Quantization of the Riemann Zeta-Function and Cosmology

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Abstract

Quantization of the Riemann zeta-function is proposed. We treat the Riemann zeta-function as a symbol of a pseudodifferential operator and study the corresponding classical and quantum field theories. This approach is motivated by the theory of p-adic strings and by recent works on stringy cosmological models. We show that the Lagrangian for the zeta-function field is equivalent to the sum of the Klein-Gordon Lagrangians with masses defined by the zeros of the Riemann zeta-function. Quantization of the mathematics of Fermat-Wiles and the Langlands program is indicated. The Beilinson conjectures on the values of L-functions of motives are interpreted as dealing with the cosmological constant problem. Possible cosmological applications of the zeta-function field theory are discussed.

1. Introduction

Recent astrophysical data require rather exotic field models that can violate the null energy condition (see [1] and refs. therein). The linearized equation for the field ϕ has the form

$$F(\Box)\phi = 0\,,$$

where $\Box$ is the d'Alembert operator and F is an analytic function. Stringy models provide a possible candidate for this type of models. In particular, in this context p-adic string models [2, 3, 4] have been considered. p-Adic cosmological stringy models are supposed to incorporate essential features of usual string models [5, 6, 7].

An advantage of the p-adic string is that it can be described by an effective field theory including just one scalar field. In this effective field theory the kinetic term contains an operator

$$p^{\Box}\,, \tag{1}$$

where p is a prime number. In the spirit of the adelic approach [3, 8] one could try an adelic product $\prod_p p^{\Box}$, but it has no obvious mathematical meaning. There is the celebrated adelic Euler formula for the zeta-function [9]

$$\zeta(s) = \prod_p (1-p^{-s})^{-1}\,,$$

therefore a natural operator should be related with the Riemann zeta-function $\zeta(s)$. Due to the Riemann hypothesis on zeros of the zeta-function the most interesting operator is the following pseudodifferential operator,

$$\zeta\Big(\frac{1}{2}+i\Box\Big)\,. \tag{2}$$

We call this operator the *quantum zeta-function*. We consider also quantization of the Riemann ξ-function and various L-functions. Euclidean version of the quantum zeta-function is $\zeta(\frac{1}{2}+i\Delta)$ where Δ is the Laplace operator.

One can wonder what is special about the zeta-function and why we do not consider an arbitrary analytic function? One of the reasons is that there is a universality of the Riemann zeta-function. Any analytic function can be approximated by the Riemann zeta-function. More precisely, any nonvanishing analytic function can be approximated uniformly by certain purely imaginary shifts of the zeta-function in the critical strip (Voronin's theorem [9]). Remind that the Riemann zeta-function has appeared in different problems in mathematical and theoretical physics, see [8, 10].

Under the assumption that the Riemann hypothesis is true we show that the Lagrangian for the zeta-function field is equivalent to the sum of the Klein-Gordon Lagrangians with masses defined by the zeros of the Riemann zeta-function at the critical line. If the Riemann hypothesis is not true then the mass spectrum of the field theory is different.

An approach to the derivation of the mass spectrum of elementary particles by using solutions of the Klein-Gordon equation with finite action is considered in [11].

We can quantize not only the Riemann zeta-function but also more general L-functions. Quantization of the mathematics of Fermat-Wiles [12] and the Langlands program [13] is indicated.

The paper is organized as follows. In Section 2 we collect an information about the Riemann zeta-function. In Section 3 we present a classical field theory with the zeta-function kinetic term and discuss solutions of the model in Minkowski space. Then we study dynamics in the Friedmann metric in some approximation and discuss cosmological properties of the constructed solutions. In Section 4 there are few comments about the corresponding quantum theory and in Sections 5 and 6 we consider modifications of the theory where instead of the zeta-function kinetic term the Riemann-Siegel function or L-function are taken.

