Renormalizable Theories from Fuzzy Higher Dimensions *

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Abstract

We consider gauge theories defined in higher dimensions where the extra dimensions form a fuzzy space (a finite matrix manifold). We reinterpret these gauge theories as four-dimensional theories with Kaluza-Klein modes. We then perform a generalized à la Forgacs-Manton dimensional reduction. We emphasize some striking features emerging such as (i) the appearance of non-abelian gauge theories in four dimensions starting from an abelian gauge theory in higher dimensions, (ii) the fact that the spontaneous symmetry breaking of the theory takes place entirely in the extra dimensions and (iii) the renormalizability of the theory both in higher as well as in four dimensions.

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1. Introduction

In the recent years a huge theoretical effort has been devoted aiming to establish a unified description of all interactions including gravity. Out of this sustained endeavor, along the the superstring theory framework [1], the non-commutative geometry one has emerged [2, 3]. An interesting development worth noting was the observation that a natural realization of non-commutativity of space appears in the string theory context of D-branes in the presence of a constant antisymmetric field [4], which brought together the two approaches. However these very interesting approaches do not address as yet the usual problem of the Standard Model of Elementary Particle Physics, i.e. the presence of a plethora of free parameters to the ad-hoc introduction of the Higgs and Yukawa sectors in the theory. These sectors might have their origin in a higher-dimensional theory according to various schemes among whose the first one was the Coset Space Dimensional Reduction (CSDR) [5, 6, 7]. The CSDR scheme has been used to reduce in four dimensions a ten-dimensional, $N = 1, E_8$ gauge theory [6, 8, 9]and might be an appropriate reduction scheme of strings over nearly Kaehler manifolds [10]. More recently the dimensional reduction of gauge theories defined in higher dimensions where the extra dimensions form a fuzzy coset (a finite matrix manifold) has been examined [11, 12]. This might lead to interesting constructions for Particle Physics models . In the present paper we would like to present the main points and results of the dimensional reduction over fuzzy coset spaces and emphasize in particular the issue of the renormalizability of gauge theories defined with such a non-commutative setup.

In the CSDR one assumes that the form of space-time is $M^D = M^4 \times S/R$ with S/R a homogeneous space (obtained as the quotient of the Lie group S via the Lie subgroup R). Then a gauge theory with gauge group G defined on M^D can be dimensionally reduced to M^4 in an elegant way using the symmetries of S/R, in particular the resulting four-dimensional gauge group is a subgroup of G. Although the reduced theory in four dimensions is power counting renormalizable the full higher-dimensional theory is non-renormalizable with dimensionful coupling. The CSDR scheme reduces dimensionally a gauge theory with gauge group G defined on $M_4 \times S/R$ to a gauge theory on M_4 imposing the principle that fields should be invariant under the S action up to a G gauge transformation. The CSDR scheme constitutes an elegant and consistent truncation of the full theory in four dimensions, keeping only the first terms of the field expansion in higher harmonics of the compact coset spaces. When keeping all the higher harmonics, i.e. the Kaluza-Klein modes in four dimensions, then the theory in general becomes non-renormalizable as expected, since the theory was originally defined in higher than four dimensions. Still it is very interesting the fact that one can discuss the dependence of the couplings of the theory on the cutoff, or the beta-function of the couplings in the Wilson renormalization scheme [13, 14, 15]. In the fuzzy-CSDR we apply the CSDR principle in the case that the extra dimensions are a finite approximation of the homogeneous spaces S/R, i.e. a fuzzy coset. Fuzzy spaces are obtained by deforming the algebra of functions on their commutative parent spaces. The algebra of functions (from the fuzzy space to complex numbers) becomes finite dimensional and non-commutative, indeed it becomes a matrix algebra. Therefore, instead of considering the algebra of functions $Fun(M^D) \sim Fun(M^4) \otimes Fun(S/R)$ we consider the algebra $A = Fun(M^4) \otimes Fun((S/R)_F)$ where $Fun(M^4)$ is the usual commutative algebra of functions on Minkowski space M^4 and $Fun((S/R)_F) = M_N$ is the finite dimensional non-commutative algebra of $N \times N$ matrices that approximates the functions on the coset S/F. On this finite dimensional algebra we still have the action of the symmetry group S; this very property allows us to apply the CSDR scheme to fuzzy cosets. The reduction of a gauge theory defined on $M^4 \times (S/R)_F$ to a gauge theory on M^4 is a two step process. One first rewrites the higher-dimensional fields, that initially depends on the commutative coordinates x and the noncommutative ones X, in terms of only the commutative coordinates x, with the fields now being also $N \times N$ matrix valued. One then imposes the fuzzy-CSDR constraints on this four-dimensional theory. We can say that the theory is a higher-dimensional theory because the fuzzy space $(S/R)_F$ is a noncommutative approximation of the coset space S/R; in particular the spatial symmetry group S of the space $(S/R)_F$ is the same as that of the commutative space S/R. However the noncommutative theory has the advantage of being power counting renormalizable because $Fun((S/R)_F)$ is a finite dimensional space; it follows that also after applying the fuzzy-CSDR scheme we obtain a power counting renormalizable theory.

