On Poisson Energy*

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Abstract

It is suggested that noncommutative geometry modifies the Einstein field equations through the introduction of a field energy associated to the gradient of the commutator. An expression is found for the new term in the quasi-classical, weak-field, WKB approximation.

1. Introduction and motivation

Within the context of a generalization of the Cartan 'moving-frame' formalism to noncommutative geometry it has been suggested [1] that noncommutative geometry modifies the Einstein field equations through the introduction of a field energy associated to the gradient of the commutator. An expression is found for the new term in the quasi-classical, weak-field, WKB approximation. In the formulation which we are here considering the Ricci tensor and therefore the Einstein tensor as well are determined as integrability conditions for the underlying associative-algebra structure. The value *in vacuo* of the commutative limit of this tensor we interpret as the energy-momentum of the symplectic structure.

In the frame formalism the Einstein field equations are most elegantly (and, once one is familiar with it, easily) written as the vanishing of a 3-form. This follows from the identity

$$G_{\alpha\beta} * \theta^{\beta} = -\frac{1}{2}\, \Omega^{\beta\gamma} * \theta_{\alpha\beta\gamma}. \tag{1..1}$$

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We write accordingly the energy-momentum of the gravitational field in terms of a vector-valued 3-form τ_S which has the property [2] that it is exact if and only if the Einstein field equations are satisfied. If this be so one sees that the total energy-momentum is given as the integral over the sphere at infinity of the Sparling 2-form

$$\sigma_\alpha = -\frac{1}{2} * \omega_{\alpha\beta}\theta^\beta. \tag{1..2}$$

Since in the noncommutative case there is a preferred frame, the one canonically aligned with respect to the eigenvectors of the conformal tensor, one can claim that the 2-form itself and not only the integral thereof is well-defined. Our assumption is that there is another 3-form

$$\tau_{PS} = \tau_{PS}(J) \tag{1..3}$$

which vanishes in the commutative limit and which is such that the sum

$$\tau = \tau_S + \tau_{PS} \tag{1..4}$$

is an exact 3-form. We cannot give an explicit formula for τ_{PS} in general. We can only express it in a simple limiting case.

Our notation is introduced and the basic equations derived in the lecture [1] of this proceedings, so they will not be repeated here. In the weak-field approximation the constraint equations are

$$\epsilon_{\alpha\beta\gamma\delta} J_0^{\alpha\epsilon} e_\epsilon I^{\beta\gamma} = 0, \tag{1..5}$$

$$e_\gamma I^{\alpha\beta} - e_{[\gamma}\Lambda^{[\alpha}_{\delta]} J_0^{\beta]\delta} = 0. \tag{1..6}$$

$I^{\alpha\beta}$ and Λ^α_β are the frame components of the perturbations of the commutator and of the frame:

$$J^{\alpha\beta} = J_0^{\alpha\beta} + i\epsilon I^{\alpha\beta}, \qquad e^\mu_\alpha = \delta^\mu_\beta(\delta^\beta_\alpha + i\epsilon\Lambda^\beta_\alpha), \tag{1..7}$$

$J_0^{\alpha\beta}$ and δ^μ_α are the unperturbed values. We assume also that $J_0^{\mu\nu}$ is constant and invertible. This, in particular, means that the unperturbed noncommutative space is flat.

In terms of the unknowns

$$\hat I_{\alpha\beta} = J_0^{-1}{}_{\alpha\gamma} J_0^{-1}{}_{\beta\delta} I^{\gamma\delta}, \qquad \hat\Lambda_{\alpha\beta} = J_0^{-1}{}_{\alpha\gamma}\Lambda^\gamma_\beta \tag{1..8}$$

Equations (1..5-1..6) be written

$$\epsilon^{\alpha\beta\gamma\delta} e_\gamma \hat I_{\alpha\beta} = 0, \tag{1..9}$$

$$e_\gamma(\hat I_{\alpha\beta} - \hat\Lambda_{[\alpha\beta]}) = e_{[\alpha}\hat\Lambda_{\beta]\gamma}. \tag{1..10}$$

