Nonrenormalizability of θ-expanded noncommutative gauge theories*

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Abstract

We show that 3-vertices in noncommutative $SU(2)$ gauge theory with fermions are not renormalizable for the θ-expanded theory. This supports other results which indicate nonrenormalizability of noncommutative gauge theories.

1. Introduction and notation

The interest in noncommutative spaces was initiated after the work of Snyder [1], who proposed to use noncommutativity of coordinates as a regulator of ultraviolet divergencies in quantum field theory. This, physically very appealing idea, has not been in all details realized till today. Recent interest in noncommutativity is due to the results in string theory, as well as to the progress in understanding of geometry of noncommutative spaces.

In principle, there are many different noncommutative (NC) generalizations (‘quantizations’) of space-time manifolds. Further, on a given noncommutative space one can define various representations of fields. Therefore one speaks of many different noncommutative field theories in the literature: the way to define a specific one depends on the feature chosen to be extended to noncommutative region. The most investigated structure among noncommutative spaces is probably the flat space. It is defined so that the coordinates $\hat{x}^\mu$ commute into a constant:

$$[\,\hat{x}^\mu, \hat{x}^\nu\,] = i\,\theta^{\mu\nu}. \tag{1}$$

The commutation rule (1) is the same as the rule obeyed by the coordinates and the conjugate momenta in phase space. It is known that the

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correspondence between functions of noncommuting variables (operators $\hat{q}_i$ and $\hat{p}_i$ in quantum mechanics or $\hat{x}^\mu$ in noncommutative field theory) with ordinary functions on $\mathbf{R}^n$ can be obtained through the Weyl prescription [2]. The multiplication is given by the Moyal product

$$\phi(x) \star \chi(x) = e^{\frac{i}{2}\,\theta^{\mu\nu}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial y^\nu}}\phi(x)\,\chi(y)|_{y\to x}\ . \tag{2}$$

Clearly, in the case of noncompact manifold this representation is infinite dimensional. However, there are noncommutative spaces which have finite-dimensional representations. They are typically related to the group manifolds, as for example the fuzzy sphere or the fuzzy $\mathbf{CP}^n$ [3]. Field theories defined on such spaces are finite by construction; what is difficult to obtain is the Minkowski space in the commutative limit. Finally, one can consider symmetries as the primary objects and quantize them. Thus for example one gets the κ-Poincare (quantum) group as a generalization of Poincare symmetry ; the underlying noncommutative space is the κ-Minkowski space and has the structure of a Lie group [4].

In this paper we are dealing with the flat noncommutative Minkowski space. A couple of different representations of scalar and gauge fields were investigated in the literature, in particular their properties under quantization. Many results concern what can be called the 'coordinate representation', the one where noncommutative fields are directly represented by their Weyl transforms:

$$\hat{\phi}(\hat{x}) \mapsto \hat{\phi}(x) = \phi(x)\,, \qquad \hat{A}_\mu(\hat{x}) \mapsto \hat{A}_\mu(x) = A_\mu(x)\,. \tag{3}$$

The lagrangian of the NC theory is obtained from the commutative one by replacement of the ordinary by the Moyal product. Therefore the quantization is done straightforwardly; the effects of noncommutativity are solely in the interaction, so they be treated by the usual perturbative methods. However, perturbation theory gives the mixing of the ultraviolet with the infrared divergencies [5], which spoils renormalizability. There are further questions regarding this representation; for example, how to interpret the field $\phi(x^\mu)$ for arbitrary values of θ, and related to this, what are the plane waves around which one does the perturbative expansion?