2. The Fuzzy sphere

The fuzzy sphere [16, 3] is a matrix approximation of the usual sphere S^2. The algebra of functions on S^2 (for example spanned by the spherical harmonics) is truncated at a given frequency and thus becomes finite dimensional. The truncation has to be consistent with the associativity of the algebra and this can be nicely achieved relaxing the commutativity property of the algebra. The fuzzy sphere is the "space" described by this non-commutative algebra. The algebra itself is that of $N \times N$ matrices. More precisely, the algebra of functions on the ordinary sphere can be generated by the coordinates of $\mathbf{R}^3$ modulo the relation $\sum_{\hat{a}=1}^{3} x_{\hat{a}} x_{\hat{a}} = r^2$. The fuzzy sphere S_F^2 at fuzziness level $N-1$ is the non-commutative manifold whose coordinate functions $iX_{\hat{a}}$ are $N \times N$ hermitian matrices proportional to the generators of the N-dimensional representation of $SU(2)$. They satisfy the condition $\sum_{\hat{a}=1}^{3} X_{\hat{a}} X_{\hat{a}} = \alpha r^2$ and the commutation relations

$$[X_{\hat{a}}, X_{\hat{b}}] = C_{\hat{a}\hat{b}\hat{c}} X_{\hat{c}} \ , \tag{1}$$

where $C_{\hat{a}\hat{b}\hat{c}} = \varepsilon_{\hat{a}\hat{b}\hat{c}}/r$ while the proportionality factor α goes as N^2 for N large. Indeed it can be proven that for $N \to \infty$ one obtains the usual commutative sphere.

On the fuzzy sphere there is a natural $SU(2)$ covariant differential calculus. This calculus is three-dimensional and the derivations $e_{\hat{a}}$ along $X_{\hat{a}}$

of a function f are given by $e_{\hat{a}}(f) = [X_{\hat{a}}, f]$. Accordingly the action of the Lie derivatives on functions is given by

$$\mathcal{L}_{\hat{a}} f = [X_{\hat{a}}, f] \ ; \tag{2}$$

these Lie derivatives satisfy the Leibniz rule and the $SU(2)$ Lie algebra relation

$$[\mathcal{L}_{\hat{a}}, \mathcal{L}_{\hat{b}}] = C_{\hat{a}\hat{b}\hat{c}} \mathcal{L}_{\hat{c}}. \tag{3}$$

In the $N \to \infty$ limit the derivations $e_{\hat{a}}$ become $e_{\hat{a}} = C_{\hat{a}\hat{b}\hat{c}} x^{\hat{b}} \partial^{\hat{c}}$ and only in this commutative limit the tangent space becomes two-dimensional. The exterior derivative is given by