The first constraint is a (well-known) cocycle condition, the statement that the symplectic 2-form is closed. The second is the origin of the particularities of our construction, it and the fact that the 'ground-state' value of $J_0^{\mu\nu}$ is an invertible matrix. It can also be written as

$$dJ^{-1} = 0. \tag{1..11}$$

We decompose $\hat{\Lambda}$ as the sum

$$\hat{\Lambda}_{\alpha\beta} = \hat{\Lambda}^{+}_{\alpha\beta} + \hat{\Lambda}^{-}_{\alpha\beta} \tag{1..12}$$

of a symmetric and antisymmetric term. The second constraint can be written then

$$e_\gamma \hat{I}_{\alpha\beta} - (e_\alpha \hat{\Lambda}^{-}_{\beta\gamma} + e_\beta \hat{\Lambda}^{-}_{\gamma\alpha} + e_\gamma \hat{\Lambda}^{-}_{\alpha\beta}) = e_\gamma \hat{\Lambda}^{-}_{\alpha\beta} + e_{[\alpha} \hat{\Lambda}^{+}_{\beta]\gamma}. \tag{1..13}$$

It implies a second cocycle condition. If we multiply by $\epsilon^{\alpha\beta\gamma\delta}$ we find that

$$\epsilon^{\alpha\beta\gamma\delta} e_\gamma \hat{\Lambda}^{-}_{\alpha\beta} = 0. \tag{1..14}$$

We can rewrite now Equation (1..13) as

$$e_\gamma (\hat{I}_{\alpha\beta} - \hat{\Lambda}^{-}_{\alpha\beta}) = e_{[\alpha} \hat{\Lambda}^{+}_{\beta]\gamma}. \tag{1..15}$$

This equation has the integrability conditions

$$e_\delta e_{[\alpha} \hat{\Lambda}^{+}_{\beta]\gamma} - e_\gamma e_{[\alpha} \hat{\Lambda}^{+}_{\beta]\delta} = 0. \tag{1..16}$$

But the left-hand side is the linearized approximation to the curvature of a (fictitious) metric with components $g_{\mu\nu} + i\epsilon \hat{\Lambda}^{+}_{\mu\nu}$. If it vanishes then the perturbation is a derivative. (We have in fact shown here that a deformation of a commutator can be always chosen antisymmetric to first order). Thus, for some 1-form A we have

$$\hat{\Lambda}^{+}_{\beta\gamma} = \frac{1}{2} e_{(\beta} A_{\gamma)}. \tag{1..17}$$

Equation (1..15) becomes therefore

$$e_\alpha (\hat{I} - \hat{\Lambda}^{-} - dA)_{\beta\gamma} = 0. \tag{1..18}$$

It follows then that for some 2-form c with constant components $c_{\beta\gamma}$

$$\hat{\Lambda}^{-} = \hat{I} - dA + c, \tag{1..19}$$

and

$$\hat{\Lambda}_{\alpha\beta} = \hat{I}_{\alpha\beta} + e_\beta A_\alpha + c_{\alpha\beta}. \tag{1..20}$$

The remaining constraints are satisfied identically.

In terms of the original 'unhatted' quantities Equation (1..20) becomes

$$\Lambda^\alpha_\beta = J_0^{-1}{}_{\beta\gamma} I^{\alpha\gamma} + J_0^{\alpha\gamma}(c_{\gamma\beta} + e_\beta A_\gamma). \tag{1..21}$$

This equation defines a map which to a given perturbation of the commutation relations associates a perturbation of the canonically flat Minkowski frame.

We are interested also in the inverse of this map. That is, given an arbitrary metric with linearization about the Minkowski metric defined by a matrix Λ^α_β we wish to know whether or not we can find a consistent perturbation $I^{\mu\nu}$ of the commutation relation. We must solve then Equation (1..21) for $I^{\mu\nu}$ in terms of Λ^α_β. That is, we write

$$I^{\alpha\beta} = J_0^{\beta\gamma}\Lambda^\alpha_\gamma - J_0^{\beta\gamma} J_0^{\alpha\delta} e_\gamma A_\delta \tag{1..22}$$

dropping the unimportant constant $c_{\gamma\delta}$. The condition (1..17) assures us that the right-hand side is antisymmetric.

