# The influence of torsion on the black hole entropy in 3D gravity*

**M. Blagojević† and B. Cvetković‡**
Institute of physics, P.O.Box 57, 11001 Belgrade, SERBIA

Abstract

We study thermodynamic properties of the black hole with torsion in the semi-classical approximation to the gravitational partition function. The entropy of the black hole differs from the standard Beckenstein-Hawking result, but its form is in complete agreement with the first law of black hole mechanics.

## 1. Introduction

It has been known since the early 1970s that black holes are thermodynamic objects, with characteristic temperatures and entropies [1]. The study of these concepts has shown that they are closely related to the quantum nature of gravity. The discovery of the BTZ black hole in three-dimensional (3D) gravity [2] was of particular importance, since it enabled us to investigate these issues in a technically much simpler context.

In the traditional approach based on general relativity (GR), 3D gravity has been studied mainly in the realm of Riemannian geometry, leading to a number of outstanding results [2, 3, 4, 5, 6, 7]. On the other hand, it has been known for nearly five decades, that there is a more natural, gauge-theoretic conception of gravity based on *Riemann-Cartan geometry*, where both the *curvature* and the *torsion* characterize the structure of gravity (see, e.g. [8, 9]). The application of these ideas to 3D gravity started in the 1990s by Mielke, Baekler and Hehl [10]. Recent developments in this direction led to several highly interesting results:

a) 3D gravity with torsion possesses the black hole solution [11, 12, 13] with interesting thermodynamic properties [14, 15, 16],
b) it can be formulated as a Chern-Simons gauge theory [12, 17],
c) it has the (anti-de Sitter) asymptotic symmetry, described by two independent Virasoro algebras with different central charges [13], and
d) there is a supersymmetric extension of the Mielke-Baeckler model, with the super-conformal symmetry in the asymptotic region [18, 19].

* Work supported by the Serbian Science Foundation, and presented by B. Cvetković.
† e-mail address: mb@phy.bg.ac.yu
‡ e-mail address: cbranislav@phy.bg.ac.yu

In the present paper, we discuss the entropy of the black hole with torsion, using both the canonical and covariant form of the action in the semiclassical approximation to the gravitational partition function [14, 16]. Although one expects that there are only technical differences between these two methods, the regularization of the covariant action and the presence of a boundary at the horizon have to be treated with special care.

The layout of the paper is as follows. In section 2, we give a brief overview of the Euclidean 3D gravity with torsion, based on the Mielke-Baekler action [10]. In sections 3 and 4, we describe the Euclidean black hole solution, and derive the canonical expressions for its energy and angular momentum. Section 5 contains basic results of our exposition [14, 16]. We derive the form of the partition function in the semiclassical approximation and obtain the expression for the entropy of the black hole with torsion. Beside the Bekenstein-Hawking term, it contains an additional contribution, stemming from the Chern-Simons term in the action. This term survives even when the torsion vanishes, in agreement with the result found in the context of Riemannian geometry [20]. Then, we show that the existence of torsion is in complete agreement with the first law of black hole mechanics. Finally, section 6 is devoted to concluding remarks, while appendix contains some technical details.

Our conventions are the same as in [14, 16]: the Greek indices refer to the coordinate frame, the Latin indices refer to the tangent frame; the middle alphabet letters $(i, j, k, \dots; \mu, \nu, \lambda, \dots)$ run over $0, 1, 2$, the first letters of the Greek alphabet $(\alpha, \beta, \gamma, \dots)$ run over $1, 2$; $\eta_{ij} = \mathrm{diag}\,(1, 1, 1)$ are the metric components in the tangent frame; totally antisymmetric tensor $\varepsilon^{ijk}$ and the related tensor density $\varepsilon^{\mu\nu\rho}$ are both normalized by $\varepsilon^{012} = +1$.

## 2. Euclidean 3D gravity with torsion

Following the analogy with Poincaré gauge theory [8, 9], Euclidean gravity with torsion in 3D can be formulated as a gauge theory of the Euclidean group $E(3) = ISO(3)$, the analytic continuation of the Poincaré group $ISO(1, 2)$. Basic gravitational variables are the triad field $b^i$ and the spin connection $\omega^i$ (1-forms), and the corresponding field strengths are the torsion $T^i = db^i + \varepsilon^i{}_{jk}\omega^j b^k$ and the curvature $R^i = d\omega^i + \frac{1}{2}\varepsilon^i{}_{jk}\omega^j\omega^k$ (2-forms). The underlying geometric structure is described by Riemann-Cartan space.