$$df = [X_{\hat{a}}, f] \theta^{\hat{a}} \tag{4}$$

with $\theta^{\hat{a}}$ the one-forms dual to the vector fields $e_{\hat{a}}$, $< e_{\hat{a}}, \theta^{\hat{b}} >= \delta^{\hat{b}}_{\hat{a}}$. The space of one-forms is generated by the $\theta^{\hat{a}}$'s in the sense that for any one-form $\omega = \sum_i f_i dh_i t_i$ we can always write $\omega = \sum_{\hat{a}=1}^{3} \omega_{\hat{a}} \theta^{\hat{a}}$ with given functions $\omega_{\hat{a}}$ depending on the functions f_i, h_i and t_i. The action of the Lie derivatives $\mathcal{L}_{\hat{a}}$ on the one-forms $\theta^{\hat{b}}$ explicitly reads

$$\mathcal{L}_{\hat{a}}(\theta^{\hat{b}}) = C_{\hat{a}\hat{b}\hat{c}} \theta^{\hat{c}} \ . \tag{5}$$

On a general one-form $\omega = \omega_{\hat{a}} \theta^{\hat{a}}$ we have $\mathcal{L}_{\hat{b}} \omega = \mathcal{L}_{\hat{b}}(\omega_{\hat{a}} \theta^{\hat{a}}) = \left[X_{\hat{b}}, \omega_{\hat{a}}\right] \theta^{\hat{a}} - \omega_{\hat{a}} C^{\hat{a}}{}_{\hat{b}\hat{c}} \theta^{\hat{c}}$ and therefore

$$(\mathcal{L}_{\hat{b}} \omega)_{\hat{a}} = \left[X_{\hat{b}}, \omega_{\hat{a}}\right] - \omega_{\hat{c}} C^{\hat{c}}{}_{\hat{b}\hat{a}} \ ; \tag{6}$$

this formula will be fundamental for formulating the CSDR principle on fuzzy cosets.

The differential geometry on the product space Minkowski times fuzzy sphere, $M^4 \times S^2_F$, is easily obtained from that on M^4 and on S^2_F. For example a one-form A defined on $M^4 \times S^2_F$ is written as

$$A = A_\mu dx^\mu + A_{\hat{a}} \theta^{\hat{a}} \tag{7}$$

with $A_\mu = A_\mu(x^\mu, X_{\hat{a}})$ and $A_{\hat{a}} = A_{\hat{a}}(x^\mu, X_{\hat{a}})$.

One can also introduce spinors on the fuzzy sphere and study the Lie derivative on these spinors. Although here we have sketched the differential geometry on the fuzzy sphere, one can study other (higher-dimensional) fuzzy spaces (e.g. fuzzy CP^M) and with similar techniques their differential geometry.

3. Actions in higher dimensions seen as four-dimensional actions (Expansion in Kaluza-Klein modes)

First we consider on $M^4 \times (S/R)_F$ a non-commutative gauge theory with gauge group $G = U(P)$ and examine its four-dimensional interpretation. $(S/R)_F$ is a fuzzy coset, for example the fuzzy sphere S^2_F. The action is

$$\mathcal{A}_{YM} = \frac{1}{4g^2} \int d^4x \, kTr \, tr_G \, F_{MN} F^{MN}, \tag{8}$$

where kTr denotes integration over the fuzzy coset $(S/R)_F$ described by $N \times N$ matrices; here the parameter k is related to the size of the fuzzy coset space. For example for the fuzzy sphere we have $r^2 = \sqrt{N^2-1}\pi k$ [3]. In the $N \to \infty$ limit kTr becomes the usual integral on the coset space. For finite N, Tr is a good integral because it has the cyclic property $Tr(f_1 \ldots f_{p-1} f_p) = Tr(f_p f_1 \ldots f_{p-1})$. It is also invariant under the action of the group S, that is infinitesimally given by the Lie derivative. In the action (8) tr_G is the gauge group G trace. The higher-dimensional field strength F_{MN}, decomposed in four-dimensional space-time and extra-dimensional components, reads as follows $(F_{\mu\nu}, F_{\mu\hat{b}}, F_{\hat{a}\hat{b}})$; explicitly the various components of the field strength are given by

$$\begin{aligned} F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu], \qquad (9) \\ F_{\mu\hat{a}} &= \partial_\mu A_{\hat{a}} - [X_{\hat{a}}, A_\mu] + [A_\mu, A_{\hat{a}}], \\ F_{\hat{a}\hat{b}} &= [X_{\hat{a}}, A_{\hat{b}}] - [X_{\hat{b}}, A_{\hat{a}}] + [A_{\hat{a}}, A_{\hat{b}}] - C^{\hat{c}}{}_{\hat{a}\hat{b}} A_{\hat{c}}. \qquad (10) \end{aligned}$$