2. Riemann Zeta-Function

Here we collect some information about the Riemann zeta-function which we shall use in the next section to study the zeta-function field theory. The Riemann zeta-function is defined as

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}\,,\qquad s=\sigma+i\tau,\ \sigma>1\,, \tag{3}$$

and there is an Euler adelic representation

$$\zeta(s) = \prod_p (1 - p^{-s})^{-1} \tag{4}$$

(we write τ instead of t which is usually used for the imaginary part of s since we shall use t as a time variable). The zeta-function admits an analytic continuation to the whole complex plane s except the point $s = 1$ where it has a simple pole. It goes as follows. We define the theta-function

$$\theta(x) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 x}, \quad x > 0\,, \tag{5}$$

which satisfies the functional equation

$$\theta(x) = \frac{1}{\sqrt{x}}\,\theta\Big(\frac{1}{x}\Big) \tag{6}$$

and is an example of a modular form. Roughly speaking the Riemann zeta-function $\zeta(s)$ is the Mellin transform of the theta-function $\theta(x)$.

The "Langlands philosophy" [13] says that all reasonable generalizations of the Riemann zeta-function are related with modular forms, see Section 6.

One introduces the Riemann ξ-function

$$\xi(s) = \frac{s(s-1)}{2}\,\pi^{-\frac{s}{2}}\,\Gamma\Big(\frac{s}{2}\Big)\,\zeta(s) \tag{7}$$

where $\Gamma(z)$ is the gamma function and obtains the expression

$$\xi(s) = \frac{s(s-1)}{2}\int_1^{\infty} (x^{\frac{s}{2}-1} + x^{-\frac{s}{2}-\frac{1}{2}})\,\omega(x)\,dx + \frac{1}{2}\,, \tag{8}$$

where

$$\omega(x) = \frac{1}{2}\,(\theta(x) - 1) = \sum_{n=1}^{\infty} e^{-\pi n^2 x}\,. \tag{9}$$

The right-hand side in (8) is meaningful for all values $s \in \mathbb{C}$, since the integral converges by virtue of the exponential decay of $\omega(x)$. The gamma-function $\Gamma(s/2)$ never vanishes. The Riemann ξ-function $\xi(s)$ is an entire function.

The zeros of $\zeta(s)$ lie in the critical strip $0 < \sigma < 1$ with the exception of the "trivial zeros" at $s = -2, -4, -6, ...$ They are situated symmetrically about the real axis $\tau = 0$ and the critical line $\sigma = 1/2$. If ρ is a nontrivial zero then $\bar{\rho}, 1 - \rho$ and $1 - \bar{\rho}$ are also zeros. If $N(T)$ is the number of zeros in the critical strip,

$$N(T) = \#\{\rho = \beta + i\gamma : 0 \le \beta \le 1,\ 0 \le \gamma \le T\}\,, \tag{10}$$

then

$$N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T)\,. \tag{11}$$

The Riemann hypothesis asserts that all nontrivial zeros ρ lie at the critical line: $\rho = \frac{1}{2} + i\tau$. There is a conjecture that all zeros are simple. The first few zeros occur approximately at the following values:
$\tau = 14.1,\ 21.0,\ 25.0,\ 30.4,\ 32.9$. The corresponding negative values are also zeros.

The zeros of the ξ-function are the same as the nontrivial zeros of the ζ-function. It is known that $\xi(\frac{1}{2} + i\tau)$ is real for real τ and is bounded. Locating zeros on the critical line of the (complex) zeta function reduces to locating zeros on the real line of the real function $\xi(\frac{1}{2} + i\tau)$.

There is the functional equation

$$\xi(s) = \xi(1-s) \tag{12}$$

and the Hadamard representation for the ξ-function

$$\xi(s) = \frac{1}{2}\, e^{as} \prod_{\rho} \left(1 - \frac{s}{\rho}\right) e^{s/\rho}\,. \tag{13}$$

Here ρ are nontrivial zeros of the zeta-function and

$$a = -\frac{1}{2}\,\gamma - 1 + \frac{1}{2} \log 4\pi \tag{14}$$

where γ is Euler's constant. The series

$$\sum_{\rho} \frac{1}{|\rho|^{1+\epsilon}} \tag{15}$$

converges for any $\epsilon > 0$ but diverges if $\epsilon = 0$.

3. Zeta-Function Classical Field Theory

3.1. Minkowski space

If $F(\tau)$ is a function of a real variable τ then we define a pseudodifferential operator $F(\Box)$ [14] by using the Fourier transform

$$F(\Box)\phi(x) = \int e^{ixk} F(k^2) \tilde{\phi}(k) dk\,. \tag{16}$$

Here $\Box$ is the d'Alembert operator

$$\Box = -\frac{\partial^2}{\partial x_0^2} + \frac{\partial^2}{\partial x_1^2} + ... + \frac{\partial^2}{\partial x_{d-1}^2}\,, \tag{17}$$

$\phi(x)$ is a function from $x \in \mathbb{R}^d$, $\tilde{\phi}(k)$ is the Fourier transform and $k^2 = k_0^2 - k_1^2 - ... - k_{d-1}^2$. We assume that the integral (16) converges, see [15] for a consideration of one-dimensional p-adic field equations.

