On the Jacobi field equations*

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Abstract

A noncommutative geometry is studied in the quasi-classical, linear field approximation. Integrability conditions for the associative structure of the algebra, which we refer to as the Jacobi equations, force restrictions on the metric which can be expressed as an effective additional source term for the Einstein tensor. All calculations are carried out in the 'quasi-classical approximation', by which we mean that the noncommutativity is considered as a first-order perturbation of the limiting classical configuration.

1. Introduction and motivation

Consider a smooth manifold M with a moving frame $\tilde{\theta}^\alpha$. Let $\mathcal{A}$ be a noncommutative deformation of the algebra $\mathcal{C}(M)$ of smooth functions on M defined by a symplectic structure J and let θ^α be a noncommutative deformation of the moving frame. The connection is assumed to satisfy both a left and right Leibniz rule, a condition intimately connected with the existence of a reality condition, and the metric to be a bilinear map, a condition connected with locality. The classical limit of the geometry is thus naturally equipped with a linear connection and a metric as well as with a Poisson structure. Under the assumptions which we shall impose the Poisson structure is non-degenerate. We shall be more precise about these extensions below. We wish to show that in the weak-field, quasi-classical approximation they imply that the metric defined by the frame cannot be arbitrary and that the Ricci tensor is fixed by Jacobi identities.

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As a measure of noncommutativity, and to recall the many parallelisms with quantum mechanics, we use the symbol $\bar{k}$, which will designate the square of a real number whose value could lie somewhere between the Planck length and the proton radius m_P^{-1}. This becomes important when we consider perturbations. We introduce a set $J^{\mu\nu}$ of elements of an associative algebra $\mathcal{A}$ defined by commutation relations

$$[x^\mu, x^\nu] = i\bar{k}J^{\mu\nu}(x^\sigma). \tag{1..1}$$

The $J^{\mu\nu}$ are of course restricted by Jacobi identities; we see below that there are two other natural requirements which also restrict them.

Let μ be a typical 'large' source mass with 'Schwarzschild radius' $G_N\mu$. We have two length scales, determined by respectively the square $G_N\hbar$ of the Planck length and by $\bar{k}$. The gravitational field is weak if the dimensionless parameter $\epsilon_{GF} = G_N\hbar\mu^2$ is small; the space-time is almost commutative if the dimensionless parameter $\epsilon = \bar{k}\mu^2$ is small. These two parameters are not necessarily related but we shall here assume that they are of the same order of magnitude.

$$\epsilon_{GF} \simeq \epsilon. \tag{1..2}$$

If noncommutativity is not directly related to gravity then it makes sense to speak of ordinary gravity as the limit $\bar{k} \to 0$ with $G_N\mu$ non vanishing. On the other hand if noncommutativity and gravity are directly related then both should vanish with $\bar{k}$. We wish here to consider an expansion in the parameter ϵ, which we have seen is a measure of the relative dimension of a typical 'space-time cell' compared with the Planck length of a typical quantity of gravitational energy.

We suppose the calculus to be defined by a special set of 1-forms, a frame, which commute with the elements of the algebra and we assume that the derivations dual to the forms are inner. The momenta p_α stand in duality to the position operators x^μ by the relation

$$[p_\alpha, x^\mu] = e^\mu_\alpha. \tag{1..3}$$

The right-hand side of this identity defines the gravitational field. The left-hand side must obey Jacobi identities. These identities yield relations between quantum mechanics in the given curved space-time and the noncommutative structure of the algebra. The three aspects of reality then, the curvature of space-time, quantum mechanics and the noncommutative structure of space-time are intimately connected. We shall consider here the even more exotic possibility that the field equations of general relativity are encoded also in the structure of the algebra so that the relation between general relativity and quantum mechanics can be understood by the relation which each of these theories has with noncommutative geometry.

In spite of the rather lengthy formalism the basic idea is simple. We start with a classical geometry described by a moving frame $\tilde{\theta}^\alpha$ and we quantize it by replacing the moving frame by a frame θ^α, as we shall described in some detail below. The easiest cases would include those frames which could be quantized without ordering problems. Let $\tilde{e}_\alpha$ be the vector fields dual to the frame $\tilde{\theta}^\alpha$ and quantize them as in (1..3) by imposing the rule

$$\tilde{e}_\alpha \mapsto e_\alpha = \mathrm{ad}\, p_\alpha \tag{1..4}$$

Finally, one must construct a noncommutative algebra consistent with the resulting differential calculus; this defines the image of the map

$$\tilde{\theta}^\alpha \longrightarrow J^{\mu\nu} \tag{1..5}$$

More details of this map will be given in Section 2.3.. The algebra we identify with 'position space'. To it we add the extra elements which are necessary in order that the derivations be inner; this is ordinary quantum mechanics. The new element is that the consistency relations in the algebra such as Jacobi identities

$$[p_\alpha, J^{\mu\nu}] = [x^{[\mu}, [p_\alpha, x^{\nu]}]]. \tag{1..6}$$

largely restrict θ^α and $J^{\mu\nu}$.

If the space is flat and the frame is the canonical flat frame then the right-hand side of (1..6) vanishes and it is possible to consistently choose the expression $J^{\mu\nu}$ to be equal to a constant. But on the other hand, if the space is curved the right-hand side does not vanish identically except, of course, in the trivial case $J^{\mu\nu} = 0$. The map (1..5) is not single valued since any constant J has flat space as inverse image. Our motivation for considering noncommutative geometry as an 'avatar' of gravity is the belief that it sheds light on the role [1] of the gravitational field as the universal regulator of ultra-violet divergences.

We resume the various possibilities, starting with a classical metric $\tilde{g}_{\mu\nu}$. The most important flow of information is from the classical metric $\tilde{g}_{\mu\nu}$ to the commutator $J^{\mu\nu}$. The first step is to associate to the metric a moving frame $\tilde{\theta}^\alpha$, which can be written in the form $\tilde{\theta}^\alpha = \tilde{\theta}^\alpha_\mu dx^\mu$. The frame is then 'quantized' according to the ordinary rules of quantum mechanics; the dual derivations $\tilde{e}_\alpha$ are replaced by inner derivations $e_\alpha = \mathrm{ad}\, p_\alpha$ of a noncommutative algebra. The commutation relations are defined by the J, obtained from the θ by solving a differential equation. The calculus is defined by the frame. The basic idea of the present article was anticipated in a more specialized treatment [2] of asymptotically-flat space-times as well as in a string-theoretical reduction [3, 4] of noncommutative geometry to a supplementary 2-form.

The problem we wish to aboard here is that of the existence and definition of an energy-momentum for the Poisson structure and of an eventual contribution of this energy-momentum to the gravitational field equations. In the formulation which we are here considering the Ricci tensor and therefore the Einstein tensor as well are determined as integrability conditions for the underlying associative-algebra structure. The value *in vacuo* of the commutative limit of this tensor we interpret as the energy-momentum of the symplectic structure. In the frame formalism the field equations are most elegantly (and, once one is familiar with it, easily) written as the vanishing of a 3-form. This follows from the identity

$$G_{\alpha\beta} * \theta^\beta = -\frac{1}{2}\,\Omega^{\beta\gamma} * \theta_{\alpha\beta\gamma} \tag{1..7}$$

We write accordingly the energy-momentum of the gravitational field in terms of a vector-valued 3-form τ_S which has the property [5] that it is exact if and only if the Einstein field equations are satisfied. If this be so one sees that the total energy-momentum is given as the integral over the sphere at infinity of the Sparling 2-form

$$\sigma_\alpha = -\frac{1}{2} * \omega_{\alpha\beta}\theta^\beta. \tag{1..8}$$

Since in the noncommutative case there is a preferred frame, the one canonically aligned with respect to the eigenvectors of the conformal tensor, one can claim that the 2-form itself and not only the integral thereof is well-defined. Our assumption is that there is another 3-form

$$\tau_{PS} = \tau_{PS}(J) \tag{1..9}$$

which vanishes in the commutative limit and which is such that the sum

$$\tau = \tau_S + \tau_{PS} \tag{1..10}$$

is an exact 3-form. We cannot give an explicit formula for τ_{PS} in general. We can only express it in certain simple limiting case.