Under an infinitesimal G gauge transformation $\lambda = \lambda(x^\mu, X^{\hat{a}})$ we have

$$\delta A_{\hat{a}} = -[X_{\hat{a}}, \lambda] + [\lambda, A_{\hat{a}}] \ , \tag{11}$$

thus F_{MN} is covariant under local G gauge transformations: $F_{MN} \to F_{MN} + [\lambda, F_{MN}]$. This is an infinitesimal abelian $U(1)$ gauge transformation if λ is just an antihermitian function of the coordinates $x^\mu, X^{\hat{a}}$ while it is an infinitesimal non-abelian $U(P)$ gauge transformation if λ is valued in Lie$(U(P))$, the Lie algebra of hermitian $P \times P$ matrices. In the following we will always assume Lie$(U(P))$ elements to commute with the coordinates $X^{\hat{a}}$. In fuzzy/non-commutative gauge theory and in Fuzzy-CSDR a fundamental role is played by the covariant coordinate,

$$\varphi_{\hat{a}} \equiv X_{\hat{a}} + A_{\hat{a}} \ . \tag{12}$$

This field transforms indeed covariantly under a gauge transformation, $\delta(\varphi_{\hat{a}}) = [\lambda, \varphi_{\hat{a}}]$. In terms of φ the field strength in the non-commutative directions reads,

$$\begin{aligned} F_{\mu\hat{a}} &= \partial_\mu \varphi_{\hat{a}} + [A_\mu, \varphi_{\hat{a}}] = D_\mu \varphi_{\hat{a}}, \qquad (13) \\ F_{\hat{a}\hat{b}} &= [\varphi_{\hat{a}}, \varphi_{\hat{b}}] - C^{\hat{c}}{}_{\hat{a}\hat{b}} \varphi_{\hat{c}} \ ; \qquad (14) \end{aligned}$$

and using these expressions the action reads

$$\mathcal{A}_{YM} = \int d^4x\, Tr\, tr_G \left(\frac{k}{4g^2} F_{\mu\nu}^2 + \frac{k}{2g^2}(D_\mu \varphi_{\hat{a}})^2 - V(\varphi) \right), \tag{15}$$

where the potential term $V(\varphi)$ is the $F_{\hat{a}\hat{b}}$ kinetic term (in our conventions $F_{\hat{a}\hat{b}}$ is antihermitian so that $V(\varphi)$ is hermitian and non-negative)

$$\begin{aligned} V(\varphi) &= -\frac{k}{4g^2} Tr\, tr_G \sum_{\hat{a}\hat{b}} F_{\hat{a}\hat{b}} F_{\hat{a}\hat{b}} \\ &= -\frac{k}{4g^2} Tr\, tr_G \left([\varphi_{\hat{a}}, \varphi_{\hat{b}}][\varphi^{\hat{a}}, \varphi^{\hat{b}}] - 4C_{\hat{a}\hat{b}\hat{c}} \varphi^{\hat{a}} \varphi^{\hat{b}} \varphi^{\hat{c}} + 2r^{-2}\varphi^2 \right). \end{aligned} \tag{16}$$

The action (15) is naturally interpreted as an action in four dimensions. The infinitesimal G gauge transformation with gauge parameter $\lambda(x^\mu, X^{\hat{a}})$ can indeed be interpreted just as an M^4 gauge transformation. We write

$$\lambda(x^\mu, X^{\hat{a}}) = \lambda^\alpha(x^\mu, X^{\hat{a}}) \mathcal{T}^\alpha = \lambda^{h,\alpha}(x^\mu) T^h \mathcal{T}^\alpha \,, \tag{17}$$