Mielke and Baekler (MB) proposed a *topological* model for 3D gravity in Riemann-Cartan spacetime [10], which is a natural generalization of Riemannian GR with a cosmological constant ($\mathrm{GR}_\Lambda$). Euclidean version of the model is defined by the action $I_E$, obtained from its Minkowskian counterpart $I_M$ by the process of analytic continuation $I_M \to iI_E$ [14]. Omitting the subscript $E$ for simplicity, the Euclidean MB action in the absence of matter reads:

$$I = 2a \int b^i R_i - \frac{\Lambda}{3} \int \varepsilon_{ijk} b^i b^j b^k \; + \alpha_3 I_{\mathrm{cs}}[\omega] + \alpha_4 \int b^i T_i \,, \qquad (2.1)$$

where the wedge product sign $\wedge$ is omitted for simplicity. The first term with $a = 1/16\pi G$ is the usual Einstein-Cartan action, the second term is a cosmological term, $I_{\text{cs}}[\omega]$ is the Chern-Simons action for the spin connection,

$$I_{\text{cs}}[\omega] = \int \left( \omega^i d\omega_i + \frac{1}{3}\varepsilon_{ijk}\omega^i\omega^j\omega^k \right) ,$$

and the fourth term is a torsion counterpart of the first one.

The variation of the action with respect to $b^i$ and $\omega^i$ yields the field equations in vacuum:

$$\begin{aligned} 2\mathcal{A}_i &:= 2aR_i + 2\alpha_4 T_i - \Lambda\varepsilon_{ijk}b^j b^k = 0 \,, \\ 2\mathcal{B}_i &:= 2aT_i + 2\alpha_3 R_i + \alpha_4\varepsilon_{ijk}b^j b^k = 0 \,. \end{aligned} \tag{2.2a}$$

In the sector $\alpha_3\alpha_4 - a^2 \neq 0$, we can rewrite these equations as:

$$2T^i = p\varepsilon^i{}_{jk}\, b^j b^k \,, \qquad 2R^i = q\varepsilon^i{}_{jk}\, b^j b^k \,, \tag{2.2b}$$

with

$$p := \frac{\alpha_3\Lambda + \alpha_4 a}{\alpha_3\alpha_4 - a^2} \,, \qquad q := -\frac{(\alpha_4)^2 + a\Lambda}{\alpha_3\alpha_4 - a^2} \,. \tag{2.2c}$$

After introducing the Levi-Civita connection $\tilde\omega^i$ by $db^i + \varepsilon^i{}_{jk}\tilde\omega^j b^k = 0$, one finds that the Riemannian piece of the curvature $R^i(\tilde\omega)$ is:

$$2R^i(\tilde\omega) = \Lambda_{\text{eff}}\ \varepsilon^i{}_{jk}\, b^j b^k \,, \qquad \Lambda_{\text{eff}} := q - \frac{1}{4}p^2 \,, \tag{2.3}$$

where $\Lambda_{\text{eff}}$ is the effective cosmological constant [14]. Thus, our spacetime has maximally symmetric metric, with the isometry group $SO(3,1)$ (the hyperbolic 3D space). In what follows, we restrict our attention to the Euclidean continuation of anti-de Sitter space, which is defined by positive $\Lambda_{\text{eff}} : \Lambda_{\text{eff}} =: 1/\ell^2 > 0$.

## 3. The black hole solution

For $\Lambda_{\text{eff}} > 0$, equation (2.3) has a well-known solution for the metric, the Euclidean BTZ black hole [2]. In Schwarzschild coordinates $x^\mu = (t, r, \varphi)$, the metric has the form

$$\begin{aligned} &ds^2 = N^2 dt^2 + N^{-2}dr^2 + r^2(d\varphi + N_\varphi dt)^2 \,, \\ &N^2 = \left( -8Gm + \frac{r^2}{\ell^2} - \frac{16G^2J^2}{r^2} \right), \qquad N_\varphi = -\frac{4GJ}{r^2}. \end{aligned} \tag{3.1}$$