1.1. Differential calculi

Let then $\mathcal{A}$ be a noncommutative $*$-algebra generated by four hermitian elements x^μ which satisfy the commutation relations (1..1). Assume that over $\mathcal{A}$ is a differential calculus which is such [6] that the module of 1-forms is free and possesses a preferred frame θ^α which commutes,

$$[x^\mu, \theta^\alpha] = 0, \tag{1..11}$$

with the algebra. The space one obtains in the commutative limit is therefore parallelizable with a global moving frame $\tilde\theta^\alpha$ defined to be the commutative limit of θ^α. We can write the differential

$$dx^\mu = e^\mu_\alpha\theta^\alpha, \qquad e^\mu_\alpha = e_\alpha x^\mu. \tag{1..12}$$

The algebra is defined by a product which is restricted by the matrix of elements $J^{\mu\nu}$; the metric is defined, we shall see below, by the matrix of elements e^μ_α. The differential calculus is defined as the largest one consistent with the module structure of the 1-forms so constructed.

Consistency requirements, essentially determined by Leibniz rules, impose relations between these two matrices which in simple situations allow us to find a one-to-one correspondence between the structure of the algebra and the metric. The input of which we shall make the most use is the Leibniz rule

$$i\hbar e_\alpha J^{\mu\nu} = [e^\mu_\alpha, x^\nu] - [e^\nu_\alpha, x^\mu]. \qquad (1..13)$$

One can see here a differential equation for $J^{\mu\nu}$ in terms of e^μ_α. In important special cases the equation reduces to a simple differential equation of one variable. The relation (1..13) can be written also as Jacobi identities

$$[p_\alpha, [x^\mu, x^\nu]] + [x^\nu, [p_\alpha, x^\mu]] + [x^\mu, [x^\nu, p_\alpha]] = 0 \qquad (1..14)$$

if one introduce the momenta p_α associated to the derivation by the relation (1..3).

Finally, we must insure that the differential is well defined. A necessary condition is that $d[x^\mu, \theta^\alpha] = 0$ from which it follows that the momenta p_α must satisfy the consistency condition

$$2p_\gamma p_\delta P^{\gamma\delta}_{\alpha\beta} - p_\gamma F^{\gamma}_{\alpha\beta} - K_{\alpha\beta} = 0. \qquad (1..15)$$

The $P^{\gamma\delta}_{\alpha\beta}$ define the product π in the algebra of forms. We write $P^{\alpha\beta}_{\gamma\delta}$ in the form

$$P^{\alpha\beta}_{\gamma\delta} = \frac{1}{2}\, \delta^{[\alpha}_\gamma \delta^{\beta]}_\delta + i\epsilon Q^{\alpha\beta}_{\gamma\delta} \qquad (1..16)$$

of a standard projector plus a perturbation. If further we decompose [7] $Q^{\alpha\beta}_{\gamma\delta}$ as the sum of two terms

$$Q^{\alpha\beta}_{\gamma\delta} = Q^{\alpha\beta}_{-\ \gamma\delta} + Q^{\alpha\beta}_{+\ \gamma\delta} \qquad (1..17)$$

symmetric (antisymmetric) and antisymmetric (symmetric) with respect to the upper (lower) indices then the condition that $P^{\alpha\beta}_{\gamma\delta}$ be a projector is satisfied to first order in $\hbar$ because of the property that

$$Q^{\alpha\beta}_{\gamma\delta} = P^{\alpha\beta}_{\zeta\eta} Q^{\zeta\eta}_{\gamma\delta} + Q^{\alpha\beta}_{\zeta\eta} P^{\zeta\eta}_{\gamma\delta}. \qquad (1..18)$$

The compatibility condition with the product

$$(P^{\alpha\beta}_{\zeta\eta})^* P^{\eta\zeta}_{\gamma\delta} = P^{\beta\alpha}_{\gamma\delta} \qquad (1..19)$$

is satisfied provided $Q^{\alpha\beta}_{\gamma\delta}$ is real.

From (1..11) it follows that

$$d[x^\mu, \theta^\alpha] = [dx^\mu, \theta^\alpha] + [x^\mu, d\theta^\alpha] = e^\mu_\beta [\theta^\beta, \theta^\alpha] - \frac{1}{2}\, [x^\mu, C^\alpha{}_{\beta\gamma}]\theta^\beta\theta^\gamma. \qquad (1..20)$$

We have here introduced the Ricci rotation coefficients $C^\alpha{}_{\beta\gamma}$. We find then that multiplication of 1-forms must satisfy

$$[\theta^\alpha, \theta^\beta] = \frac{1}{2}\, \theta^\beta_\mu [x^\mu, C^\alpha{}_{\gamma\delta}] \theta^\gamma \theta^\delta. \tag{1..21}$$

Consistency requires then that

$$\theta^{[\beta}_\mu [x^\mu, C^{\alpha]}{}_{\gamma\delta}] = 0. \tag{1..22}$$

Because of the condition (1..11) consistency also requires that

$$\theta^{(\alpha}_\mu [x^\mu, C^{\beta)}{}_{\gamma\delta}] = Q_-^{\alpha\beta}{}_{\gamma\delta}. \tag{1..23}$$

We have in general four consistency relations which must be satisfied in order to obtain a noncommutative extension. They are the Leibniz rule (1..13), the Jacobi identity and the conditions (1..22) and (1..23) on the differential. The first two constraints follow from Leibniz rules but they are not completely independent of the differential calculus since one involves the momentum operators. The condition (1..23) follows in general from the expression [8]

$$C^\alpha{}_{\beta\gamma} = -4i\epsilon p_\delta Q_-^{\alpha\delta}{}_{\beta\gamma} \tag{1..24}$$

for the rotation coefficients. It follows also from general considerations that the rotation coefficients must satisfy the gauge condition

$$e_\alpha C^\alpha{}_{\beta\gamma} = 0. \tag{1..25}$$

1.2. Metrics

The metric is a map

$$g:\ \Omega^1(\mathcal{A}) \otimes \Omega^1(\mathcal{A}) \to \mathcal{A}. \tag{1..26}$$

Using the frame it is defined by

$$g(\theta^\alpha \otimes \theta^\beta) = g^{\alpha\beta}, \tag{1..27}$$

and bilinearity of the metric implies that $g^{\alpha\beta}$ are complex numbers. In the present formalism [6] the metric is 'real' if it satisfies the condition

$$\bar{g}^{\beta\alpha} = S^{\alpha\beta}_{\gamma\delta} g^{\gamma\delta}. \tag{1..28}$$

'Symmetry' of the metric can be defined either using the projection

$$P^{\alpha\beta}_{\gamma\delta} g^{\gamma\delta} = 0, \tag{1..29}$$

or the flip

$$S^{\alpha\beta}_{\gamma\delta} g^{\gamma\delta} = c g^{\alpha\beta}. \tag{1..30}$$