The frame components of the perturbation of the metric are

$$g_{1\alpha\beta} = -\Lambda_{(\alpha\beta)} = -J^{-1}{}_{0(\alpha\gamma} I_{\beta)}{}^\gamma - J_{0(\alpha}{}^\gamma e_{\beta)} A_\gamma. \tag{1..23}$$

As it is shown in [1], under the transformation of coordinates

$$x'^\mu = x^\mu + i\epsilon B^\mu \tag{1..24}$$

A_α transform as

$$\delta A_\alpha = J_0^{-1}{}_{\alpha\beta} B^\beta, \tag{1..25}$$

while $I^{\alpha\beta}$ are unchanged. Therefore we can fix the sistem of coordinates in such a way that $A = 0$. In this system we have

$$g_{1\alpha\beta} = -J^{-1}{}_{0(\alpha\gamma} I_{\beta)}{}^\gamma = -J_{0(\alpha\gamma} \hat{I}^\gamma{}_{\beta)}. \tag{1..26}$$

In order to simplify the analyze of the linearized solution and the properties of waves, it is often convenient also to suppress some of the modes to facilitate the description of those remaining. For example the physical components of the electromagnetic theory are the two transverse modes which describe the dynamics of the two polarization states of the photon. In a plasma there is a third, longitudinal mode which describes plasma waves and there are complicated mixtures known as Alfène waves. Because of the linearity these modes can all be studied separately. Although there is no gravitational equivalent of a plasma there are new additional modes when a wave enters a physical medium, for example an elastic body and these additional modes must be analyzed along with the two dynamical modes of the graviton.

2. The WKB approximation

As an example a noncommutative extension of a classical metric we blur the WKB geometry. We have but to calculate Λ and I in this particular case. We suppose that the algebra $\mathcal{A}$ is a tensor product

$$\mathcal{A} = \mathcal{A}_0 \otimes \mathcal{A}_\omega \tag{2..27}$$

of a 'slowly-varying' factor $\mathcal{A}_0$ in which all amplitudes lie and a 'rapidly-varying' phase factor which is of order-of-magnitude $i\epsilon$ so that only functions linear in this factor can appear. By 'slowly-varying' we mean an element f with a classical limit $\tilde{f}$ such that $\partial_\alpha \tilde{f} \lesssim \mu \tilde{f}$. The generic element f of the algebra is of the form then

$$f(x^\lambda, \phi) = f_0(x^\lambda) + i\epsilon f_1(x^\lambda) = f_0(x^\lambda) + i\epsilon \bar{f}_1(x^\lambda)\, e^{i\omega\phi} \tag{2..28}$$

where f_0 and f_1 belong to $\mathcal{A}_0$. Because of the condition on ϵ the factor order does not matter and these elements form an algebra. The frequency parameter ω is so chosen that for an element f of $\mathcal{A}_0$ the estimate

$$[\phi, f] \simeq \bar{k}\mu \tag{2..29}$$

holds. The commutator $[f, e^{i\omega\phi}]$ is thus of order of magnitude

$$[f, e^{i\omega\phi}] \simeq \bar{k}\mu\omega. \tag{2..30}$$

We shall require also that the energy of the wave be such that it contribute not as source to the background field. This inequality can be written

$$\epsilon\omega^2 \ll \mu^2. \tag{2..31}$$

In the WKB case the Ansatz concerns the derivatives. We suppose that Λ^α_β and $I^{\alpha\beta}$ are of the form

$$\Lambda^\alpha_\beta = \bar{\Lambda}^\alpha_\beta e^{i\omega\phi}, \qquad I^{\alpha\beta} = \bar{I}^{\alpha\beta} e^{i\omega\phi}, \tag{2..32}$$

where $\bar{\Lambda}^\alpha_\beta$ and $\bar{I}^{\alpha\beta}$ belong to $\mathcal{A}_0$. Therefore also we have

$$g_1^{\mu\nu} = \bar{g}^{\mu\nu} e^{i\omega\phi}. \tag{2..33}$$

We introduce the normal $\xi_\alpha = e_\alpha\phi$ to the surfaces of constant phase. We have then

$$e_\alpha I_{\beta\gamma} = (i\omega\xi_\alpha \bar{I}_{\beta\gamma} + e_\alpha \bar{I}_{\beta\gamma})\, e^{i\omega\phi} \tag{2..34}$$

$$e_\alpha \Lambda_{\beta\gamma} = (i\omega\xi_\alpha \bar{\Lambda}_{\beta\gamma} + e_\alpha \bar{\Lambda}_{\beta\gamma})\, e^{i\omega\phi} \tag{2..35}$$