The zeros of $N^2$, $r_+$ and $r_- = -i\rho_-$, satisfy the relations: $r_+\rho_- = 4GJ\ell$, $r_+^2 - \rho_-^2 = 8Gm\ell^2$, and both $\varphi$ and $t$ are periodic:

$$0 \le \varphi < 2\pi \,, \qquad 0 \le t < \beta \,, \qquad \beta = \frac{2\pi\ell^2 r_+}{r_+^2 + \rho_-^2} \,.$$

Any surface with constant $r$ has the topology of a 2-torus, and the whole black hole manifold $\mathcal{M}$ is a *solid torus*. The black hole horizon $r = r_+$ is a circle at the core of the solid torus, so that the manifold does not contain the region $r < r_+$, corresponding to the "inner" part of the Minkowskian black hole. There are strong arguments in favor of the assumption that the horizon should be regarded as an additional, inner boundary of $\mathcal{M}$, beside the (asymptotic) boundary located at spatial infinity (see section 5).

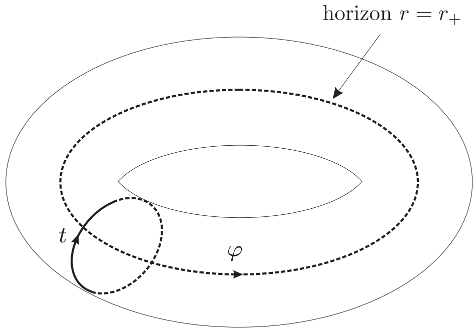


Figure 1: The Euclidean black hole manifold

For later convenience, we introduce the black hole angular velocity:

$$\Omega = N_\varphi(r_+) = -\frac{\rho_-}{\ell r_+}\,.$$

Starting with the black hole metric (3.1), the solution for the black hole with torsion takes the form:

$$b^0 = N dt\,, \qquad b^1 = N^{-1}dr \qquad b^2 = r\,(d\varphi + N_\varphi dt) \tag{3.2a}$$

$$\omega^i = \tilde\omega^i + \frac{p}{2}\,b^i\,, \tag{3.2b}$$

where $\tilde\omega^i$ is the Levi-Civita connection:

$$\tilde\omega^0 = N d\varphi\,, \quad \tilde\omega^1 = -N^{-1}N_\varphi dr\,, \quad \tilde\omega^2 = -\frac{r}{\ell^2}dt + rN_\varphi d\varphi\,. \tag{3.2c}$$

## 4. Energy and angular momentum

Asymptotic conditions and their symmetries are of essential importance for the physical content of gauge theories, as they give rise to the conserved charges, which characterize dynamical behavior of the system.

General asymptotic structure of both Minkowskian and Euclidean 3D gravity with torsion and its relation to the conformal symmetry at spatial infinity, is well understood [13, 15]. In order to determine the form of energy and angular momentum for the black hole, it is sufficient to assume that the fields $b^i$ and $\omega^i$ are restricted to the family of black hole configurations (3.2), defined by parameters $m$ and $J$ [14].

Asymptotic conditions in the canonical formalism ensure the existence of well-defined canonical generators [21, 22]. The canonical generator $G$ is a spatial integral of the Hamiltonian constraints, and it acts on phase-space variables via the Poisson bracket operation, defined in terms of functional derivatives. In general, $G$ does not have well-defined functional derivatives, but the problem can be resolved by adding a suitable *surface term* $\Gamma$. The improved canonical generator $\tilde{G} = G + \Gamma$ is not only differentiable, but also a finite and *conserved* phase-space functional [14].

The value of the improved generator $\tilde{G}$ defines the *asymptotic charge*. Since $\tilde{G} \approx \Gamma$, the charge is completely determined by the boundary term. In particular, energy and angular momentum are defined as the values of $\Gamma$ for $\xi^0 = 1$ and $\xi^2 = 1$, respectively. For the black hole configuration (3.2), one obtains [14]:

$$E = m + \frac{\alpha_3}{a}\left(\frac{pm}{2} - \frac{J}{\ell^2}\right), \qquad M = J + \frac{\alpha_3}{a}\left(\frac{pJ}{2} + m\right). \tag{4.1}$$