We choose the frame to be orthonormal in the commutative limit; we can write therefore

$$g^{\alpha\beta} = \eta^{\alpha\beta} - i\epsilon h^{\alpha\beta}. \tag{1..31}$$

In the linear approximation, the condition of the reality of the metric becomes

$$h^{\alpha\beta} + \bar{h}^{\alpha\beta} = T^{\beta\alpha}_{\gamma\delta}\eta^{\gamma\delta}. \tag{1..32}$$

We introduce also

$$g^{\mu\nu} = g(dx^\mu \otimes dx^\nu) = e^\mu_\alpha e^\nu_\beta g^{\alpha\beta}. \tag{1..33}$$

To analyze the relations between these matrices it is best to consider a small perturbation of flat space. We set

$$g^{\mu\nu} = g_0^{\mu\nu} - i\epsilon g_1^{\mu\nu}, \qquad e^\mu_\alpha = \delta^\mu_\alpha + i\epsilon\Lambda^\mu_\alpha, \qquad g^{\alpha\beta} = \eta^{\alpha\beta} - i\epsilon h^{\alpha\beta}. \tag{1..34}$$

To first order then we have the relations

$$g_1^{\mu\nu} = h^{\mu\nu} - g_0^{\alpha\beta}\Lambda^{(\mu}_\alpha \delta^{\nu)}_\beta. \tag{1..35}$$

We have included here the first term although it becomes important only when the perturbation is around a non-flat background. We shall return to this in Section 2.

1.3. Connections

To define a linear connection one needs a 'flip' [9, 10],

$$\sigma(\theta^\alpha \otimes \theta^\beta) = S^{\alpha\beta}_{\gamma\delta}\theta^\gamma \otimes \theta^\delta, \tag{1..36}$$

which in the present notation is equivalent to a 4-index set of complex numbers $S^{\alpha\beta}_{\gamma\delta}$ which we can write as

$$S^{\alpha\beta}_{\gamma\delta} = \delta^\beta_\gamma \delta^\alpha_\delta + i\epsilon T^{\alpha\beta}_{\gamma\delta}. \tag{1..37}$$

The covariant derivative is given by

$$D\xi = \sigma(\xi \otimes \theta) - \theta \otimes \xi. \tag{1..38}$$

In particular

$$D\theta^\alpha = -\omega^\alpha{}_\gamma \otimes \theta^\gamma = -(S^{\alpha\beta}_{\gamma\delta} - \delta^\beta_\gamma\delta^\alpha_\delta)p_\beta\theta^\gamma \otimes \theta^\delta = -i\epsilon T^{\alpha\beta}_{\gamma\delta}p_\beta\theta^\gamma \otimes \theta^\delta, \tag{1..39}$$

so the connection-form coefficients are linear in the momenta

$$\omega^\alpha{}_\gamma = \omega^\alpha{}_{\beta\gamma}\theta^\beta = i\epsilon p_\delta T^{\alpha\delta}_{\beta\gamma}\theta^\beta. \tag{1..40}$$

On the left-hand side of the last equation is a quantity $\omega^\alpha{}_\gamma$ which measures the variation of the metric; on the right-hand side is the array $T^{\alpha\delta}_{\beta\gamma}$ which is directly related to the anti-commutation rules for the 1-forms, and more important the momenta p_δ which define the frame. As $\hbar \to 0$ the right-hand side remains finite and

$$\omega^\alpha{}_\gamma \to \tilde{\omega}^\alpha{}_\gamma. \tag{1..41}$$

The identification is only valid in the weak-field approximation. The connection is torsion-free if the components satisfy the constraint

$$\omega^\alpha{}_{\eta\delta} P^{\eta\delta}_{\beta\gamma} = \frac{1}{2} C^\alpha_{\beta\gamma}. \tag{1..42}$$

The connection is metric if

$$\omega^\alpha{}_{\beta\gamma} g^{\gamma\delta} + \omega^\delta{}_{\gamma\eta} S^{\alpha\gamma}_{\beta\zeta} g^{\zeta\eta} = 0, \tag{1..43}$$

or linearized,

$$T^{(\alpha\gamma}{}_\delta{}^{\beta)} = 0. \tag{1..44}$$

1.4. Geometry to algebra

The rotation coefficients are directly related to the commutators of the momentum generators. We have seen that the former are given by the expression (1..24) and the latter by (1..15), which we write in the form

$$[p_\alpha, p_\beta] = -4i\epsilon Q^{\gamma\delta}_{-\ \alpha\beta} p_\gamma p_\delta - K_{\alpha\beta}. \tag{1..45}$$

But the momentum generators are directly related to the position generators by the duality relations. There is therefore a direct connection between the rotation coefficients and the commutators $J^{\mu\nu}$. This relation can be derived directly without explicitly referring to the momenta.

One can express the commutator of an arbitrary function f with x^λ as a derivative:

$$[x^\lambda, f] = i\hbar J^{\lambda\sigma} \partial_\sigma f (1 + o(\epsilon)). \tag{1..46}$$

Then the Leibniz rule and the Jacobi identity can be written in leading order as

$$e_\alpha J^{\mu\nu} = \partial_\sigma e^{[\mu}_\alpha J^{\sigma\nu]}\,, \tag{1..47}$$

$$\epsilon_{\kappa\lambda\mu\nu} J^{\gamma\lambda} e_\gamma J^{\mu\nu} = 0. \tag{1..48}$$

Written in terms of the frame components, these two constraint equations become

$$e_\gamma J^{\alpha\beta} - C^{[\alpha}{}_{\gamma\delta} J^{\beta]\delta} = 0, \tag{1..49}$$

$$\epsilon_{\alpha\beta\gamma\delta} J^{\alpha\epsilon} e_\epsilon J^{\beta\gamma} = 0. \tag{1..50}$$

We have used here the definition of the rotation coefficients:

$$C^{\alpha}{}_{\beta\gamma} = \theta^{\alpha}_{\mu} e_{[\beta} e^{\mu}_{\gamma]} = -e^{\nu}_{\beta} e^{\mu}_{\gamma} \partial_{[\nu} \theta^{\alpha}_{\mu]}. \tag{1..51}$$

This can be written also, not forgetting that $\epsilon_{\alpha\beta\gamma\delta}\epsilon^{\alpha\epsilon\zeta\eta} = -\delta^{\epsilon\zeta\eta}_{\beta\gamma\delta}$, in terms of the dual quantities

$$J^{*}_{\alpha\beta} = \frac{1}{2}\epsilon_{\alpha\beta\gamma\delta} J^{\gamma\delta} \tag{1..52}$$

as

$$e_{\alpha} J^{*}_{\beta\gamma} + C^{\delta}{}_{\alpha[\beta} J^{*}_{\gamma]\delta} + C^{\delta}{}_{\alpha\delta} J^{*}_{\beta\gamma} = 0. \tag{1..53}$$

Similarly, from (1..48)

$$\epsilon_{\alpha\beta\gamma\delta} J^{\gamma\eta}(e_{\eta} J^{\alpha\beta} + J^{\alpha\zeta} C^{\beta}{}_{\eta\zeta}) = 0. \tag{1..54}$$

Inserting (1..50) into (1..54) one finds the relation

$$\epsilon_{\alpha\beta\gamma\delta} J^{\alpha\zeta} J^{\beta\eta} C^{\gamma}{}_{\eta\zeta} = 0. \tag{1..55}$$

This equation can be written also in terms of the $J^{*}_{\alpha\beta}$ as

$$C^{\alpha}{}_{[\alpha\gamma} J^{*}_{\beta]\delta} J^{\delta\gamma} = 0. \tag{1..56}$$