and the expression for the perturbation to the Riemann tensor becomes

$$\begin{aligned} R_{\alpha\beta\gamma\delta} &= \frac{1}{2}\, i\epsilon(i\omega)^2 \xi_{[\alpha}\bar{g}_{\beta][\delta}\xi_{\gamma]}\, e^{i\omega\phi} \\ &+ \frac{1}{2}\, i\epsilon(i\omega)\left(e_{[\alpha}\bar{g}_{\beta][\delta}\,\xi_{\gamma]} + e_{[\gamma}\bar{g}_{\delta][\beta}\,\xi_{\alpha]}\right) e^{i\omega\phi} \\ &+ \frac{1}{2}\, i\epsilon(i\omega)\left(e_\alpha \xi_{[\gamma}\,\bar{g}_{\delta]\beta} - e_\beta \xi_{[\gamma}\,\bar{g}_{\delta]\alpha}\right) e^{i\omega\phi} + o(i\omega)\,. \end{aligned} \tag{2..36}$$

It depends only on the first term linear in I in the expression (1..23). Classically, the vacuum equations are given by the condition that the Einstein tensor vanish. In the WKB approximation this yields in fact two equations. The leading order term, proportional to ω^2 is a dispersion relation ('E' for Einstein)

$$G^{(2)}_{\alpha\beta} = \frac{1}{2}\, i\epsilon(i\omega)^2 \mathrm{Disp}_{E\alpha\beta} = 0 \tag{2..37}$$

with

$$\mathrm{Disp}_{E\alpha\beta} = -\xi^2\psi_{\alpha\beta} + \xi^\gamma\psi_{\gamma(\alpha}\xi_{\beta)} - \xi^\gamma\xi^\delta\psi_{\gamma\delta}g_{0\alpha\beta}, \tag{2..38}$$

and

$$\psi_{\alpha\beta} = g_{1\alpha\beta} - \frac{1}{2}\, g_1{}^\gamma_\gamma g_{0\alpha\beta}. \tag{2..39}$$

If $\xi^2 = 0$ then it follows that

$$\xi^\gamma\psi_{\gamma\beta} = 0. \tag{2..40}$$

The second term in the expansion (2..36), proportional to ω, yields a conservation law

$$G^{(1)}_{\alpha\beta} = \frac{1}{2}\, i\epsilon(i\omega)\mathrm{Cons}_{E\alpha\beta} = 0 \tag{2..41}$$

with

$$\mathrm{Cons}_{E\alpha\beta} = 2\,\xi^\gamma e_\gamma\psi_{\alpha\beta} + e_\gamma\xi^\gamma\psi_{\alpha\beta}. \tag{2..42}$$

This second equation can be interpreted [3, 4] as a conservation of graviton number *in vacuo*. One easily sees that from it follows

$$\mathrm{Cons}_{E\beta\gamma}\psi^{\beta\gamma} = e_\alpha(\psi_{\beta\gamma}\psi^{\beta\gamma}\xi^\alpha) = 0. \tag{2..43}$$

In a viscous fluid the gravitons are absorbed and generate heat [5]. We shall show that the Poisson structure behaves similarly. An essential difference in the noncommutative case is that the gauge has been fixed by the commutation relations; setting for example $\psi^\gamma_\gamma = 0$ would be a physical constraint. In the lowest order as $\omega \to \infty$ the momenta commute amongst themselves and we are justified in neglecting the contribution $h^{\alpha\beta}$ to the metric. However in the next-to-leading order this is not necessarily the case. We shall return to this point in Section 3..

2.1. Dispersion relations

We have here no action; the field equations, we claim, are integrability conditions for the differential calculus. One of these conditions is the co-cycle condition (1..11), which is similar in structure to the condition that the Ricci tensor vanish; the two are however not equivalent. In the WKB approximation Equation (1..11) yields to leading order

$$\xi_\alpha \hat{I}_{\beta\gamma} + \xi_\beta \hat{I}_{\gamma\alpha} + \xi_\gamma \hat{I}_{\alpha\beta} = 0. \tag{2..44}$$

We introduce the vector $\check{\xi}^\alpha$ and the bivector $V_{\alpha\beta}$ defined as

$$\check{\xi}^\alpha = J_0^{\alpha\beta}\xi_\beta, \qquad V_{\alpha\beta} = \xi_{[\alpha}\check{\xi}_{\beta]}. \tag{2..45}$$

If we multiply condition (2..44) by $J_0^{\beta\gamma}$ we obtain as constraint the equation

$$\hat{I}_{\alpha\beta}\check{\xi}^\beta = \frac{1}{2}\,\xi_\alpha Y \tag{2..46}$$

where, to alleviate the formulae, we have introduced the element

$$Y = J_0^{\gamma\delta}\hat{I}_{\delta\gamma} = \bar{Y}e^{i\omega\phi}. \tag{2..47}$$

