problem. One is to consider only $U(N)$ gauge theories and that one we will not follow here. The other one is to go to the enveloping algebra [11] approach and we continue analyzing this one.

### 4.1. Enveloping Algebra Approach

To start with, we define the basis in the enveloping algebra (we choose symmetric ordering)

$$\begin{aligned} :T^a: &\ =\ T^a, \\ :T^aT^b: &\ =\ \frac{1}{2}\,(T^aT^b + T^bT^a), \\ :T^{a_1}\dots T^{a_l}: &\ =\ \frac{1}{l!}\sum_{\sigma\in S_l}(T^{\sigma(a_1)}\dots T^{\sigma(a_l)}). \end{aligned}$$

Gauge parameter $\Lambda_\alpha$ is said to be enveloping algebra-valued

$$\begin{aligned}\Lambda_\alpha(x) &= \sum_{l=1}^{\infty}\sum_{\text{basis}} \alpha_l^{a_1\dots a_l}(x) : T^{a_1}\dots T^{a_l} \\ &= \alpha^a(x) : T^a : +\ \alpha_2^{a_1a_2}(x) : T^{a_1}T^{a_2} : + \dots . \end{aligned} \tag{47}$$

In that case algebra (46) will close since we work in the enveloping algebra. Now one can proceed and define a covariant derivative $\mathcal{D}_\mu\psi(x) = \partial_\mu\psi(x) - iV_\mu \star \psi(x)$ by its transformation law

$$\delta_\alpha(\mathcal{D}_\mu\psi(x)) = i\Lambda_\alpha \star \mathcal{D}_\mu\psi(x). \tag{48}$$

The noncommutative gauge field $V_\mu$ has to be enveloping algebra-valued as well

$$V_\mu = \sum_{l=1}^{\infty} \sum_{\text{basis}} V^l_{\mu\, a_1 \dots a_l} : T^{a_1} \dots T^{a_l} : .$$

From all this it looks like we again obtain a theory with infinitely many degrees of freedom. Fortunately, in this approach there is a solution to this problem and it is given in terms of the Seiberg-Witten (SW) map [12].

### 4.2. Seiberg-Witten Map

The basic idea of this map is to suppose that the noncommutative gauge parameter (field) can be expressed in terms of the commutative gauge parameter and field, for example $\Lambda_\alpha = \Lambda_\alpha(x; \alpha, A^0_\mu)$. Then one uses (46) to calculate explicitly this dependence. Inserting $\Lambda_\alpha = \Lambda_\alpha(x; \alpha, A^0_\mu)$ in (46) gives[5]

$$(\Lambda_\alpha \star \Lambda_\beta - \Lambda_\beta \star \Lambda_\alpha) \star \psi + i(\delta_\alpha \Lambda_\beta - \delta_\beta \Lambda_\alpha) \star \psi = \delta_{-i[\alpha,\beta]} \psi. \tag{49}$$

The equation (49) has to be solved perturbatively, therefore one has to expand the $\star$-product and also expand $\Lambda_\alpha$ in orders of the deformation parameter $\theta$ as

$$\Lambda_\alpha = \alpha + \Lambda^1_\alpha + \dots + \Lambda^k_\alpha + \dots .$$

Here $\alpha$ is just the gauge parameter of the undeformed (commutative) gauge theory. Then $\Lambda^1_\alpha$ is the solution of the unhomogeneous equation

$$\delta^\star_\alpha \Lambda^1_\beta - \delta^\star_\beta \Lambda^1_\alpha \; - \; i[\alpha, \Lambda^1_\beta] - i[\Lambda^1_\alpha, \beta] - \Lambda^1_{-i[\alpha,\beta]} = -\frac{1}{2}\, \theta^{\mu\nu} \{\partial_\mu \alpha, \partial_\nu \beta\}. \tag{50}$$

The solution up to first order in $\theta$ is given by

$$\Lambda_\alpha = \alpha - \frac{1}{4}\, \theta^{\mu\nu} \{A^0_\mu, \partial_\nu \alpha\} + \mathcal{O}(\theta^2), \tag{51}$$

where $A^0_\mu$ is the gauge field in the undeformed gauge theory. Note that this solution is not unique, one can always add solutions of the homogeneous equation to it [13].