Or, using (1..50) it can be written

$$\epsilon_{\alpha\beta\gamma\delta} J^{\alpha\zeta} e_{\zeta} J^{\beta\gamma} = 0. \tag{1..57}$$

One can solve (1..50) for the rotation coefficients. One obtains

$$J^{\gamma\eta} e_{\eta} J^{\alpha\beta} = J^{\alpha\eta} J^{\beta\zeta} C^{\gamma}{}_{\eta\zeta} \tag{1..58}$$

or, provided J^{-1} exists, as

$$C^{\alpha}{}_{\beta\gamma} = J^{\alpha\eta} e_{\eta} J^{-1}_{\beta\gamma}. \tag{1..59}$$

This can be rewritten as

$$C^{\alpha}{}_{\beta\gamma} = J^{\alpha\delta} e_{\delta} J^{\zeta\eta} J^{-1}_{\zeta\beta} J^{-1}_{\eta\gamma} \tag{1..60}$$

and also, using (1..50) as

$$C^{\alpha}{}_{\beta\gamma} = J^{\alpha\delta}(C^{\zeta}{}_{\epsilon\delta} J^{\epsilon\eta} - C^{\eta}{}_{\epsilon\delta} J^{\epsilon\zeta}) J^{-1}_{\zeta\beta} J^{-1}_{\eta\gamma}. \tag{1..61}$$

Anticipating a notation from Section 2. we introduce

$$\hat{C}_{\alpha\beta\gamma} = J^{-1}_{\alpha\delta} C^{\delta}{}_{\beta\gamma}. \tag{1..62}$$

We find then that

$$\hat{C}_{\alpha\beta\gamma} = e_\alpha J^{-1}_{\beta\gamma} \tag{1..63}$$

and that

$$\hat{C}_{\alpha\beta\gamma} + \hat{C}_{\beta\gamma\alpha} + \hat{C}_{\gamma\alpha\beta} = 0, \tag{1..64}$$

an equation which we can write as a de Rham cocyle condition

$$dJ^{-1} = 0. \tag{1..65}$$

It follows that in the quasi-classical approximation, the linear connection and the curvature can be directly expressed in terms of the commutation relations. In particular if the latter are constants then the curvature vanishes. Similar to Equation (1..49) one can derive the identity

$$e_\alpha J^{-1}_{\beta\gamma} = J^{-1}_{\alpha\delta} C^\delta{}_{\beta\gamma} \tag{1..66}$$

for the derivative of the inverse if it exists. The two are consistent because of the cocycle condition (1..65). If we consider J^{-1} as a Maxwell field strength then there is a source given by

$$e^\alpha J^{-1}_{\alpha\beta} = \hat{C}^\alpha{}_{\alpha\beta} = J^{-1\alpha\gamma} C_{\alpha\beta\gamma}. \tag{1..67}$$

It follows also from the condition (1..25) that the commutator must necessarily satisfy the constraint

$$e_\alpha \left(J^{\alpha\eta} e_\eta J^{-1}_{\beta\gamma} \right) = 0. \tag{1..68}$$

This can also be written as

$$\left(e_\alpha J^{\alpha\zeta} + J^{\alpha\eta} C^\zeta{}_{\alpha\eta} \right) e_\zeta J^{-1}_{\beta\gamma} = 0. \tag{1..69}$$

We have assumed that the noncommutativity is small and we have derived some relations to first-order in the parameter ϵ. We shall now make an analogous assumption concerning the gravitational field; we shall assume that ϵ_{GF} is also small and that we consider only the equations to first-order in it as well. With these two assumptions the equations become relatively easy to solve.

If we equate the Expression (1..59) for the rotation coefficients with that (1..51) in terms of the components of the frame we find after a few simple applications of the Leibniz rule that

$$(dJ^{-1})_{\alpha\beta\gamma} = e^\mu_{[\beta} e_{\gamma]} J^{-1}_{\alpha\mu.} \tag{1..70}$$

The cocyle condition (1..65) is equivalent to the condition

$$e^\mu_{[\beta} e_{\gamma]} J^{-1}_{\alpha\mu.} = 0. \tag{1..71}$$

An interesting particular solution is given by constants:

$$J^{-1}_{\alpha\mu.} = J^{-1}_{0\alpha\mu.}. \tag{1..72}$$

It follows then that

$$J^{\mu\nu} = J_0^{\mu\alpha} e_\alpha^\nu, \qquad J_0^{(\mu\alpha} e_\alpha^{\nu)} = 0. \tag{1..73}$$

One verifies that

$$C^\alpha{}_{\alpha\gamma} = J^{\alpha\eta} e_\eta J^{-1}_{\alpha\gamma} = e_\eta J^{\alpha\eta} J^{-1}_{\alpha\gamma} \tag{1..74}$$

and so the left-hand side vanishes if and only if

$$e_\beta J^{\alpha\beta} = 0. \tag{1..75}$$

It follows also that

$$D_\beta J^{\alpha\beta} = e_\beta J^{\alpha\beta} + \omega^{[\alpha}{}_{\beta\gamma} J^{\gamma\beta]} \tag{1..76}$$

$$= C^\beta{}_{\beta\gamma} J^{\gamma\alpha} + \omega^{[\alpha}{}_{\beta\gamma} J^{\gamma\beta]} \tag{1..77}$$

$$= \omega^\beta{}_{[\beta\gamma]} J^{\gamma\alpha} + \omega^{[\alpha}{}_{\beta\gamma} J^{\gamma\beta]} \tag{1..78}$$

$$= -\frac{1}{2} C^\alpha{}_{\beta\gamma} J^{\beta\gamma}. \tag{1..79}$$

Equation (1..50) can be written also using the covariant derivative as

$$D_\gamma J^{\alpha\beta} = C^{[\alpha}{}_{\gamma\delta} J^{\beta]\delta} + \omega^{[\alpha}{}_{\gamma\delta} J^{\delta\beta]} \tag{1..80}$$

$$= \omega^{[\beta}{}_{\delta\gamma} J^{\alpha]\delta}. \tag{1..81}$$

These last formulae are difficult to interpret; they might become clearer in association with torsion.

2. The quasi-classical, weak-field approximation

In the weak-field approximation a frame can be written in the form

$$\theta^\alpha = (\delta^\alpha_\mu - i\epsilon\Lambda^\alpha_\mu) dx^\mu. \tag{2..1}$$

We do not assume here that $\Lambda_{\alpha\mu}$ is symmetric in order to include the possibility of a frame rotation. We now must assure that the four constraints of Section 1. are satisfied; we must express these constraints in the appropriate approximation. We suppose that as $e^\lambda_\alpha \to \delta^\lambda_\alpha$ we have also

$$J^{\mu\nu} \to J_0^{\mu\nu}. \tag{2..2}$$

Were we to choose $e^\lambda_{0\alpha}$ to be a flat frame then the assumption would mean that $J_0^{\mu\nu}$ 'spontaneously' breaks Lorentz invariance. Since Lorentz invariance is broken for every non-flat frame by definition, it would be a stronger assumption to suppose that $J_0^{\mu\nu} = 0$. A particular classical solution might have Killing vectors. We must assure that the additional symplectic structure respects them. Be this not the case, then we must look for a noncommutative extension in a more general class of classical metrics. The constant background $J_0^{\mu\nu}$ is, except in the case we are here considering, a very poor starting point for perturbation.