If we multiply condition (2..44) by ξ^γ we obtain a dispersion relation ('L' is for Lie)

$$\mathrm{Disp}_{L\alpha\beta} = 0 \tag{2..48}$$

with

$$\mathrm{Disp}_{L\alpha\beta} = \xi^2\hat{I}_{\alpha\beta} + \xi^\gamma\hat{I}_{\gamma[\alpha}\xi_{\beta]}. \tag{2..49}$$

Equation(2..49) is similar to (2..39) but not identical. The latter has 10 equations for 10 unknowns whereas the former is a system of 6 equations. This difference can be minimized by pointing out that whereas the equations (2..40) are real constraints the corresponding equations derived from Disp_L are in fact identities. To compare the two relations we rewrite Disp_E in terms of Disp_L. If we introduce ('P' is for Poisson)

$$\mathrm{Disp}_{P\alpha\beta} = \mathrm{Disp}_{E\alpha\beta} + J_{0(\alpha\zeta}\mathrm{Disp}_L{}^\zeta{}_{\beta)}, \tag{2..50}$$

then using (2..46) we find

$$\mathrm{Disp}_{P\alpha\beta} = V^\zeta{}_{(\alpha}\hat{I}_{\beta)\zeta} - g_{0\alpha\beta}\xi^2 Y = -\xi^\gamma\check{\xi}_{(\alpha}\hat{I}_{\beta)\gamma} + (\xi_\alpha\xi_\beta - g_{0\alpha\beta}\xi^2)Y. \tag{2..51}$$

It is obvious that $\xi^\alpha\check{\xi}_\alpha = 0$. Therefore, if ξ is an eigenvector of J_0 then by definition $V^{\alpha\beta} = 0$. From the expression for the components of the Riemann tensor given in Section 3. we see that it then vanishes identically, that is the space stays unperturbed. No gravitational radiation can propagate along the eigenvalues of the symplectic structure: the two principle null directions of J_0 are precisely the directions along which there can be no radiation. One of the characteristics of the noncommutative 'lattice' that it does not admit wave propagation along its preferred directions. We interpret this as the result of the formation of a gap as in the case of an ordinary lattice, combined with the fact that the preferred directions must be null. We saw in the last section the classical result that radiation can propagate only along the null cones. This condition is necessary but not sufficient. The background symplectic structure described by J_0 places further restriction, much like a lattice influences (as well as permits) the propagation of sound waves.

If $\check{\xi}$ is not proportional to ξ then we interpret it as the propagation vector of a secondary wave engendered by the interaction of the gravitational wave with the 'lattice'. If we multiply the cocycle condition by $\check{\xi}^\gamma$ we find that

$$\check{\xi}^\gamma \hat{I}_{\gamma[\alpha}\xi_{\beta]} = 0 \tag{2..52}$$

whatever the value of ξ^2. The perturbation $\hat{I}_{\gamma\delta}$ maps $\check{\xi}$ to ξ. It is interesting to note that the condition (2..52) is equivalent to the gauge condition

$$e_\alpha C^\alpha{}_{\beta\gamma} = 0. \tag{2..53}$$

We saw in [1] that this condition is necessary for the existence of momenta.

In order to analyze further the oscillation modes we split the metric perturbation as

$$g_{1\alpha\beta} = g_1{}^{TT}_{\alpha\beta} + \xi_{(\alpha} g_1{}^L_{\beta)} + g_1{}^T g_{0\alpha\beta} \tag{2..54}$$

of the trace-free transverse part $g_1{}^{TT}_{\alpha\beta}$, the longitudinal part $g_1{}^L_\alpha$ and the trace $g_1{}^T$. By definition

$$\xi^\alpha g_1{}^{TT}_{\alpha\beta} = 0, \qquad g_0^{\alpha\beta} g_1{}^{TT}_{\alpha\beta} = 0, \qquad \xi^\alpha g_1^T{}_\alpha = 0. \tag{2..55}$$

From(2..54) we have

$$g_{1\alpha\beta} g_1{}^{\alpha\beta} = J_0^{\alpha\beta} \hat{I}_{\beta\gamma} J_0^{\gamma\delta} \hat{I}_{\delta\alpha} = g_1^{TT}{}_{\alpha\beta} g_1^{TT\alpha\beta} + 2\xi^2 (g_1^L)^2 + 4(g_1^T)^2. \tag{2..56}$$