For gauge theories, there is yet another representation of NC fields by functions on $\mathbf{R}^4$. The relation between them follows from the closure property of gauge symmetry and is given by the Seiberg-Witten (SW) map [6, 7] as an expansion in the noncommutativity parameter $\theta^{\mu\nu}$. To first order it reads:

$$\begin{aligned}\hat{A}_\rho(x) &= A_\rho(x) - \tfrac{1}{4}\,\theta^{\mu\nu}\left\{A_\mu(x), \partial_\nu A_\rho(x) + F_{\nu\rho}(x)\right\} + \dots\\ \hat{\psi}(x) &= \psi(x) - \tfrac{1}{2}\,\theta^{\mu\nu}A_\mu(x)\partial_\nu\psi(x) + \tfrac{i}{4}\,\theta^{\mu\nu}A_\mu(x)A_\nu(x)\psi(x) + \dots .\end{aligned} \tag{4}$$

The SW expansion is not given uniquely, which is an important fact. Classical noncommutative action is obtained by the insertion of (4) in the commutative action, also as a series in θ.

Noncommutative Minkowski space can be represented in the 'matrix base', too. If the kinetic part of the lagrangian is neglected (i.e., for large θ), the scalar field theory is equivalent to the matrix model [2]. There are also interesting new results for the full theory regarding renormalizability [8]. Renormalizability can be restored by adding a coordinate-dependent counterterm and thus breaking the translational symmetry. There are other interesting proposals to restore renormalizability by changing the original lagrangian. One of them, which preserves the space-time symmetries is given in [9]; however, the additional term changes the symmetry content of the theory.

2. Divergencies in $SU(2)$

Expansions (4) are general, valid for any gauge group. Here we restrict our discussion to the $SU(2)$ gauge theory with massless fermions in the fundamental representation. To linear order the noncommutative action is given by:

$$S = S_0 + S_{1,A} + S_{1,\psi}, \tag{5}$$

with

$$\begin{aligned} S_0 &= \int d^4x \, \left(\bar\psi(i\,\gamma^\mu D_\mu)\,\psi - \tfrac14\, F^{\mu\nu a}\, F^a_{\mu\nu}\right), \\ S_{1,A} &= 0, \\ S_{1,\psi} &= \tfrac12\,\theta^{\rho\sigma}\int d^4x\, \left(-i\,\bar\psi\,\gamma^\mu F_{\mu\rho}\, D_\sigma\psi - \tfrac{i}{2}\,\bar\psi\, F_{\rho\sigma}\gamma^\mu D_\mu\psi\right) \\ &= -\tfrac18\,\theta^{\rho\sigma}\Delta^{\mu\nu\alpha}_{\rho\sigma\beta}\int d^4x\,\bar\psi F_{\mu\nu}\gamma^\beta(i\partial_\alpha + A_\alpha)\psi, \end{aligned} \tag{6}$$

$\Delta^{\alpha\beta\gamma}_{\sigma\rho\mu} = \delta^\alpha_\sigma\delta^\beta_\rho\delta^\gamma_\mu - \delta^\alpha_\rho\delta^\beta_\sigma\delta^\gamma_\mu + (\text{cyclic } \alpha\beta\gamma) = -\epsilon^{\alpha\beta\gamma\lambda}\,\epsilon_{\sigma\rho\mu\lambda}$. The linear bosonic term $S_{1,A}$ does not vanish in the general case and it is proportional to the symmetric symbols, d_{abc}. The corrections $S_{1,A}$ and $S_{1,\psi}$ can be treated as couplings.

The fact that the 'coupling constant' $\theta^{\mu\nu}$ is dimensionfull implies immediately nonrenormalizability, unless there is some additional symmetry. The idea that nonuniqueness of the expansion (4) might play the role of a symmetry appeared in [10]. They found that the divergencies in the photon propagator in noncommutative electrodynamics can be, in θ-linear order, absorbed in a field redefinition; moreover, that such redefinition can be done in all orders. Let us formulate this more precisely. Write the SW expansion as

$$\hat A_\mu = \sum A^{(n)}_\mu\,, \qquad \hat\psi = \sum \psi^{(n)}\,, \tag{7}$$

where $A^{(n)}_\mu$, $\psi^{(n)}$ denote the terms of the n-th order in θ. Then the transformation

$$A^{(n)}_\mu \to A^{(n)}_\mu + \mathbf{A}^{(n)}_\mu\,, \qquad \psi^{(n)} \to \psi^{(n)} + \mathbf{\Psi}^{(n)}\,, \tag{8}$$