One can introduce a natural field theory related with the real valued function $F(\tau) = \xi(\frac{1}{2} + i\tau)$ defined by means of the zeta-function. We consider the following Lagrangian

$$\mathcal{L} = \phi\,\xi\Big(\frac{1}{2} + i\Box\Big)\phi\,. \tag{18}$$

The integral

$$\xi\Big(\frac{1}{2} + i\Box\Big)\phi(x) = \int e^{ixk}\xi\Big(\frac{1}{2} + ik^2\Big)\tilde{\phi}(k)dk \tag{19}$$

converges if $\phi(x)$ is a decreasing function since $\xi(\frac{1}{2} + i\tau)$ is bounded. Note that the operator (19) does not belong to the class of the standard pseudodifferential operators considered in [14].

By using (8) the quantized ξ-function can be expressed as

$$\xi\Big(\frac{1}{2} + i\Box\Big) = \frac{1}{2} - \Big(\Box^2 + \frac{1}{4}\Big)\int_1^\infty x^{-\frac{3}{4}}\cos\Big[\frac{\Box}{2}\log x\Big]\omega(x)dx\,. \tag{20}$$

The operator $\xi(\frac{1}{2} + i\Box)$ (or $\zeta(\frac{1}{2} + i\Box)$) is the first quantization of the Riemann zeta-function. Similarly one can define operators $\zeta(\sigma + i\Box)$ and also $\zeta(\sigma + i\Delta)$ and $\zeta(1 + \epsilon - \Delta), \epsilon > 0$ where Δ is the Laplace operator. We will obtain the second quantization of the Riemann zeta-function when we quantize the field $\phi(x)$.

Suppose that the Riemann hypothesis is true and all zeros are simple. Let us prove the following

Proposition 1. *The Lagrangian (18) is equivalent to the following Lagrangian*

$$\mathcal{L}' = \sum_{\epsilon,n}\eta_{\epsilon n}\psi_{\epsilon n}(\Box + \epsilon\, m_n^2)\psi_{\epsilon n}\,, \tag{21}$$

where the notations are defined below.

Let

$$\rho_n = \frac{1}{2} + i\,m_n^2, \qquad \bar{\rho}_n = \frac{1}{2} - i\,m_n^2, \quad m_n > 0, \quad n = 1, 2, ... \tag{22}$$

be the zeros at the critical line.

We shall show that the zeros m_n^2 of the Riemann zeta-function become the masses of elementary particles in the Klein-Gordon equation.

From the Hadamard representation (13) we get

$$\xi\Big(\frac{1}{2}+i\tau\Big)=\frac{C}{2}\prod_{n=1}^{\infty}\Big(1-\frac{\tau^2}{m_n^4}\Big) \tag{23}$$

because

$$-\frac{1}{2}\gamma-1+\frac{1}{2}\log 4\pi+\sum_{n=1}^{\infty}\frac{1}{(1/4)+m_n^4}=0\,. \tag{24}$$

Here

$$C=\prod_n\frac{1}{1+\frac{1}{4m_n^4}}\,. \tag{25}$$

Remind that

$$m_n^2\sim 2\pi n/\log n,\qquad n\longrightarrow\infty\,. \tag{26}$$

It will be convenient to write the formula (23) in the form

$$\xi\Big(\frac{1}{2}+i\tau\Big)=\frac{C}{2}\prod_{\epsilon,n}\Big(1+\frac{\tau}{\epsilon\, m_n^2}\Big)\,, \tag{27}$$

where $\epsilon=\pm 1$ and a regularization is assumed. Then our Lagrangian (18) takes the form

$$\mathcal{L}=\phi\,\xi\Big(\frac{1}{2}+i\Box\Big)\phi=\frac{C}{2}\phi\prod_{\epsilon,n}\Big(1+\frac{\Box}{\epsilon\, m_n^2}\Big)\phi\,. \tag{28}$$