Now one can solve the SW map for the matter field using (45) and (51). Matter field is also expanded in orders of $\theta$ as $\psi = \psi^0 + \psi^1 + \dots$ and $\psi^0$ is the matter field in the undeformed gauge theory. The first order solution is given by

$$\psi = \psi^0 - \frac{1}{2}\, \theta^{\mu\nu} A^0_\mu (\partial_\nu \psi^0) + \frac{i}{8}\, \theta^{\mu\nu} [A^0_\mu, A^0_\nu] \psi^0 + \mathcal{O}(\theta^2). \tag{52}$$

[5] One should notice that now $\delta_\alpha \Lambda_\beta \neq 0$ because $\Lambda_\beta$ depends on the commutative gauge field $A^0_\mu$ as well and $\delta_\alpha A^0_\mu = \partial_\mu \alpha - i[A^0_\mu, \alpha]$.

The same procedure works for the gauge field $V_\mu$ which transforms as follows

$$\delta^\star_\alpha V_\mu = \partial_\mu \Lambda_\alpha + i[\Lambda_\alpha \overset{\star}{,} V_\mu]. \tag{53}$$

The solution up to first order in $\theta$ is

$$V_\rho = A^0_\rho + \frac{1}{4}\,\theta^{\mu\nu}\Big(\{F^0_{\mu\rho}, A^0_\nu\} - \{A^0_\mu, \partial_\nu A^0_\rho\}\Big) + \mathcal{O}(\theta^2). \tag{54}$$

Field strength tensor is calculated from

$$\begin{aligned} F_{\mu\nu} &= \partial_\mu V_\nu - \partial_\nu V_\mu - i[V_\mu \overset{\star}{,} V_\nu] \\ &= F^0_{\mu\nu} + \frac{1}{4}\,\theta^{\rho\sigma}\Big(2\{F^0_{\rho\mu}, F^0_{\sigma\nu}\} + \{\mathcal{D}^0_\rho F^0_{\mu\nu}, A^0_\sigma\} - \{A^0_\rho, \partial_\sigma F^0_{\mu\nu}\}\Big), \end{aligned} \tag{55}$$

where $F^0_{\mu\nu} = \partial_\mu A^0_\nu - \partial_\nu A^0_\mu - i[A^0_\mu, A^0_\nu]$ is the field strength tensor in the undeformed gauge theory and $\mathcal{D}^0_\rho F^0_{\mu\nu} = \partial_\rho F^0_{\mu\nu} - i[A^0_\rho, F^0_{\mu\nu}]$

All this gives enough information to write down the action for the Seiberg-Witten gauge theory. This action can be expanded in orders of the deformation parameter and analysed perturbatively. One sees that no new fields appear (unlike in the "twisted" approach), but only new interactions and the deformation parameter enters as a coupling constant. Also formulation of a NC Standard Model (SM) using this approach has been done in [14]. Then, since new interactions appear, some of the processes that are forbidden in commutative SM become allowed and can be analyzed [15]. Also, renormalization of these theories has been discussed in various ways [16].

To conclude, this approach enables us to analyze some properties of gauge theories on NC spaces in perturbative way, starting from the commutative gauge theory in the zeroth order. Unlike the "twisted" approach, it does not lead to new fields, but only to new interactions. On the other hand, in the "twisted" approach we know how to formulate deformed gauge symmetry in well defined mathematical language, that is in terms of Hopf algebras. This tells us that we have really changed the commutative gauge symmetry in well defined way. This is not the case with the SW approach, where the deformed gauge symmetry finally reduces to the undeformed gauge symmetry. Which of these two approaches (if any) is more applicable to our real world remains an open question and a subject of further research.

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