does not change noncommutative fields $\hat A_\mu$, $\hat\psi$ if $\mathbf{A}^{(n)}_\mu$, $\mathbf{\Psi}^{(n)}$ are gauge covariant expressions containing exactly n factors θ [10, 11]. The change of the action induced by (8) is

$$
\begin{aligned}
\Delta\, S^{(n,A)} &= \textstyle\int d^4x\, (D_\nu F^{\mu\nu})\mathbf{A}^{(n)}_\mu\,, \\
\Delta\, S^{(n,\psi)} &= \textstyle\int d^4x\, \Big(\bar\psi\, i\not{D}\mathbf{\Psi}^{(n)} + \bar{\mathbf{\Psi}}^{(n)} i\not{D}\psi\Big)\,.
\end{aligned}
\tag{9}
$$

Thus, along with S all actions $S + \Delta S^{(n)}$ describe the same physical theory. One might conclude that, if divergent terms in the effective action are of the form (9), the theory is renormalizable; the divergencies can be absorbed by the redefinition. The new, redefined fields are the physical ones. This generalizes the usual notion of renormalizability and gives an additional freedom to the theory.

A more detailed analysis of noncommutative electrodynamics with fermions was done in [12] to first order in θ. The conclusion was that all propagators and vertices, with the exception of the 4-fermion vertex, are renormalizable if fermions are massless. This result was confirmed to θ^2-order for the propagators in [13]. The divergencies of the one-loop effective action for $SU(2)$ gauge theory were calculated in [14]. The result for the covariantized divergent θ-linear 2-point function is

$$
\Gamma_2' = \frac{1}{(4\pi)^2\epsilon}\,\frac{i}{8}\,\theta^{\mu\nu}\epsilon_{\mu\nu\rho\sigma}\int d^4x\,\bar\psi\,\gamma_5\,\gamma^\sigma D^2 D^\rho\psi\,. \tag{10}
$$

The divergent 3-point function reads

$$
\begin{aligned}
\Gamma_3' \;=\; -\frac{1}{(4\pi)^2\epsilon}\,\frac{1}{2}\,\theta^{\mu\nu}\int d^4x\,\Big(&-\frac{37\,i}{4}\,\bar\psi\,\gamma^\alpha(F_{\nu\alpha}D_\mu + F_{\mu\nu}D_\alpha + F_{\alpha\mu}D_\nu\Big)\psi \\
&-\frac{6\,i}{4}\,\bar\psi\,\gamma^\alpha F_{\mu\nu}D_\alpha\psi - \frac{3\,i}{4}\,\bar\psi\,\gamma^\alpha(D_\alpha F_{\mu\nu})\,\psi \\
&+2\,i\bar\psi\,\gamma_\mu F_{\nu\alpha}\,D^\alpha\psi + i\,\bar\psi\,\gamma_\mu(D^\alpha F_{\nu\alpha})\psi \\
&+\frac{5}{8}\,\epsilon_{\nu\alpha\beta\rho}\,\bar\psi\,\gamma_5\,\gamma^\rho(D_\mu F^{\alpha\beta})\psi - \frac{1}{16}\,\epsilon_{\mu\nu\alpha\beta}\,\bar\psi\,\gamma_5\,\gamma^\sigma(D_\sigma F^{\alpha\beta})\psi \\
&+\frac{1}{8}\,\epsilon_{\mu\nu\alpha\beta}(2\,\bar\psi\,\gamma_5\,\gamma^\beta F^{\rho\alpha}D_\rho\psi + \bar\psi\,\gamma_5\,\gamma^\beta(D_\rho F^{\rho\alpha})\psi)\Big),
\end{aligned}
\tag{11}
$$

and finally, the 4-fermion vertex has the same form as in electrodynamics:

$$
\Gamma_{4\psi}' = \frac{1}{(4\pi)^2\epsilon}\,\frac{9}{32}\,\theta^{\mu\nu}\epsilon_{\mu\nu\rho\sigma}\int d^4x\,\bar\psi\,\gamma_5\,\gamma^\sigma\psi\,\bar\psi\,\gamma^\rho\psi\,. \tag{12}
$$