We shall now consider fluctuations around a particular given solution to the problem we have set. We suppose that is we have a reference solution comprising a frame $e^\lambda_{0\alpha} = \delta^\lambda_\alpha$ and a commutation relation $J_0^{\mu\nu}$ which we perturb to

$$J^{\alpha\beta} = J_0^{\alpha\beta} + i\epsilon I^{\alpha\beta}, \qquad e^\mu_\alpha = \delta^\mu_\beta(\delta^\beta_\alpha + i\epsilon\Lambda^\beta_\alpha). \tag{2..3}$$

The term $J_0^{\alpha\beta}$ is not necessarily constant but we suppose that it is slowly varying with respect to the second. We consider the first term as being determined by the asymptotic conditions of the problem, the global, distant boundary conditions. The second term will be in general smaller than the first but we shall assume that its derivatives are on the average larger.

2.1. The constraints

The two constraint equations (1..49) and (1..50) become

$$\epsilon_{\alpha\beta\gamma\delta} J_0^{\alpha\epsilon} e_\epsilon I^{\beta\gamma} = 0, \tag{2..4}$$

$$e_\gamma I^{\alpha\beta} - e_{[\gamma}\Lambda^{[\alpha}_{\delta]} J_0^{\beta]\delta} = 0. \tag{2..5}$$

In terms of the unknowns

$$\hat I_{\alpha\beta} = J_0^{-1}{}_{\alpha\gamma} J_0^{-1}{}_{\beta\delta} I^{\gamma\delta}, \qquad \hat\Lambda_{\alpha\beta} = J_0^{-1}{}_{\alpha\gamma}\Lambda^\gamma_\beta \tag{2..6}$$

they can be written

$$\epsilon^{\alpha\beta\gamma\delta} e_\gamma \hat I_{\alpha\beta} = 0, \tag{2..7}$$

$$e_\gamma(\hat I_{\alpha\beta} - \hat\Lambda_{[\alpha\beta]}) = e_{[\alpha}\hat\Lambda_{\beta]\gamma}. \tag{2..8}$$

The first constraint is a (well-known) cocycle condition, the statement that the symplectic 2-form is closed. The second is the origin of the particularities of our construction, it and the fact that the 'ground-state' value of $J^{\mu\nu}$ is an invertible matrix.

We decompose also $\hat\Lambda$ as the sum

$$\hat\Lambda_{\alpha\beta} = \hat\Lambda^+_{\alpha\beta} + \hat\Lambda^-_{\alpha\beta} \tag{2..9}$$

of a symmetric and antisymmetric term. The second constraint can be written then

$$e_\gamma \hat{I}_{\alpha\beta} - (e_\alpha \hat{\Lambda}^-_{\beta\gamma} + e_\beta \hat{\Lambda}^-_{\gamma\alpha} + e_\gamma \hat{\Lambda}^-_{\alpha\beta}) = e_\gamma \hat{\Lambda}^-_{\alpha\beta} + e_{[\alpha} \hat{\Lambda}^+_{\beta]\gamma}. \tag{2..10}$$

It implies a second cocyle condition. If we multiply by $\epsilon^{\alpha\beta\gamma\delta}$ we find that

$$\epsilon^{\alpha\beta\gamma\delta} e_\gamma \hat{\Lambda}^-_{\alpha\beta} = 0. \tag{2..11}$$

We can rewrite now Equation (2..10) as

$$e_\gamma (\hat{I}_{\alpha\beta} - \hat{\Lambda}^-_{\alpha\beta}) = e_{[\alpha} \hat{\Lambda}^+_{\beta]\gamma}. \tag{2..12}$$

This equation has the integrability conditions

$$e_\delta e_{[\alpha} \hat{\Lambda}^+_{\beta]\gamma} - e_\gamma e_{[\alpha} \hat{\Lambda}^+_{\beta]\delta} = 0. \tag{2..13}$$

But the left-hand side is the linearized approximation to the curvature of a (ficticious) metric with components $g_{\mu\nu} + i\epsilon \hat{\Lambda}^+_{\mu\nu}$. If it vanishes then the perturbation is a derivative. (We have in fact shown here that a deformation of a commutator can be always chosen antisymmetric to first order). For some 1-form A

$$\hat{\Lambda}^+_{\beta\gamma} = \frac{1}{2} e_{(\beta} A_{\gamma)}. \tag{2..14}$$

Equation (2..12) becomes therefore

$$e_\alpha (\hat{I} - \hat{\Lambda}^- - dA)_{\beta\gamma} = 0. \tag{2..15}$$

It follows then that for some 2-form c with constant components $c_{\beta\gamma}$

$$\hat{\Lambda}^- = \hat{I} - dA + c, \qquad \hat{\Lambda}_{\alpha\beta} = \hat{I}_{\alpha\beta} + e_\beta A_\alpha + c_{\alpha\beta}. \tag{2..16}$$

The remaining constraints are satisfied identically. The most important relation is Equation (2..16) which, in terms of the original 'unhatted' quantities, becomes

$$\Lambda^\alpha_\beta = J_0^{-1}{}_{\beta\gamma} I^{\alpha\gamma} + J_0^{\alpha\gamma} (c_{\gamma\beta} + e_\beta A_\gamma). \tag{2..17}$$

This equation defines a map which to a given perturbation of the commutation relations associates a perturbation of the canonically flat Minkowski frame. We are interested here in the image of this map.

We can also introduce

$$\hat{I} = \frac{1}{2} \hat{I}_{\alpha\beta} \theta^\alpha \theta^\beta, \qquad \hat{\Lambda}^- = \frac{1}{2} \hat{\Lambda}^-_{\alpha\beta} \theta^\alpha \theta^\beta \tag{2..18}$$

and write

$$d\hat{I} = 0, \qquad d\hat{\Lambda}^- = 0. \tag{2..19}$$

The first equation is a particular case of Equation (1..65).

2.2. The metric

We recall that a perturbation of a frame

$$e^\mu_\alpha = e^\mu_{0\beta} + i\epsilon\Lambda^\mu_\alpha \tag{2..20}$$

engenders a perturbation

$$g^{\mu\nu} = g^{\mu\nu}_0 - i\epsilon g_1{}^{\mu\nu}, \qquad g_1{}^{\mu\nu} = -g^{\alpha\beta}_0 e^{(\mu}_{0\alpha}\Lambda^{\nu)}_\beta \tag{2..21}$$

of the metric. The frame components of the perturbation are

$$g_{1\alpha\beta} = -J^{-1}{}_{0(\alpha\gamma}I_{\beta)}{}^\gamma - J_{0(\alpha}{}^\gamma e_{\beta)}A_\gamma. \tag{2..22}$$

Since

$$J^{\alpha\gamma}_0 \hat{I}_{\gamma\beta} = J^{-1}{}_{0\beta\gamma}I^{\alpha\gamma} \tag{2..23}$$

one can write (2..22) also as

$$g_1{}^{\alpha\beta} = -J^{(\alpha\gamma}_0(\hat{I}_\gamma{}^{\beta)} + e^{\beta)}A_\gamma). \tag{2..24}$$

2.3. The algebra to geometry map

We can now be more precise about the map (1..5). Let θ^α be a frame which is a small perturbation of a flat frame and let $J^{\alpha\beta}$ be the frame components of a small perturbation of a constant 'background' J_0. Us interests the map (2..17)

$$I^{\alpha\beta} \longrightarrow \Lambda^\alpha_\beta. \tag{2..25}$$

We recall that we are considering only first-order fluctuations around a given frame and that these fluctuations are redundently parameterized by the array Λ^α_β. We can rewrite the map (1..5) as a map

$$\Lambda^\alpha_\beta \longrightarrow I^{\alpha\beta}. \tag{2..26}$$

It can be defined as an inverse of the map defined in Equation (2..25).