## 5. Partition function and the black hole entropy

Thermodynamic properties of black holes are closely related to the quantum nature of gravity [23, 24, 25, 26]. In this section, we examine the influence of torsion on the partition function of the Euclidean 3D gravity with torsion. Let us consider the functional integral (in units $\hbar = 1$)

$$Z[\beta, \Omega] = \int DbD\omega \exp\left(-\tilde{I}[b, \omega, \beta, \Omega]\right), \tag{5.1a}$$

where $\beta$ and $\Omega$ are, respectively, the inverse temperature and the angular velocity of the black hole, defined by certain boundary conditions on the fields $(b^i, \omega^i)$, and $\tilde{I}$ is the Euclidean action (2.3), corrected by suitable boundary terms, $\tilde{I} = I + B$. The *Lagrangian* boundary conditions define the set of allowed field configurations $\mathcal{C}_L$, such that:

(i) $\mathcal{C}_L$ contains black holes with the values of $(m, J)$ belonging to a small region around some $(m, J)_0$,

(ii) $\beta$ and $\Omega$ remain constant on the boundary, and

(iii) the improved action $\tilde{I} = I + B$ is differentiable.

If we recall the definition of $\beta$ and $\Omega$ from section 3, we can see that they are determined in terms of $r_+$ and $\rho_-$ (the functions of $m$ and $J$), so that the conditions (i) and (ii) may become incompatible. However, this is not a sincere problem. Indeed, it can be resolved by treating $\beta$ and $\Omega$ as *independent parameters*, determined in terms of $r_+$ and $\rho_-$ only "on-shell". Geometrically, an "off-shell" extension of $\beta$ and $\Omega$ leads to conical singularities, but at the end, after imposing the field equations, they disappear [25]. Interpreted in this way, the above boundary conditions correspond to the *grand canonical ensemble*.

Although the partition function (5.1a) cannot be calculated exactly, the semiclassical approximation around the black hole solution (3.2) gives a highly interesting information regarding the role of torsion in quantum dynamics. Indeed, this approximation corresponds to the thermodynamic limit, in which the partition function has the general form

$$Z[\beta,\mu]=e^{-\beta F}\,,\qquad F=E-\mu M-TS\,, \tag{5.1b}$$

where $F$ is the free energy, $\beta=1/T$, $E$ and $S$ are the thermodynamic values of energy and entropy, and $\mu$ is the chemical potential corresponding to the conserved charge $M$. Since $E$ and $M$ are already known from the canonical analysis, it follows that, in the semiclassical approximation, the value of $\tilde I$ at the black hole configuration (3.2) defines the black hole entropy:

$$\tilde I_{\rm bh}=\beta(E-\mu M)-S\,. \tag{5.1c}$$

## 5.1. The canonical approach

In this subsection, we use the canonical form of the MB action, as in [2]:

$$I_{\rm c}=\int dt\int d^2x\left(\pi_i{}^\mu\dot b^i{}_\mu+\Pi_i{}^\mu\dot\omega^i{}_\mu-\hat{\mathcal H}_T\right)\,, \tag{5.2a}$$

where $\hat{\mathcal H}_T$ is the total Hamiltonian (A1). The Lagrangian boundary conditions (i)–(iii) can be easily extended to the related *Hamiltonian* boundary conditions, which define the set of allowed phase-space configurations $\mathcal C_H$. The action $I_{\rm c}$ does not have well-defined functional derivatives, since its variation on $\mathcal C_H$ produces not only the field equations, but also some boundary terms. The improved action has the general form

$$\tilde I_{\rm c}=I_{\rm c}+B\,, \tag{5.2b}$$

where the boundary term $B$ is chosen so as to make $\tilde I_{\rm c}$ differentiable. The partition function is now given as the functional integral over $\mathcal C_H$.

For a thermodynamic system in equilibrium, the ensemble must be time independent. If one calculates the value of $\tilde I_{\rm c}$ on the set of static triads and connections, one finds that $I_{\rm c}$ vanishes on shell ($\hat{\mathcal H}_T\approx 0$), so that the complete information about the black hole thermodynamics is contained in the boundary term $B$.

**Boundary terms.** The boundary term $B$ is constructed to cancel the unwanted boundary terms in $\delta I_{\rm c}$, arising from integrations by parts: $\delta I_{\rm c}+\delta B\approx 0$, where $\approx$ is an "on-shell" equality.