Apparently, Γ_2' and Γ_3' are of the 'allowed' form, given by (9). Therefore we could conclude that, as in $U(1)$, the only obstacle to renormalization is the term (12). However, to prove this we have to construct the field redefinition which removes the divergencies explicitly. From (10-11) we see that only the fermionic field need to be redefined. The most general first order redefinition is

$$\mathbf{\Psi}^{(1)} = \theta^{\mu\nu}\Big(\kappa_1 F_{\mu\nu} + i\kappa_2 \sigma_{\mu\rho} F_\nu{}^\rho + i\kappa_3 \epsilon_{\mu\nu\rho\sigma}\gamma_5 F^{\rho\sigma} + \kappa_4 \sigma_{\mu\nu} D^2 + \kappa_5 \sigma_{\rho\mu} D_\nu D^\rho\Big)\psi. \tag{13}$$

It has five free parameters, $\kappa_1, \ldots, \kappa_5$. The corresponding change in the classical lagrangian reads:

$$\begin{aligned}\Delta\,\mathcal{L}^{(1)} \;=\; & i\,\theta^{\alpha\beta}\Big((\kappa_1 - \frac{1}{2}\,\kappa_5)(\bar\psi\,\gamma^\mu (D_\mu F_{\alpha\beta})\psi + 2\,\bar\psi\,\gamma^\mu F_{\alpha\beta} D_\mu\psi)\\ & - \kappa_2(\bar\psi\,\gamma^\rho (D_\alpha F_{\beta\rho})\psi + 2\,\bar\psi\,\gamma^\rho F_{\beta\rho} D_\alpha\psi)\\ & + \kappa_2(\bar\psi\,\gamma_\alpha (D_\mu F_\beta{}^\mu)\psi + 2\,\bar\psi\,\gamma_\alpha F_\beta{}^\mu D_\mu\psi)\\ & + i\kappa_2 \epsilon_{\mu\alpha\rho\sigma}\,\bar\psi\,\gamma_5\,\gamma^\sigma (D^\mu F_\beta{}^\rho)\psi - i\kappa_3 \epsilon_{\alpha\beta\rho\sigma}\,\bar\psi\,\gamma_5\,\gamma^\mu (D_\mu F_{\rho\sigma})\psi\\ & + \kappa_4 \epsilon_{\rho\alpha\beta\sigma}\,\bar\psi\,\gamma_5\,\gamma^\sigma (2\,D^2 D^\rho\psi - 2\,i\,F^{\rho\mu} D_\mu\psi - i(D_\mu F^{\mu\rho})\psi)\\ & + 2\,\kappa_4(\bar\psi\,\gamma_\beta (D_\mu F_\alpha{}^\mu)\psi + 2\,\bar\psi\,\gamma_\beta F_{\alpha\mu} D^\mu\psi)\Big).\end{aligned} \tag{14}$$

Note that, as κ_1 and κ_5 appear only in the combination $\kappa_1 - \frac{\kappa_5}{2}$, one of these parameters is superfluous; we fix $\kappa_5 = 0$. If we perform the change $\psi \to \psi + \mathbf{\Psi^{(1)}}$ in the renormalized Lagrangian [15], for the following choice of the parameters

$$\kappa_1 = \frac{g}{16\,(4\pi)^2\epsilon}\;, \quad \kappa_2 = \frac{5\,g}{8\,(4\pi)^2\epsilon}\;, \quad \kappa_3 = \frac{g}{32\,(4\pi)^2\epsilon}\;, \quad \kappa_4 = \frac{g}{16\,(4\pi)^2\epsilon}\;,$$