where $\mathcal{T}^\alpha$ are hermitian generators of $U(P)$, $\lambda^\alpha(x^\mu, X^{\hat{a}})$ are $n \times n$ antihermitian matrices and thus are expressible as $\lambda(x^\mu)^{\alpha,h} T^h$, where T^h are antihermitian generators of $U(n)$. The fields $\lambda(x^\mu)^{\alpha,h}$, with $h = 1, \ldots n^2$, are the Kaluza-Klein modes of $\lambda(x^\mu, X^{\hat{a}})^\alpha$. We now consider on equal footing the indices h and α and interpret the fields on the r.h.s. of (17) as one field valued in the tensor product Lie algebra $\mathrm{Lie}(U(n)) \otimes \mathrm{Lie}(U(P))$. This Lie algebra is indeed $\mathrm{Lie}(U(nP))$ (the $(nP)^2$ generators $T^h \mathcal{T}^\alpha$ being $nP \times nP$ antihermitian matrices that are linear independent). Similarly we rewrite the gauge field A_ν as

$$A_\nu(x^\mu, X^{\hat{a}}) = A_\nu^\alpha(x^\mu, X^{\hat{a}}) \mathcal{T}^\alpha = A_\nu^{h,\alpha}(x^\mu) T^h \mathcal{T}^\alpha, \tag{18}$$

and interpret it as a $\mathrm{Lie}(U(nP))$ valued gauge field on M^4, and similarly for $\varphi_{\hat{a}}$. Finally $Tr\, tr_G$ is the trace over $U(nP)$ matrices in the fundamental representation.

Up to now we have just performed a ordinary fuzzy dimensional reduction. Indeed in the commutative case the expression (15) corresponds to rewriting the initial lagrangian on $M^4 \times S^2$ using spherical harmonics on S^2. Here the space of functions is finite dimensional and therefore the infinite tower of modes reduces to the finite sum given by Tr.

4. Non-trivial Dimensional reduction in the case of Fuzzy Extra Dimensions

Next we reduce the number of gauge fields and scalars in the action (15) by applying the Coset Space Dimensional Reduction (CSDR) scheme. Since

$SU(2)$ acts on the fuzzy sphere $(SU(2)/U(1))_F$, and more in general the group S acts on the fuzzy coset $(S/R)_F$, we can state the CSDR principle in the same way as in the continuum case, i.e. the fields in the theory must be invariant under the infinitesimal $SU(2)$, respectively S, action up to an infinitesimal gauge transformation

$$\mathcal{L}_{\hat{b}}\phi = \delta_{W_{\hat{b}}}\phi = W_{\hat{b}}\phi, \tag{19}$$

$$\mathcal{L}_{\hat{b}}A = \delta_{W_{\hat{b}}}A = -DW_{\hat{b}}, \tag{20}$$

where A is the one-form gauge potential $A = A_\mu dx^\mu + A_{\hat{a}}\theta^{\hat{a}}$, and $W_{\hat{b}}$ depends only on the coset coordinates $X^{\hat{a}}$ and (like A_μ, A_a) is antihermitian. We thus write $W_{\hat{b}} = W_{\hat{b}}^\alpha \mathcal{T}^\alpha$, $\alpha = 1, 2 \ldots P^2$, where $\mathcal{T}^i$ are hermitian generators of $U(P)$ and $(W_b^i)^\dagger = -W_b^i$, here † is hermitian conjugation on the $X^{\hat{a}}$'s.

In terms of the covariant coordinate $\varphi_{\hat{d}} = X_{\hat{d}} + A_{\hat{d}}$ and of

$$\omega_{\hat{a}} \equiv X_{\hat{a}} - W_{\hat{a}} \ , \tag{21}$$

the CSDR constraints assume a particularly simple form, namely

$$[\omega_{\hat{b}}, A_\mu] = 0, \tag{22}$$

$$C_{\hat{b}\hat{d}\hat{e}}\varphi^{\hat{e}} = [\omega_{\hat{b}}, \varphi_{\hat{d}}]. \tag{23}$$

In addition we have a consistency condition following from the relation $[\mathcal{L}_{\hat{a}}, \mathcal{L}_{\hat{b}}] = C_{\hat{a}\hat{b}}{}^{\hat{c}}\mathcal{L}_{\hat{c}}$:

$$[\omega_{\hat{a}}, \omega_{\hat{b}}] = C_{\hat{a}\hat{b}}{}^{\hat{c}}\omega_c, \tag{24}$$

where $\omega_{\hat{a}}$ transforms as $\omega_{\hat{a}} \to \omega'_{\hat{a}} = g\omega_{\hat{a}}g^{-1}$. One proceeds in a similar way for the spinor fields [11, 12].