We saw that by transformation of coordinates we can put $A = 0$. In these coordinates

$$g_{1\alpha\beta} = -J_{0(\alpha}{}^\zeta \hat{I}_{\zeta\beta)}, \tag{2..57}$$

and we obtain

$$\begin{aligned} g_{0\alpha\beta} g_1^{\alpha\beta} &= 4g_1^T = -2Y, \\ \xi_\alpha \xi_\beta g_1^{\alpha\beta} &= \xi^2 g_1^T = -\xi^2 Y. \end{aligned} \tag{2..58}$$

Comparing these two equations, we obtain

$$\xi^2 g_1^T = -\xi^2 Y = 0. \tag{2..59}$$

The oscillation modes are null or traceless. As the expression $\xi^2 Y$ is proportional to the scalar curvature R, we see that the space-time but is always perturbed in such a way that the corresponding curvature scalar is 0. Also, from (2..54) we have

$$g_{1\alpha\beta}\xi^{\alpha} = -\frac{1}{2}\,Y\xi_{\beta}. \tag{2..60}$$

We will analyze the two possibilities $g^T = 0$ and $\xi^2 = 0$ separately; in Einstein gravity both conditions are fulfilled simultaneously. For the traceless mode we have

$$\xi^{\beta}J_{0\beta\gamma}\hat{I}^{\gamma}{}_{\alpha} = 0, \tag{2..61}$$

and therefore

$$\begin{aligned}\xi^{\beta}J_{0\alpha\gamma}\hat{I}^{\gamma}{}_{\beta} &= \xi^{\beta}J_{0(\beta\gamma}\hat{I}^{\gamma}{}_{\alpha)}\\ &= -\xi^{\beta}g_{1\alpha\beta} = -\xi^2 g^L_{1\ \alpha}.\end{aligned} \tag{2..62}$$

If $\xi^2 \neq 0$, we can solve for the longitudinal part of the metric:

$$g^L_{1\ \alpha} = -\frac{1}{\xi^2}J_{0\alpha\gamma}\hat{I}^{\gamma\beta}\xi_{\beta}. \tag{2..63}$$

In principle as we see from (2..56), both longitudinal and transverse-traceless parts are non-zero.

In the null case, $\xi^2 = 0$, we have

$$\xi^{\alpha}\xi_{\beta}\hat{I}_{\gamma\alpha} = \xi^{\alpha}\xi_{\gamma}\hat{I}_{\beta\alpha}. \tag{2..64}$$

The Ricci curvature in this case simplifies. We obtain

$$\begin{aligned}R_{\beta\gamma} &= \frac{1}{2}\,i\epsilon(i\omega)^2\Big(-J_{0(\beta\zeta}\xi^{\zeta}\xi^{\alpha}\hat{I}_{\gamma)\alpha} + J_0^{\alpha\zeta}\xi_{\zeta}\xi_{(\beta}\hat{I}_{\gamma)\alpha}\Big)\\ &= \frac{1}{2}\,i\epsilon(i\omega)^2\Big(g_{1\,(\beta\alpha}\xi^{\alpha}\xi_{\gamma)} + 2\xi_{\beta}\xi_{\gamma}Y\Big) = \frac{1}{2}\,i\epsilon(i\omega)^2\xi_{\beta}\xi_{\gamma}Y.\end{aligned} \tag{2..65}$$

3. Poisson Energy

We interpret Equation (2..50) as the leading-order term in the WKB approximation of a hypothetical equality

$$G_{P\alpha\beta} = G_{E\alpha\beta} + J_{0(\alpha\zeta}G_L{}^{\zeta}{}_{\beta)}, \tag{3..66}$$

valid in general. The last term is to be considered as the field equations which arise as integrability conditions for the Lie algebra structure defined by the commutation relations. When it vanishes the equality between the remaining two terms can be then interpreted as an equation which defines the Einstein tensor $G_{E\alpha\beta}$ in terms of the energy-momentum tensor

$$T_{P\alpha\beta} = -\frac{1}{8\pi G_N}\,G_{P\alpha\beta} \tag{3..67}$$

of a new source of gravitational energy which we call the Poisson energy. We can write then the estimate

$$G_{P\alpha\beta} = G^{(2)}_{P\ \alpha\beta} + G^{(1)}_{P\ \alpha\beta} + G^{(0)}_{P\ \alpha\beta} \tag{3..68}$$

where

$$G^{(2)}_{P\alpha\beta} = \frac{1}{2}\, i\epsilon(i\omega)^2 \mathrm{Disp}_{P\alpha\beta}. \tag{3..69}$$