We are interested also in the inverse map. That is, given an arbitrary metric with linearization about the Minkowski metric defined by a matrix Λ^α_β we wish to know whether or not we can find a consistent perturbation $I^{\mu\nu}$ of the commutation relation. We must solve then Equation (2..17) for $I^{\mu\nu}$ in terms of Λ^α_β. That is, we write

$$I^{\alpha\beta} = J^{\beta\gamma}_0\Lambda^\alpha_\gamma - J^{\beta\gamma}_0 J^{\alpha\delta}_0 e_\gamma A_\delta \tag{2..27}$$

dropping the unimportant constant $c_{\gamma\delta}$. The condition (2..14) assures us that the right-hand side is antisymmetric.

In principle, because we use the frame formalism covariance is assured at least to the semi-classical approximation. (We are not sure exactly what this would mean in general.) However since we base our construction on the commutator of two generators (coordinates) it is of interest to show this explicitly. We consider therefore a first order transformation of the form

$$x'^{\mu} = x^{\mu} + i\epsilon B^{\mu} \tag{2..28}$$

and we introduce the notation $J^{\mu\nu} = J_0^{\mu\nu} + i\epsilon J_1^{\mu\nu}$. The transformation of the commutator is given by

$$J_1'^{\mu\nu} = J_1^{\mu\nu} + J_0^{[\mu\sigma}\partial_\sigma B^{\nu]}\,. \tag{2..29}$$

The basic equations (1..47) and (1..48) are to first order written as

$$e_\alpha J_1^{\mu\nu} = \partial_\sigma e_\alpha^{[\mu} J_0^{\sigma\nu]} \tag{2..30}$$

$$\epsilon_{\kappa\lambda\mu\nu} J_0^{\gamma\lambda} e_\gamma J_1^{\mu\nu} = 0. \tag{2..31}$$

in the old coordinate system and as

$$e_\alpha J_1'^{\mu\nu} = \partial_\sigma e_\alpha^{[\prime\mu} J_0^{\sigma\nu]} \tag{2..32}$$

$$\epsilon_{\kappa\lambda\mu\nu} J_0^{\gamma\lambda} e_\gamma J_1'^{\mu\nu} = 0. \tag{2..33}$$

in the new. We must show that the two are equivalent. We recall the definitions

$$[p_\alpha, x^\mu] = e_\alpha^\mu = \delta_\alpha^\mu + i\epsilon\Lambda_\alpha^\mu, \tag{2..34}$$

$$\begin{aligned}[p_\alpha, x'^\mu] &= e_\alpha'^\mu = \delta_\alpha^\mu + i\epsilon\Lambda_\alpha'^\mu = \delta_\alpha^\mu + i\epsilon(\Lambda_\alpha^\mu + e_\alpha B^\mu) \\ &= [p_\alpha, x^\mu] + i\epsilon e_\alpha B^\mu.\end{aligned} \tag{2..35}$$

The basic equations in the new coordinates become then

$$e_\alpha J_1'^{\mu\nu} = \quad \partial_\sigma e_\alpha'^{[\mu} J_0^{\sigma\nu]} \tag{2..36}$$

$$\epsilon_{\kappa\lambda\mu\nu} J_0^{\gamma\lambda} e_\gamma J_1^{\mu\nu} = 0. \tag{2..37}$$

But we shall not use these equations but rather work exclusively using frame components. The system written in frame components

$$e_\gamma I^{\alpha\beta} - C_1{}^{[\alpha}{}_{\gamma\delta} J_0^{\beta]\delta} = 0, \tag{2..38}$$

$$\epsilon_{\alpha\beta\gamma\delta} J_0^{\alpha\epsilon} e_\epsilon I^{\beta\gamma} = 0 \tag{2..39}$$

remains untouched by the coordinate transformation. We must verify however that the components are well defined. We have respectively in the two coordinate systems

$$J^{\alpha\beta} = J^{\mu\nu}\theta^\alpha_\mu\theta^\beta_\nu = J_0^{\alpha\beta} + i\epsilon(J_1^{\alpha\beta} - \Lambda^{[\alpha}_\mu J_0^{\mu\beta]}), \tag{2..40}$$

$$J'^{\alpha\beta} = J'^{\mu\nu}\theta'^\alpha_\mu\theta'^\beta_\nu = J_0^{\alpha\beta} + i\epsilon(J_1'^{\alpha\beta} - \Lambda'^{[\alpha}_\mu J_0^{\mu\beta]}). \tag{2..41}$$

To show that the two right-hand sides are equal we calculate

$$\begin{aligned}
\delta J^{\alpha\beta} &= J'^{\alpha\beta} - J^{\alpha\beta} \\
&= J'^{\mu\nu}\theta'^\alpha_\mu\theta'^\beta_\nu - J^{\mu\nu}\theta^\alpha_\mu\theta^\beta_\nu \\
&= i\epsilon(J_1'^{\alpha\beta} - J_1^{\alpha\beta} - (\Lambda'^{[\alpha}_\mu - \Lambda^{[\alpha}_\mu)J_0^{\mu\beta]}) \\
&= 0.
\end{aligned} \tag{2..42}$$

The frame components of the perturbation and the perturbation of the frame components are related in a simple way. From (2..40) we have

$$J_1^{\alpha\beta} = I^{\alpha\beta} + J_0^{[\alpha\gamma}\Lambda^{\beta]}_\gamma. \tag{2..43}$$

Introducing the solution (2..27) into the last equation we obtain

$$J_1^{\alpha\beta} = -I^{\alpha\beta} + J_0^{\alpha\gamma}J_0^{\beta\delta}e_{[\gamma}A_{\delta]}. \tag{2..44}$$

From the last equation we can see that under coordinate transformation

$$\delta A_\alpha = J_0^{-1}{}_{\alpha\beta}B^\beta. \tag{2..45}$$

We see than that for any perturbation of Minkowski space there is a compatible noncommutative algebra, defined by the solution to Equation (2..27) and differential calculus, defined by the frame which characterizes the perturbation, such that the resulting noncommutative geometry has the given geometry as limit. In fact we have reduced the problem to finding the solution of a Maxwell-Einstein problem with the components of the metric perturbation simple linear combinations of a monopole field strength.