Now if we define the fields $\psi_{\epsilon n}$ as

$$\psi_{\epsilon_0 n_0}\;=\;\frac{C}{2\,m_{n_0}^2}\prod_{\epsilon,n\neq\epsilon_0,n_0}\left(1+\frac{\Box}{\epsilon\, m_n^2}\right)\phi\,, \tag{29}$$

and the real constants $\eta_{\epsilon n}$ by

$$\eta_{\epsilon n}\;=\;\frac{1}{i\,\xi'(\frac{1}{2}-i\epsilon\, m_n^2)}\,, \tag{30}$$

then it is straightforward to see that the Lagrangians (18) and (21) are equivalent. The proposition is proved. $\Box$

If the Riemann hypothesis is not true then one can obtain a modification of the proposition by adding terms which come from the complex zeros off the critical line. If $\rho=\beta+i\gamma,(\beta\neq 1/2)$ is a zero of the zeta-function then we obtain the following operator

$$(\Box+\omega)(\Box+\bar{\omega})(\Box-\omega)(\Box-\bar{\omega})$$

where $\omega=\gamma+i(1/2-\beta)$. Similar but different Lagrangians are considered in [1, 16].

3.2. The state parameter

The energy and pressure corresponding to the individual space homogeneous (depending only on time $x_0 = t$) fields $\psi = \psi_{\epsilon n}(t)$ in (21) are [17]

$$E = \eta(\,\dot\psi^2 - \epsilon\, m^2\psi^2), \qquad P = \eta(\,\dot\psi^2 + \epsilon\, m^2\psi^2)\,. \tag{31}$$

Solutions of the equation of motion for ψ

$$\ddot\psi - \epsilon\ m^2\psi = 0 \tag{32}$$

are

$$\psi\ =\ Ae^{mt} + Be^{-mt}, \qquad \epsilon = +1\,, \tag{33}$$

$$\psi\ =\ F\sin(m(t-t_0)), \qquad \epsilon = -1\,, \tag{34}$$

where A, B, F and t_0 are real constants. The state parameter $w = P/E$ corresponding to the solution with $\epsilon = +1$ is

$$w = -\frac{1}{2AB}\,(A^2e^{2mt} + B^2e^{-2mt})\,,$$

and with $\epsilon = -1$ is

$$w = \cos(2m(t-t_0))\,.$$

We see that if $\epsilon = +1$ the state parameter w can be positive or negative depending on the value of AB. In the case $\epsilon = -1$ the state parameter oscillates. The behavior of the state parameter w is important for cosmological applications, see for example [18].

3.3. Zeta-Function Field Theory in Curved Space-Time

We couple the zeta-function field with gravity. We consider the Lagrangian

$$\mathcal{L} = R + \phi F(\Box)\phi + \Lambda\,. \tag{35}$$

Here R is the scalar curvature of the metric tensor $g_{\mu\nu}$ with the signature $(-+++)$, Λ is a (cosmological) constant and $\Box$ is the d'Alembertian

$$\Box = \frac{1}{\sqrt{-g}}\,\partial_\mu(\sqrt{-g}\,g^{\mu\nu})\partial_\nu\,.$$

Such a Lagrangian for various choices of the function $F(\Box)$ was considered in [1, 18]. For the zeta-function field theory, when $F(\Box) = \xi(\frac{1}{2} + i\Box)$ there is a problem of how to define rigorously the operator $\xi(\frac{1}{2} + i\Box)$ in the curved space-time. One can prove, by using a regularized Hadamard product, similarly to the previous proposition, that the Lagrangian (35) is equivalent to the following Lagrangian

$$\mathcal{L}' = R + \sum_{\epsilon,n}\eta_{\epsilon n}\psi_{\epsilon n}(\Box + \epsilon\, m_n^2)\psi_{\epsilon n} + \Lambda\,. \tag{36}$$

Then one can study various cosmological solutions along the lines of [18]. As compare to models considered in [18] in (36) there is no an extra exponent of an entire function. This permits to find deformations of the model (35) that admits exact solutions.