With the known form (A1) of $\hat{\mathcal H}_T$, the general variation of $I_{\rm c}$ at fixed $r$ is given by

$$\delta I_{\rm c}\big|^r\approx 2\int dtd\varphi\left[b^i{}_0(a\delta\omega_{i2}+\alpha_4\delta b_{i2})+\omega^i_0(a\delta b_{i2}+\alpha_3\delta\omega_{i2})\right]^r\,. \tag{5.3}$$

The boundary terms at infinity and at the horizon are determined by the requirement

$$\delta I_{\mathrm{c}}\big|^{r\to\infty} - \delta I_{\mathrm{c}}\big|^{r_+} + \delta(B^\infty + B^{r_+}) \approx 0\,.$$

**Spatial infinity.** The boundary term at spatial infinity has the form

$$\begin{aligned}
&\delta I_{\mathrm{c}}\big|^{r\to\infty} = -\int_0^\beta dt\left[\delta\int d^2x\,\hat{\mathcal{H}}_T\right]^{r\to\infty} \approx -\delta B^\infty\,,\\
&B^\infty := \beta E\,.
\end{aligned} \tag{5.4}$$

Here, $E$ is the canonical energy given in (4.1), and the time period $\beta$ is kept fixed, as we are in the grand canonical ensemble.

**The horizon.** If we consider Minkowski black hole as a macroscopic object, it seems quite natural to treat only its "outer" part $r > r_+$ as physical. In the Euclidean formalism, in spite of the fact that the "inner" region $r < r_+$ is absent from the black hole manifold, there are convincing arguments that one still needs boundary terms at the horizon [24, 25, 26]. These arguments are based on the observation that the Killing vector field $\partial_t$ is not well defined at $r = r_+$. One can overcome the problem by removing the line $r = r_+$ from the black hole manifold, but this modifies the topology: the horizon becomes an additional (inner) boundary of the manifold.

After accepting such an assumption, we have to introduce appropriate boundary conditions at the horizon, in order to further improve the differentiability of the action. The values of the triad field and Riemannian connection on the horizon satisfy the following relations:

$$b^a{}_0 - \Omega b^a{}_2 = 0\,,\qquad \tilde\omega^a{}_0 - \Omega\tilde\omega^a{}_2 = -\frac{2\pi}{\beta}\,\delta^a_2\,, \tag{5.5}$$

for $a = 0, 2$. Having in mind that the boundary conditions should correspond to the grand canonical ensemble, we promote these "on-shell" relations into the "off-shell" boundary conditions on the horizon, with $\Omega$ and $\beta$ as independent parameters.

Using the general relation (5.3) and the boundary conditions (5.5), one obtains the boundary term on the horizon:

$$\begin{aligned}
&\delta I_{\mathrm{c}}\big|^{r_+} \approx \delta B^{r_+}\,,\\
&\quad B^{r_+} := -\beta\Omega M - \left[\frac{2\pi r_+}{4G} + 4\pi^2\alpha_3\left(pr_+ - 2\frac{\rho_-}{\ell}\right)\right].
\end{aligned} \tag{5.6}$$

The value of the improved canonical action at the black hole solution is equal to the sum of boundary terms (5.4) and (5.6):

$$(\tilde I_{\mathrm{c}})_{\mathrm{bh}} = \beta(E - \Omega M) - \left[\frac{2\pi r_+}{4G} + 4\pi^2\alpha_3\left(pr_+ - 2\frac{\rho_-}{\ell}\right)\right]. \tag{5.7}$$

We shall return to the thermodynamic interpretation of this result soon.

### 5.2. The covariant approach

One could expect that transition to the *covariant*, Lagrangian form of the action in the partition function represents only a technical step, which cannot change the final result for the entropy, contained in (5.7). However, here we have at least two issues that deserve a careful analysis. (a) In the semiclassical calculation of the partition function (5.1a), one needs the value of the covariant action at the black hole configuration, which is a divergent expression. The existence of this divergence can be taken care of by a convenient regularization procedure, but one should verify that this procedure is consistent with the adopted boundary conditions [27]. (b) Although one expects a boundary at the horizon in the Hamiltonian formalism [2, 26], it seems that its presence in the covariant formalism of $\mathrm{GR}_\Lambda$ can be ignored [27]. These issues need a consistent explanation in the context of 3D gravity with torsion.