2.4. The curvature

To the lowest order in noncommutativity one obtains the standard expression for the frame components of the linearized Riemann tensor in terms of $g_{1\alpha\beta}$. That is,

$$R_{\alpha\beta\gamma\delta} = \tfrac{1}{2}\, i\epsilon\left(e_\alpha e_{[\gamma}g_{1\delta]\beta} - e_\beta e_{[\gamma}g_{1\delta]\alpha}\right). \tag{2..46}$$

The right-hand side can be expressed in terms of the perturbation of the symplectic structure. Using the expression (2..24) for the perturbation of the metric we find that

$$R_{\alpha\beta\gamma\delta} = -\frac{1}{2}\, i\epsilon \left(J_0^{-1\zeta}{}_{[\gamma} e_{\delta]} e_{[\alpha} I_{\beta]\zeta} + J_0^{-1\zeta}{}_{[\alpha} e_{\beta]} e_{[\gamma} I_{\delta]\zeta} \right). \tag{2..47}$$

We see that as expected, the curvature does not depend on A_α.
With use of the cocycle condition the equation can be written as a divergence

$$R_{\alpha\beta\gamma\delta} = -\frac{1}{2}\, i\epsilon e_\zeta \left(J_{0\zeta[\gamma} e_{\delta]} \hat{I}_{\alpha\beta} + J_{0\zeta[\alpha} e_{\beta]} I_{\gamma\delta} \right). \tag{2..48}$$

For the linearized Ricci curvature one finds the expression

$$R_{\beta\gamma} = \frac{1}{2}\, i\epsilon e^\zeta \left(J_{0\zeta(\beta} e^\alpha \hat{I}_{\gamma)\alpha} + J_0^{\alpha\zeta} e_{(\beta} \hat{I}_{\gamma)\alpha} \right). \tag{2..49}$$

One more contraction yields the expression

$$R = 2i\epsilon J_0{}^{\zeta\alpha} e_\zeta e^\beta \hat{I}_{\alpha\beta} \tag{2..50}$$

for the Ricci scalar. Using again the cocycle condition permits us to write this in the form

$$R = i\epsilon J_0^{\zeta\alpha} e_\gamma e^\gamma \hat{I}_{\alpha\zeta}. \tag{2..51}$$

We see than that for any perturbation of Minkowski space there is a compatible noncommutative algebra, defined by the solution to Equation (2..27) and differential calculus, defined by the frame which characterizes the perturbation, such that the resulting noncommutative geometry has the given geometry as limit.

3. Ground-state examples

To further illustrate the formalism we shall discuss the two most important types of broken symmetry and give an example for each. We shall also give two examples of q-deformed symmetries. None of these examples can be completely solved; they illustrate some of the of the difficulties involved in finding a solution. By the expression ‘ground state’ we mean a convenient solution about which we can apply the perturbation procedure and analyse the structure of the solutions in a neighborhood. In the examples this coincides with what one would normally call a ground state.

3.1. Plane symmetric ground state

The simplest anisotropic homogeneous solution to Einstein equations is the Kasner metric:

$$ds^2 = -dt^2 + t^{2q_1}(dx^1)^2 + t^{2q_2}(dx^2)^2 + t^{2q_3}(dx^3)^2. \tag{3..52}$$

The vacuum equations with vanishing cosmological constant impose the constraints on the parameters q_i

$$q_1 + q_2 + q_3 = 1, \qquad q_1^2 + q_2^2 + q_3^2 = 1. \tag{3..53}$$

The metric (3..52) is a member of a 1-parameter family of solutions. A partial noncommutative extension of the Kasner space expressed in terms of the momentum generators has been found [11]. We shall give a complete noncommutative version of it for the parameter values $q_i = (0, 0, 1)$ at which it is flat. This solution can be used as a background around which a family of perturbative solutions can be found using the same technique as in the previous section.

The moving frame is given by

$$\theta^0 = dt, \qquad \theta^i = (t^Q)^i_j dx^j, \tag{3..54}$$

where Q is a symmetric 3×3 matrix. It can be simply written also in the coordinates $y^i = (t^Q)^i_j x^j$ as

$$\theta^0 = dt, \qquad \theta^i = dy^i - Q^i_j t^{-1} y^j dt, \tag{3..55}$$

The Ricci rotation coefficients for the Kasner frame are given by the nonvanishing value

$$C^a{}_{b0} = Q^a_b t^{-1}, \tag{3..56}$$

and the nonvanishing components of the Ricci curvature tensor are

$$R^0{}_0 = -\mathrm{Tr}\,(Q - Q^2)\, t^{-2}, \qquad R^a{}_b = -(1 - \mathrm{Tr}\, Q)\, Q^a_b\, t^{-2}. \tag{3..57}$$

We impose the commutation relations

$$[x, y] = i\bar{k} J^{12}, \qquad [t, z] = i\bar{k} J(\tau), \tag{3..58}$$

with $\tau = \tau(t)$. The Jacobi identities are satisfied if

$$J^{12} = c \tag{3..59}$$

with c a constant which we shall set equal to one. The algebra is the tensor product

$$\mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2. \tag{3..60}$$

of a factor generated by (x, y) and a factor generated by (t, z). The tensor product structure, we shall see, is respected by the differential calculus; the classical limit is just the metric product of two manifolds. The algebra just defined is too restrictive to describe a general element of the Kasner family. It can be explicitly and easily solved however and it is a convenient ground state.

From the definitions follow the commutation relations

$$[p_0, \tau] = 1, \qquad [p_b, x^a] = (\tau^Q)^a_b. \tag{3..61}$$

If the momenta are expressed in terms of the position generators the Leibniz rules are satisfied automatically.

The first factor is generated by the elements (x, y). We set

$$i\bar{k}p_1 = -y, \qquad i\bar{k}p_2 = x. \tag{3..62}$$

Then we have

$$\theta^1 = dx, \qquad \theta^2 = dy. \tag{3..63}$$

and we have completely described the geometry of the first factor.
For the second factor we suppose $p_3 = p_3(\tau)$ so that

$$[p_3, z] = \dot{\tau}p_3' J^{03}, \qquad f' = \partial_\tau f. \tag{3..64}$$

and

$$[p_3, \tau] = 0, \qquad [p_3, x] = 0, \qquad [p_3, y] = 0. \tag{3..65}$$

We have further

$$\theta^3(e_3) = \tau^{-q_3}[p_3, z] = 1 \tag{3..66}$$

from which we conclude that $[p_3, z] = \tau^{q_3}$. We write this as a differential equation

$$\dot{\tau}p_3' J^{03} = \tau^{q_3} \tag{3..67}$$

for $p_3(\tau)$. By construction

$$[p_3, t] = 0, \qquad [p_3, x] = 0, \qquad [p_3, y] = 0. \tag{3..68}$$

We define

$$i\bar{k}p_0 = z, \qquad i\bar{k}p_3(\tau) = \int_0^{t(\tau)} (J^{03})^{-1}\tau^{q_3} dt. \tag{3..69}$$

The frame is given by

$$\theta^0 = dt, \qquad \theta^3 = \tau^{-q_3} dz. \tag{3..70}$$

We have thus completely described the geometry of the second factor. There are two free quantities, the functions $J(t)$ and $\tau(t)$, as well as the eigenvalues of the matrix Q. The solution is however a very particular one. One can find more general solutions by adding a perturbation. From this point of view the most interesting Kasner solution is highly nonperturbative.

3.2. Spherically symmetric ground state

The noncommutative version of a spacelike compact surface, be it chosen in a way which truly screens points, must have a finite-dimensional structure algebra since the number of degrees of freedom must be finite. This is certainly a difficult problem in general and it will be aborded in a subsequent article [12] devoted to a noncommutative extension of the newtonian limit. However when one imposes spherical symmetry the problem simplifies and was indeed solved [13] several years ago. We recall that in the Wick-rotated version both factors behave as spheres. We must construct a differential calculus over the product which has the Schwarzschild metric as compatible metric. Over each factor one can easily construct a differential calculus but the product of these two would have as metric simply the product metric. We wish the metric to be such that if restricted to the first factor it have a well determined dependence on the radial generator from the second factor. We saw in the first section that both the metric and the differential calculus are defined by the frame.

A fuzzy (noncommutative) version of the 2-sphere can be constructed if one allows the differential calculus to be that of the (parallelizable) circle bundle S^3 over it. Let w be the parameter which parametrizes the extra circle. We have then five possible generators for the algebra, the two which describe the 2-sphere, the circle variable and the time t and radial parameter r. With the three generators (w, t, r) an obvious algebra can be constructed for each. Since the w plays a secondary role one could omit it and quantize by making t and r conjugate variables. Since r takes discrete values it would seem that the logical variable to choose as conjugate would be w. The time would be left as commutative variable. This is acceptable if the metric is static. One can show however that to within terms of second order the radial parameter is continuous.