The Poisson energy, given in the leading order by (2..51), can be also obtained from the expressions for the curvature tensor in the WKB approximation:

$$\begin{aligned}
R^{(2)}_{\alpha\beta\gamma\delta} &= -\frac{1}{2}\, i\epsilon(i\omega)^2 (V_{\alpha\beta}\hat{I}_{\gamma\delta} + V_{\gamma\delta}\hat{I}_{\alpha\beta}),\\
R^{(2)}_{\beta\gamma} &= \frac{1}{2}\, i\epsilon(i\omega)^2 V^{\alpha}{}_{(\beta}\hat{I}_{\gamma)\alpha},\\
R^{(2)} &= i\epsilon(i\omega)^2\xi^2 Y
\end{aligned} \tag{3..70}$$

which are valid when $\mathrm{Disp}_{L\alpha\beta} = 0$ and represent the leading order, quadratic in ω. We see easily that

$$g_0^{\alpha\beta}\mathrm{Disp}_{P\alpha\beta} = -2\xi^2 Y \tag{3..71}$$

and

$$\xi^{\alpha}\mathrm{Disp}_{P\alpha\beta} = 0\,. \tag{3..72}$$

The Poisson energy-momentum is conserved.

To understand the expression for the Poisson energy better we will calculate it to second order in $i\epsilon$. In fact, it is possible to obtain a general expression when noncommutativity is small. From Ricci rotation coefficients which are in this case given by [1],

$$C^{\alpha}{}_{\beta\gamma} = J^{\alpha\eta} e_{\eta} J^{-1}_{\beta\gamma} \tag{3..73}$$

for the Ricci curvature tensor we obtain

$$\begin{aligned}
2R_{\beta\zeta} = {} & J_{(\zeta\delta} e^{\alpha} e^{\delta} J^{-1}_{\beta)\alpha} + J^{\alpha\delta} e_{(\zeta} e_{\delta} J^{-1}_{\alpha\beta)}\\
& - J_{(\zeta}{}^{\eta} e^{\alpha} J^{-1}_{\eta\gamma} J^{\gamma\delta} e_{\delta} J^{-1}_{\beta)\alpha} + J^{\alpha\delta} e_{\delta} J^{-1}_{\eta\beta} J^{\eta\gamma} e_{\gamma} J^{-1}_{\alpha\zeta}\\
& + J_{\eta\delta} e^{\delta} J^{-1}_{\beta\alpha} J^{\eta\gamma} e_{\gamma} J^{-1\alpha}_{\zeta} + J^{\alpha\eta} e_{(\zeta} J^{-1}_{\eta\gamma} J^{\gamma\delta} e_{\delta} J^{-1}_{\beta)\alpha}\\
& - \frac{1}{2}\, J_{\zeta\delta} e^{\delta} J^{-1\alpha\eta} J_{\beta\gamma} e^{\gamma} J^{-1}_{\alpha\eta} + J^{\alpha\delta} e_{\delta} J^{-1}_{\alpha\eta} J_{(\zeta\gamma} e^{\gamma} J^{-1\eta}_{\beta)}.
\end{aligned} \tag{3..74}$$

We have here neglected the ordering on the right-hand side. In an attempt to clarify the right-hand side we give an approximate expression valid to second order in the perturbation $i\epsilon I^{\alpha\beta}$ to the constant background $J_0^{\alpha\beta}$ when the perturbation is of the exponential form described in Section 2.. For this we shall need only the linear perturbation of the Poisson structure.

Since

$$e_\delta J^{-1}_{\beta\gamma} = -J^{-1}_{\beta\eta} e_\delta J^{\eta\zeta}\, J^{-1}_{\zeta\gamma} \tag{3..75}$$

to first order in $i\epsilon$ we have

$$e_\delta J^{-1}_{\beta\gamma} = i\epsilon e_\delta \hat{I}_{\beta\gamma}. \tag{3..76}$$