**Regularization.** The value of the action (2.1) at the black hole configuration is *divergent*:

$$I_{\rm bh} \approx \frac{4\pi a\beta}{\ell^2}\left(r_\infty^2 - r_+^2\right) , \tag{5.8}$$

where $r_\infty \to \infty$. Note that this result is of the same form as in $\mathrm{GR}_\Lambda$ [27]. One can define a natural regularization procedure by subtracting the value of $I_{\rm bh}$ at the black hole vacuum, where $m = J = 0$ (see also [28]). The regularized action reads

$$I_{\rm reg} = I - \frac{4\pi a\beta}{\ell^2} r_\infty^2 , \tag{5.9}$$

and its value at the black hole configuration is finite: $I_{\rm reg} \approx -4\pi a\beta r_+^2/\ell^2$.

Our approach to the black hole thermodynamics again relies on the construction of the improved action, in accordance with the adopted boundary conditions (i)–(iii). The construction is now modified by a new element—the regularization procedure, whose consistency with the boundary conditions should be verified.

**Boundary terms.** Similarly as in the previous subsection, we find that the improved covariant action has the following form:

$$\begin{aligned} &\tilde I = I_{\rm reg} + B^\infty + B^{r^+} , \\ &B^\infty = \beta m , \qquad B^{r^+} = -2\pi^2\alpha_3\left(pr_+ - 2\frac{\rho_-}{\ell}\right) . \end{aligned} \tag{5.10}$$

In $\mathrm{GR}_\Lambda$, where $\alpha_3 = 0$, the boundary term stemming from the horizon vanishes, and the boundary at the horizon can be safely ignored [27]. More generally, this is true whenever the Chern-Simons term in the action is absent ($\alpha_3 = 0$). On the other hand, whenever $\alpha_3 \neq 0$, even in the absence of torsion ($p = 0$) [20], the boundary at the horizon yields a nontrivial contribution, and its presence cannot be disregarded.

The value of the action (5.10) at the black hole is

$$\tilde{I}_{\mathrm{bh}} = -\frac{\pi r_+}{4G} - 2\pi^2\alpha_3\left(pr_+ - 2\frac{\rho_-}{\ell}\right) . \tag{5.11}$$

The last term represents the contribution from the boundary at the horizon. Using the generalized Smarr formula in Riemann-Cartan spacetime

$$\beta(E - \Omega M) \approx \frac{\pi r_+}{4G} + 2\pi^2\alpha_3\left(pr_+ - 2\frac{\rho_-}{\ell}\right) ,$$

we obtain that the value of the improved covariant action coincides with the value of the improved canonical action (5.7), i.e. $\tilde{I}_{\mathrm{bh}} \equiv (\tilde{I}_{\mathrm{c}})_{\mathrm{bh}}$.

### 5.3. Entropy

We now return to the thermodynamic interpretation of the results (5.7) and (5.11), based on equation (5.1c). It tells us that $\beta$ is the inverse temperature, $\Omega$ is the chemical potential corresponding to $M$, and

$$S = \frac{2\pi r_+}{4G} + 4\pi^2\alpha_3\left(pr_+ - 2\frac{\rho_-}{\ell}\right) \tag{5.12}$$

is the entropy of the black hole with torsion [14, 15, 16].

For $\alpha_3 = 0$, $S$ reduces to the Bekenstein-Hawking value. The additional term, proportional to $\alpha_3$, stems from the Chern-Simons contribution to the action (2.1). The first piece of this term, proportional to $pr_+$, can be interpreted as the contribution of the torsion degrees of freedom at the outer horizon, while the second piece is associated to the "inner horizon" $\rho_-$. The Chern-Simons contribution survives even in Riemannian theory (with vanishing torsion), in agreement with [18].

Using the rules of Eucilidean continuation defined in [14], the entropy can be easily expressed in terms of the Minkowskian parameters as:

$$S = \frac{2\pi r_+}{4G} + 4\pi^2\alpha_3\left(pr_+ - 2\frac{r_-}{\ell}\right) . \tag{5.13}$$

### 5.4. The first law of thermodynamics

If the black hole solution is an extremum of the action $I$ on the set of allowed phase-space configurations, $\delta I|_{\mathrm{bh}} \approx -\delta(B^\infty + B^{r_+})|_{\mathrm{bh}}$, we obtain the relation

$$T\delta S = \delta E - \Omega\delta M , \tag{5.14}$$

which represents the first law of black hole thermodynamics (compare with arguments given in [29]). The result (5.14) is also confirmed by a direct calculation, based on our expressions for $S, E$ and $M$. Thus, the existence of torsion in 3D is in complete agreement with the first law of black hole thermodynamics.