3.3. A q-deformed ground state

The quantum euclidean planes were introduced by Faddeev *et al.* [14]. There is a problem constructing real differential calculi over them which can only be solved by dropping the condition of q-covariance. We shall briefly discuss the two simplest cases, the quantum line and quantum euclidean space.

3.3.1. The q-deformed line

The algebra $\mathbb{R}^1_q$, called the q-deformed line is generated by an hermitian x and a unitary Λ which satisfy the commutation relation

$$x\Lambda = q\lambda x\,. \tag{3..71}$$

One can think of the unitary operator as a generator of finite translations. One can represent $\mathbb{R}^1_q$ on a Hilbert space $\mathcal{R}_q$ with basis $|k\rangle$ by

$$x|k\rangle = q^k|k\rangle, \qquad \Lambda|k\rangle = |k+1\rangle. \tag{3..72}$$

This explains the origin of the expression 'dilatator'. The spectrum of Λ is continuous. We set $x = q^y$. Then the element y has the representation

$$y|k\rangle = k|k\rangle \tag{3..73}$$

on the basis elements. As operators on $\mathbb{R}_q$ one finds the representations

$$e_1|k\rangle = -z^{-1}|k+1\rangle + z^{-1}\beta|k\rangle, \qquad \bar{e}_1|k\rangle = z^{-1}|k-1\rangle + z^{-1}\bar{\beta}|k\rangle \tag{3..74}$$

for the derivations with two arbitrary complex parameters β and $\bar{\beta}$. The z is a renormalization parameter. If we choose

$$\beta = \alpha, \qquad \bar{\beta} = \bar{\alpha} \tag{3..75}$$

then we can write

$$e_1 = z^{-1}\alpha + p_1, \qquad \bar{e}_1 = z^{-1}\bar{\alpha} + \bar{p}_1. \tag{3..76}$$

The limit $q \to 1$ is rather difficult to control. From the relations of the algebra and the two differential calculi one might expect $\Lambda \to 1$. This is consistent with the limiting relations $e_1 x = \bar{e}_1 x = x$ and the intuitive idea that x is an exponential function on the line. We mention this algebra and its differential calculus because it is a possible model for the (t, r) plane discussed in the previous example. It has also attracted the attention of workers [15] in the field of loop quantum cosmology, independently it would seem and for the same reason. This manifest convergence of views was apparently first noticed some time later [16]. For more details concerning a model with an extra commutative time added we refer to Cerchiai *et al.* [17, 18].

3.3.2. The q-deformed euclidean space

We mention this algebra and its differential calculus because it is a possible model for a static ground state. For more details we refer to Fiore [19] and to Fiore & Madore [20]. The geometry of the higher-dimensional quantum planes has been also examined [21].

The conditions $[x^i, \theta^a] = 0$ become

$$x^i \theta^a_j = \hat{R}^{-1ki}{}_{lj} \theta^a_k x^l \tag{3..77}$$

in terms of the braid matrix and yield equations with solution

$$\begin{aligned}
\theta^- &= \Lambda^{-1} y^{-1} \xi^-, \\
\theta^0 &= \Lambda^{-1} r^{-1} (\sqrt{q}(q+1) y^{-1} x^+ \xi^- + \xi^0), \\
\theta^+ &= -\Lambda^{-1} r^{-2} (\sqrt{q} q (q+1) y^{-1} (x^+)^2 \xi^- + (q+1) x^+ \xi^0 - y \xi^+).
\end{aligned}$$

An analogous expression can be found for the frame $\bar\theta^a$ of the differential calculus $\bar\Omega^*(\mathbb{R}^3_q)$. From the relations

$$[(\theta^a)^*, f^*] = -[\theta^a, f]^* = 0, \qquad f \in \mathbb{R}^3_q, \tag{3..78}$$

it follows that $(\theta^a)^*$ can be written in terms of $\bar\theta^b$. We choose the second frame so that the relation

$$(\theta^a)^* = \bar\theta^b g_{ba} \tag{3..79}$$

is satisfied.

Consider the elements $p_a \in \mathbb{R}^3_q$ with

$$\begin{aligned} p_- &= +h^{-1} q \Lambda y^{-1} x^+, \\ p_0 &= -h^{-1} \sqrt{q} \Lambda y^{-1} r, \\ p_+ &= -h^{-1} \Lambda y^{-1} x^-. \end{aligned} \tag{3..80}$$

By direct calculation one verifies that they define the derivations dual to the frame.

Since Λ is unitary the hermitian adjoints p_a^* are given by

$$p_\pm^* = -\Lambda^{-2} g^{\pm b} p_b, \qquad p_0^* = \Lambda^{-2} g^{0b} p_b. \tag{3..81}$$

The fact that the p_a are not anti-hermitian is related to the fact that the differential d is not real. We have chosen this rather odd normalization so that the p_a satisfy the commutation relations

$$\begin{aligned} p_- p_0 &= q p_0 p_-, \\ p_+ p_0 &= q^{-1} p_0 p_+, \\ [p_+, p_-] &= h (p_0)^2. \end{aligned} \tag{3..82}$$

These equations can be rewritten more compactly in the form

$$P^{ab}{}_{cd} p_a p_b = 0. \tag{3..83}$$

This is Equation (1..15) with

$$C^a{}_{bc} = 0, \quad F_{ab} = 0. \tag{3..84}$$

It is easy to check that

$$g^{ab} p_a p_b = q h^{-2} \Lambda^2. \tag{3..85}$$

If one introduces the corresponding elements $\bar p_a$ which yield the derivations dual to the frame $\bar\theta^a$ one finds that the involution on the p_a can be written

$$p_a^* = -g^{ab} \bar p_b. \tag{3..86}$$

It follows that $(df)^* = \bar{d}f^*$. The metric is conformally flat with conformal factor r^{-2}. If one uses spherical polar coordinates then one sees immediately that the space is $S^2 \times \mathbb{R}$ with $\log r$ as the preferred coordinate along the line. The radius of the sphere is equal to one.

To write the commutative limit of the frame it is most convenient to use the real coordinates $x^r = (x, y, z)$ and use the components

$$\theta^1 = \frac{1}{\sqrt{2}}(\theta^- + \theta^+), \qquad \theta^2 = \theta^0, \qquad \theta^3 = \frac{i}{\sqrt{2}}(\theta^- - \theta^+) \tag{3..87}$$

of the frame. A short calculation yields

$$\begin{aligned}
\theta^1 &= y^{-1}dx - y^{-1}r^{-1}(x - iz)dr,\\
\theta^2 &= y^{-1}dr - iy^{-1}r^{-1}(xdz - zdx),\\
\theta^3 &= y^{-1}dz - iy^{-1}r^{-1}(x - iz)dr
\end{aligned}$$

and by direct calculation one can verify that in fact the line element is indeed

$$ds^2 = (\theta^1)^2 + (\theta^2)^2 + (\theta^3)^2 = r^{-2}(dx^2 + dy^2 + dz^2). \tag{3..88}$$

The conformal factor r has the effect of introducing a function of q in the expression for the connection. This might be used to arrange the compatibility condition but it is not possible to choose $r\theta^a$ as a frame since it does not commute with Λ. This element, which does not lie in the center but nevertheless has vanishing exterior derivative, is the origin of the problem.

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