For a homogeneous, isotropic and the spatially flat Universe with the metric

$$ds^2 = -dt^2 + a^2(t)\left(dx_1^2 + dx_2^2 + dx_3^2\right), \tag{37}$$

we obtain the Friedmann equations

$$H^2 = \frac{1}{6}\sum_{\epsilon,n}\eta_{\epsilon n}(\dot\psi_{\epsilon n}^2 - \epsilon\, m_n^2\psi_{\epsilon n}^2) - \Lambda/3\,, \tag{38}$$

$$\dot H = -\frac{1}{2}\sum_{\epsilon,n}\eta_{\epsilon n}\dot\psi_{\epsilon n}^2\,, \qquad \ddot\psi_{\epsilon n} + 3H\dot\psi_{\epsilon n} - \epsilon\, m_n^2\psi_{\epsilon n} = 0\,,$$

where the Hubble parameter $H = \dot a/a$. For a mode with $\epsilon = -1$, $\eta_{\epsilon n} > 0$ it is proved in [19] that there is a cosmological initial space-time singularity. The problem of the cosmological singularity is considered in [20].

Let us consider examples of solutions of these equations. Consider the case when there is only one mode.

Example 1. Scalar field with the cosmological constant. The system of equations

$$H^2 = \frac{1}{6}\eta\,(\dot\psi^2 - \epsilon\, m^2\psi^2) - \Lambda/3\,, \quad \dot H = -\frac{1}{2}\eta\,\dot\psi^2\,, \quad \ddot\psi + 3H\dot\psi - \epsilon\, m^2\psi = 0$$

where $\eta\,\epsilon < 0$ and $\Lambda = -\epsilon\, m^2/3$ has the solution without singularities

$$\psi = -2c(t+t_0)/\eta, \quad H = -2c^2(t+t_0)/\eta\,, \quad a = A\exp\Big[-\frac{2c^2}{\eta}\big(\frac{1}{2}t^2 + t_0 t\big)\Big]\,.$$

Here $c^2 = -\eta\epsilon m^2/6$ while t_0 and A are arbitrary constants. We do not discuss here the question of stability of this particular solution.

Example 2. Massless scalar field. The system of equations

$$H^2 = \frac{1}{6}\eta\,\dot\psi^2\,, \qquad \dot H = -\frac{1}{2}\eta\,\dot\psi^2\,, \qquad \ddot\psi + 3H\dot\psi = 0$$

where $\eta > 0$ and $t > 0$ has the solution

$$\psi = \pm\sqrt{\frac{2}{3\eta}}\log\frac{t}{t_0}, \qquad H = \frac{1}{3t}\,, \qquad a = At^{1/3}\,.$$

Here $t_0 > 0$ and A are arbitrary constants.

For the constant Hubble parameter, $H = H_0$, one takes the solution of the last equation in (38) in the form $\psi = \exp(\alpha t)$. Then the spectrum equation is

$$\alpha^2 + 3H_0\alpha = \epsilon\, m^2\,, \tag{39}$$

and we have a deformation of the mass spectrum.

The Hubble parameter H_0 is related with the cosmological constant Λ as $H_0 = \sqrt{\Lambda/6}$. If the zeta-function field theory would be a fundamental theory then we obtain a relation between the Riemann zeros and the cosmological constant. This gives an additional support to the proposal [21] that the Beilinson conjectures [22] on the values of L-functions of motives can be interpreted as dealing with the cosmological constant problem.

4. The Zeta-Function Quantum Field Theory

To quantize the zeta-function classical field $\phi(x)$ which satisfies the equation in the Minkowski space

$$F(\Box)\phi(x) = 0\,, \tag{40}$$

where $F(\Box) = \xi(\frac{1}{2} + i\Box)$ we can try to interpret $\phi(x)$ as an operator valued distribution in a space $\mathcal{H}$ which satisfies the equation (40). We suppose that there is a representation of the Poincare group and a vacuum vector $|0\rangle$ in $\mathcal{H}$ though the space $\mathcal{H}$ might be equipped with indefinite metric and the Lorentz invariance can be violated. The Wightman function

$$W(x-y) = \langle 0|\phi(x)\phi(y)|0\rangle$$

is a solution of the equation

$$F(\Box)W(x) = 0\,. \tag{41}$$

By using Proposition we can write the formal Kallen-Lehmann [17] representation

$$W(x) = \sum_{\epsilon n} \int e^{ixk} f_{\epsilon n}(k)\delta(k^2 + \epsilon m_n^2)dk\,.$$

A mathematical meaning of this formal expression requires a further investigation.