## 6. Concluding remarks

We discussed thermodynamic properties of the black hole with torsion by calculating the gravitational partition function in the semiclassical approximation, using first the canonical and then the covariant form of the action.

(1) In both the canonical and covariant formalisms, the correct value for the black hole entropy is obtained provided the black hole manifold has two boundaries, one at infinity and one at the horizon.

(2) The entropy of the black hole with torsion differs from the Bekenstein-Hawking result by an additional contribution, stemming from the Chern-Simons term in the action. This is true even in Riemannian theory, where the torsion vanishes.

(3) The existence of torsion is in complete agreement with the first law of black hole thermodynamics.

## A Hamiltonian and constraints

Introducing the canonical momenta $(\pi_i{}^\mu, \Pi_i{}^\mu)$, corresponding to the Lagrangian variables $(b^i{}_\mu, \omega^i{}_\mu)$, the action (2.1) leads to the following primary constraints:

$$\begin{aligned}
&\phi_i{}^0 \equiv \pi_i{}^0 \approx 0, \quad \phi_i{}^\alpha \equiv \pi_i{}^\alpha - \alpha_4 \varepsilon^{0\alpha\beta} b_{i\beta} \approx 0, \\
&\Phi_i{}^0 \equiv \Pi_i{}^0 \approx 0, \quad \Phi_i{}^\alpha \equiv \Pi_i{}^\alpha - \varepsilon^{0\alpha\beta}(2a b_{i\beta} + \alpha_3 \omega_{i\beta}) \approx 0.
\end{aligned}$$

The canonical Hamiltonian is linear in unphysical variables:

$$\begin{aligned}
&\mathcal{H}_c = b^i{}_0 \mathcal{H}_i + \omega^i{}_0 \mathcal{K}_i + \partial_\alpha D^\alpha\,, \\
&\mathcal{H}_i = -\varepsilon^{0\alpha\beta} \mathcal{A}_{i\alpha\beta}\,, \qquad \mathcal{K}_i = -\varepsilon^{0\alpha\beta} \mathcal{B}_{i\alpha\beta}\,, \\
&D^\alpha = \varepsilon^{0\alpha\beta}\left[\omega^i{}_0(2a b_{i\beta} + \alpha_3 \omega_{i\beta}) + \alpha_4 b^i{}_0 b_{i\beta}\right]\,.
\end{aligned}$$

The total Hamiltonian reads:

$$\begin{aligned}
&\mathcal{H}_T = \hat{\mathcal{H}}_T + \partial_\alpha \bar{D}^\alpha\,, \\
&\hat{\mathcal{H}}_T = b^i{}_0 \bar{\mathcal{H}}_i + \omega^i{}_0 \bar{\mathcal{K}}_i + u^i{}_0 \pi_i{}^0 + v^i{}_0 \Pi_i{}^0\,,
\end{aligned} \tag{A1a}$$

where

$$\begin{aligned}
&\bar{\mathcal{H}}_i = \mathcal{H}_i - \nabla_\beta \phi_i{}^\beta + \varepsilon_{imn} b^m{}_\beta \left(p\phi^{n\beta} + q\Phi^{n\beta}\right)\,, \\
&\bar{\mathcal{K}}_i = \mathcal{K}_i - \nabla_\beta \Phi_i{}^\beta - \varepsilon_{imn} b^m{}_\beta \phi^{n\beta}\,. \\
&\bar{D}^\alpha = b^i{}_0 \pi_i{}^\alpha + \omega^i{}_0 \Pi_i{}^\alpha\,.
\end{aligned} \tag{A1b}$$

The constraints $(\pi_i{}^0, \Pi_i{}^0, \bar{\mathcal{H}}_i, \bar{\mathcal{K}}_i)$ are first class, while $(\phi_i{}^\alpha, \Phi_i{}^\alpha)$ are second class.

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