Therefore we find that to within second order

$$\begin{aligned} 2R_{\beta\zeta} &= i\epsilon(i\omega)^2\Big(\xi^\alpha \check{\xi}_{(\zeta}\hat{I}_{\beta)\alpha} + \xi_{(\zeta}\check{\xi}^\alpha \hat{I}_{\alpha\beta)} + J_{0(\zeta}{}^\delta \xi^\alpha \check{\xi}^\gamma \hat{I}_{\gamma\delta}\hat{I}_{\beta)\alpha} \\ &\qquad + \check{\xi}^\alpha \check{\xi}^\eta \hat{I}_{\eta\beta}\hat{I}_{\alpha\zeta} + \check{\xi}_\eta \check{\xi}^\eta \hat{I}_{\beta\alpha}\hat{I}_\zeta{}^\alpha - J_0^{\alpha\delta}\xi_{(\zeta}\check{\xi}^\gamma \hat{I}_{\gamma\delta}\hat{I}_{\beta)\alpha} \\ &\qquad - \frac{1}{2}\,\check{\xi}_\zeta \check{\xi}_\beta \hat{I}^{\alpha\eta}\hat{I}_{\alpha\eta} + \check{\xi}^\alpha \check{\xi}_{(\zeta}\hat{I}_{\alpha\eta}\hat{I}_{\beta)}{}^\eta\Big) \qquad\qquad (3..77) \\ &= i\epsilon(i\omega)^2\Big(\xi^\alpha \check{\xi}_{(\zeta}\hat{I}_{\beta)\alpha} - \xi_\beta \xi_\zeta Y + \check{\xi}_{(\zeta}\xi^\eta \hat{I}_{\eta\beta)} Y \\ &\qquad + \frac{3}{4}\,\xi_\beta \xi_\eta Y^2 - \check{\xi}^2 \hat{I}_{\beta\eta}\hat{I}^\eta{}_\gamma + \frac{1}{2}\,\check{\xi}_\beta \check{\xi}_\zeta \hat{I}_{\alpha\eta}\hat{I}^{\eta\alpha}\Big). \end{aligned}$$

and

$$R = -i\epsilon(i\omega)^2 \xi^2 Y + \frac{1}{8}\,(i\epsilon)^2(i\omega)^2(7\xi^2 Y^2 - 2\check{\xi}^2 \hat{I}_{\alpha\eta}\hat{I}^{\eta\alpha}). \tag{3..78}$$

This expression is in fact ambiguous since it contains some second-order terms and neglects others. It becomes well-defined when averaged over several wavelengths. We use that is the approximations

$$\langle \hat{I}^{\alpha\beta}\rangle = 0, \qquad \langle \hat{I}^{\alpha\beta}\hat{I}^{\gamma\delta}\rangle = \frac{1}{2}\,\bar{\hat{I}}^{\alpha\beta}\bar{\hat{I}}^{\gamma\delta}. \tag{3..79}$$

We find then

$$\langle R_{\beta\zeta}\rangle = \frac{1}{4}\,(i\epsilon)^2(i\omega)^2\Big(\check{\xi}_{(\zeta}\xi^\eta \bar{\hat{I}}_{\eta\beta)}\bar{Y} + \frac{3}{4}\,\xi_\beta \xi_\eta \bar{Y}^2 - \check{\xi}^2 \bar{\hat{I}}_{\beta\eta}\bar{\hat{I}}^\eta{}_\gamma + \frac{1}{2}\,\check{\xi}_\beta \check{\xi}_\zeta \bar{\hat{I}}_{\alpha\eta}\bar{\hat{I}}^{\eta\alpha}\Big) \tag{3..80}$$

and

$$\begin{aligned} \langle G_{\beta\zeta}\rangle = \frac{1}{4}\,(i\epsilon)^2(i\omega)^2\Big(&\check{\xi}_{(\zeta}\xi^\eta \bar{\hat{I}}_{\eta\beta)}\bar{Y} + \frac{3}{4}\,\xi_\beta \xi_\eta \bar{Y}^2 - \check{\xi}^2 \bar{\hat{I}}_{\beta\eta}\bar{\hat{I}}^\eta_\gamma \\ &+ \frac{1}{2}\,\check{\xi}_\beta \check{\xi}_\zeta \bar{\hat{I}}_{\alpha\eta}\bar{\hat{I}}^{\eta\alpha} - \frac{1}{4}\,g_{0\beta\zeta}(7\xi^2 \bar{Y}^2 - 2\check{\xi}^2 \bar{\hat{I}}_{\alpha\eta}\bar{\hat{I}}^{\eta\alpha})\Big). \end{aligned} \tag{3..81}$$

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