Quantization of the fields $\psi_{\epsilon n}$ with the Lagrangian (21) can be performed straightforwardly. We will obtain ordinary scalar fields as well as ghosts and tachyons. Remind that tachyon presents in the Veneziano amplitude. It was removed by using supersymmetry. In Section 6 we shall discuss an approach of how to use a Galois group and quantum L-functions instead of supersymmetry to improve the spectrum.

5. Quantization of the Riemann-Siegel Function

One introduces also another useful function

$$Z(\tau) == e^{i\vartheta(\tau)}\zeta\Big(\frac{1}{2}+i\tau\Big) \tag{42}$$

where

$$e^{i\vartheta(\tau)} = \pi^{-i\tau/2}\frac{\Gamma(\frac{1}{4}+\frac{i\tau}{2})}{|\Gamma(\frac{1}{4}+\frac{i\tau}{2})|}\,.$$

The function $Z(\tau)$ is called the Riemann-Siegel (or Hardy) function [9]. It is known that $Z(\tau)$ is real for real τ , $|Z(\tau)| = |\zeta(\tau)|$ and there is a bound

$$Z(\tau) = O(|\tau|^{\frac{1}{6}+\epsilon}), \quad \epsilon > 0\,. \tag{43}$$

Note that the Riemann hypothesis implies the Lindelöf hypothesis: For any fixed $\epsilon > 0$ and $\sigma \geq 1/2$,

$$\zeta(\sigma+\tau) = O(|\tau|^{\epsilon}), \qquad |\tau| \longrightarrow \infty\,.$$

Values g_n such that

$$\vartheta(g_n) = \pi n, \qquad n = 0, 1, ... \tag{44}$$

are known as Gram points. Gram's empirical "law" is the tendency for zeros of the Riemann-Siegel function to alternate with Gram points:

$$(-1)^n Z(g_n) > 0\,. \tag{45}$$

An excellent approximation for the Gram point is given by the formula

$$g_n \simeq 2\pi\exp\Big[1+W\Big(\frac{8n+1}{8e}\Big)\Big]\,, \tag{46}$$

where W is the Lambert function, the inverse function of $f(W) = We^W$. Note that the Lambert function appears also in the consideration of the spectrum of the nonlocal cosmological model in [18]. We have the asymptotic behavior

$$W(x) \sim \log x - \log\log x, \qquad x \longrightarrow \infty\,,$$

hence from (46) we get

$$g_n \sim 2\pi n/\log n$$

(compare with the asymptotic behavior of the Riemann zeros (26)).

One can introduce a natural field theory related with the real valued functions $Z(\tau)$ defined by means of the zeta-function by considering the following Lagrangian

$$\mathcal{L} = \phi Z(\Box)\phi\,.$$

The integral (16) converges if $\phi(x)$ is a decreasing function since there is the bound (43).

A generalization of the Riemann zeta-function is studied in [23].

It would be interesting also to study the corresponding "heat equation"

$$\frac{\partial u}{\partial t} = F(\Delta)u\,,$$

where Δ is the Laplace operator. The p-adic heat equation has been considered in [24].

6. Quantum L-functions

In this section we briefly discuss L-functions and their quantization. The role of (super) symmetry group here is played by the Galois group, see [3, 4, 21].

For any character χ to modulus q one defines the corresponding Dirichlet L-function [9] by setting

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}\,, \qquad (\sigma > 1).$$

If χ is primitive then $L(s,\chi)$ has an analytic continuation to the whole complex plane. One introduces the function

$$\xi(s,\chi) = \left(\frac{\pi}{k}\right)^{-(s+a)/2} \Gamma\left(\frac{s}{2}+\frac{a}{2}\right) L(s,\chi)\,,$$

where k and a are some parameters. The ξ-function is an entire function and it has a representation similar to (13)

$$\xi(s,\chi) = Ae^{Bs}s^k \prod_{\rho} \left(1-\frac{s}{\rho}\right) e^{s/\rho}\,, \qquad (47)$$

where A, B and k are constants. The zeros ρ lie in the critical strip and symmetrically distributed about the critical line $\sigma = 1/2$. It is important to notice that unless χ is real the zeros will not necessary be symmetric about the real line. Therefore if we quantize the L- function by considering the pseudodifferential operator

$$L\Big(\frac{1}{2}+i\Box,\chi\Big)\,,$$

and the Lagrangian

$$\mathcal{L} = \phi\, L\Big(\frac{1}{2}+i\Box,\chi\Big)\,\phi\,,$$

then we could avoid the appearance of tachyons and/or ghosts by choosing an appropriate character χ. To get the Hermitian Lagrangian we could take the real or imaginary part of the previous expression.