the renormalized lagrangian reduces to

$$\begin{aligned}\mathcal{L} + \mathcal{L}_{ct} + \Delta\,\mathcal{L}^{(1)} = & -\frac{1}{4}\,F^a_{\mu\nu}\,F^{\mu\nu a}\Big(1 + \frac{40\,g^2}{3\,(4\pi)^2\epsilon}\Big)\\ & + i\,\bar\psi\,\partial\!\!\!/\psi\Big(1 - \frac{3\,g^2}{2\,(4\pi)^2\epsilon}\Big) + g\mu^{\frac{\epsilon}{2}}\,\bar\psi A^a_\mu \gamma^\mu\,\frac{\sigma^a}{2}\,\psi\Big(1 - \frac{3\,g^2}{2\,(4\pi)^2\epsilon}\Big)\\ & - \frac{i}{4}\,g\,\mu^{\frac{\epsilon}{2}}\theta^{\alpha\beta}\,\bar\psi\,\gamma^\mu (F_{\beta\mu} D_\alpha + F_{\alpha\beta} D_\mu + F_{\mu\alpha} D_\beta)\psi\Big(1 + \frac{21\,g^2}{(4\pi)^2\epsilon}\Big)\\ & + i\,g\,\mu^{\frac{\epsilon}{2}}\theta^{\alpha\beta}(\bar\psi\,\gamma_\alpha (D^\mu F_{\beta\mu})\psi + 2\,\bar\psi\,\gamma_\alpha\,F_{\beta\mu} D^\mu\psi)\frac{g^2}{(4\pi)^2\epsilon}\,.\end{aligned} \tag{15}$$

If one introduced the bare fields and the bare couplings as

$$\begin{aligned}\psi_0 &= \sqrt{1 - \frac{3\,g^2}{2\,(4\pi)^2\epsilon}}\,\psi\,,\\ A^\mu_0 &= \sqrt{1 + \frac{40\,g^2}{3\,(4\pi)^2\epsilon}}\,A^\mu\,,\end{aligned} \tag{16}$$

$$
\begin{aligned}
g_0 &= g\mu^{\frac{\epsilon}{2}}\Big(1 - \frac{20\, g^2}{3\,(4\pi)^2\epsilon}\Big),\\
\theta_0^{\alpha\beta} &= \theta^{\alpha\beta}\Big(1 + \frac{45\, g^2}{2\,(4\pi)^2\epsilon}\Big),
\end{aligned}
$$

the renormalized lagrangian would be

$$
\begin{aligned}
\mathcal{L} + \mathcal{L}_{ct} + \Delta\,\mathcal{L}^{(1)} = &-\frac{1}{4}\, F^a_{\mu\nu 0}F_0^{\mu\nu a} + i\,\bar\psi_0\,\not{\partial}\psi_0 + g_0\,\bar\psi_0\,\gamma_\mu\, A_0^{\mu a}\,\frac{\sigma^a}{2}\,\psi_0\\
&- \frac{i}{4}\, g_0\,\theta_0^{\alpha\beta}\bar\psi_0(F_{\beta\mu 0}D_\alpha + F_{\alpha\beta 0}D_\mu + F_{\mu\alpha 0}D_\beta)\psi_0 \qquad (17)\\
&+ i\, g\mu^{\frac{\epsilon}{2}}\,\theta^{\alpha\beta}(\bar\psi\,\gamma_\alpha(D^\mu F_{\beta\mu})\,\psi + 2\,\bar\psi\,\gamma_\alpha F_{\beta\mu}D^\mu\psi)\frac{g^2}{(4\pi)^2\epsilon}\,.
\end{aligned}
$$

But the last term, divergent, remains: it cannot be absorbed by the SW the redefinition or by the standard renormalizion procedure.

3. Conclusion

We saw that the SW redefinition does not possess enough free parameters to absorb the divergencies of the 3-point functions. Thus, apart from the 4-fermion vertex, 3-vertices are also not renormalizable. This result differs from the corresponding one for noncommutative $U(1)$, [12]. The negative result about renormalizability given here is conclusive, as clearly it cannot be altered in higher orders in θ. The presence of fermions prevents renormalizability, except of course in the supersymmetric theories. On the other hand, the behavior of the pure gauge theories is still not fully clear and deserves further investigation.

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