There could exist a nontrivial real zero of the L-function which corresponds to a massless excitation. Hardy and Littlewood noted that the location of nontrivial real zeros of the L-function is related with the Goldbach conjecture which was proved by Vinogradov [9].

The Taniyama-Shimura conjecture relates elliptic curves and modular forms. It asserts that if E is an elliptic curve over $\mathbb{Q}$, then there exists a wight-two cusp form f which can be expressed as the Fourier series

$$f(z) = \sum a_n e^{2\pi nz}$$

with the coefficients a_n depending on the curve E. Such a series is a modular form if and only if its Mellin transform, i.e. the Dirichlet L-series

$$L(s,f) = \sum a_n n^{-s}$$

has a holomorphic extension to the full s-plane and satisfies a functional equation. For the elliptic curve E we obtain the L-series $L(s,E)$. The Taniyama-Shimura conjecture was proved by Wiles and Taylor for semistable elliptic curves and it implies Fermat's Last Theorem [12].

Let E be an elliptic curve defined over $\mathbb{Q}$, and $L(s) = L_K(s,E)$ be the L-function of E over the field K. The Birch and Swinnerton-Dyer conjecture asserts that

$$ord_{s=1} L(s) = r\,,$$

where r is the rank of the group $E(K)$ of points of E defined over K. The quantum L-function in this case has the form

$$L(1+i\Box) = A(i\Box)^r + ...$$

where the leading term is expressed in terms of the Tate-Shafarevich group.

Wiles's work can be viewed as establishing connections between the automorphic forms and the representation theory of the adelic groups and the Galois representations. Therefore it can be viewed as part of the Langlands program in *number theory* and the representation theory [13] (for a recent consideration of the *geometrical* Langlands program see [25]).

Let G be the Galois group of a Galois extension of $\mathbb{Q}$ and α a representation of G in $\mathbb{C}^n$. There is the Artin L-function $L(s,\alpha)$ associated with α. Artin conjectured that $L(s,\alpha)$ is entire, when α is irreducible, and moreover it is automorphic: there exists a modular form f such that

$$L(s,\alpha) = L(s,f)\,.$$

Langlands formulated the conjecture that $L(s,\alpha)$ is the L-function associated to an automorphic representation of $GL(n,\mathbb{A})$ where $\mathbb{A} = \mathbb{R} \times \prod_p' \mathbb{Q}_p$

is the ring of adeles of $\mathbb{Q}$ (restricted product). Here $\mathbb{Q}_p$ is the field of p-adic numbers. Let π be an automorphic cuspidal representation of $GL(n, \mathbb{A})$ then there is a L-function $L(s, \pi)$ associated to π. Langlands conjecture (general reciprocity law) states: Let K be a finite extension of $\mathbb{Q}$ with Galois group G and α be an irreducible representation of G in $\mathbb{C}^n$. Then there exists an automorphic cuspidal representation π_α of $GL(n, \mathbb{A}_K)$ such that

$$L(s, \alpha) = L(s, \pi_\alpha)\,.$$

Quantization of the L-function can be performed similarly to the quantization of the Riemann zeta-function discussed above by considering the corresponding pseudodifferential operator $L(\sigma + i\Box)$ with some σ. One speculates that we can not observe the structure of space-time at the Planck scale but feel only its motive [21].

There is a conjecture that the zeros of L-functions are distributed like the eigenvalues of large random matrices from a gaussian ensemble, see [26]. The limit of large matrices in gauge theory (the master field) is derived in [27].

It would be interesting to explore for which manifolds the mass spectrum obtained in [11] can be represented by means of zeros of an L-function. Inverse problem for the L-functions: Which set of points in the complex plane can be the set of zeros of an L-function?

Investigation of the mass spectrum and field theories for quantum L-functions we leave for future work.

A preliminary version of this work is in the arxiv [28], Recently, an interesting consideration of a nonlinear field theory which includes the quantum zeta function $\zeta(\Box)$ was performed in [29].

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