4.1. Solving the CSDR constraints for the fuzzy sphere

We consider $(S/R)_F = S_F^2$, i.e. the fuzzy sphere, and to be definite at fuzziness level $N-1$ ($N \times N$ matrices). We study here the basic example where the gauge group is $G = U(1)$. In this case the $\omega_{\hat{a}} = \omega_{\hat{a}}(X^{\hat{b}})$ appearing in the consistency condition (24) are $N \times N$ antihermitian matrices and therefore can be interpreted as elements of Lie$(U(N))$. On the other hand the $\omega_{\hat{a}}$ satisfy the commutation relations (24) of Lie$(SU(2))$. Therefore in order to satisfy the consistency condition (24) we have to embed Lie$(SU(2))$ in Lie$(U(N))$. Let T^h with $h = 1, \ldots, (N)^2$ be the generators of Lie$(U(N))$ in the fundamental representation, we can always use the convention $h = (\hat{a}, u)$ with $\hat{a} = 1, 2, 3$ and $u = 4, 5, \ldots, N^2$ where the $T^{\hat{a}}$ satisfy the $SU(2)$ Lie algebra,

$$[T^{\hat{a}}, T^{\hat{b}}] = C^{\hat{a}\hat{b}}{}_{\hat{c}}T^{\hat{c}} \ . \tag{25}$$

Then we define an embedding by identifying

$$\omega_{\hat{a}} = T_{\hat{a}}. \tag{26}$$

The constraint (22), $[\omega_{\hat{b}}, A_\mu] = 0$, then implies that the four-dimensional gauge group K is the centralizer of the image of $SU(2)$ in $U(N)$, i.e.

$$K = C_{U(N)}(SU((2))) = SU(N-2) \times U(1) \times U(1) \ ,$$

where the last $U(1)$ is the $U(1)$ of $U(N) \simeq SU(N) \times U(1)$. The functions $A_\mu(x, X)$ are arbitrary functions of x but the X dependence is such that $A_\mu(x, X)$ is Lie(K) valued instead of Lie$(U(N))$, i.e. eventually we have a four-dimensional gauge potential $A_\mu(x)$ with values in Lie(K). Concerning the constraint (23), it is satisfied by choosing

$$\varphi_{\hat{a}} = r\varphi(x)\omega_{\hat{a}} \ , \tag{27}$$

i.e. the unconstrained degrees of freedom correspond to the scalar field $\varphi(x)$ which is a singlet under the four-dimensional gauge group K.

The choice (26) defines one of the possible embedding of Lie$(SU(2))$ in Lie$(U(N))$. For example we could also embed Lie$(SU(2))$ in Lie$(U(N))$ using the irreducible N-dimensional rep. of $SU(2)$, i.e. we could identify $\omega_{\hat{a}} = X_{\hat{a}}$. The constraint (22) in this case implies that the four-dimensional gauge group is $U(1)$ so that $A_\mu(x)$ is $U(1)$ valued. The constraint (23) leads again to the scalar singlet $\varphi(x)$.

In general, we start with a $U(1)$ gauge theory on $M^4 \times S_F^2$. We solve the CSDR constraint (24) by embedding $SU(2)$ in $U(N)$. There exist p_N embeddings, where p_N is the number of ways one can partition the integer N into a set of non-increasing positive integers [16]. Then the constraint (22) gives the surviving four-dimensional gauge group. The constraint (23) gives the surviving four-dimensional scalars and eq. (27) is always a solution but in general not the only one. By setting $\phi_{\hat{a}} = \omega_{\hat{a}}$ we obtain always a minimum of the potential. This minimum is given by the chosen embedding of $SU(2)$ in $U(N)$.

5. Discussion and Conclusions

Non-commutative Geometry has been regarded as a promising framework for obtaining finite quantum field theories and for regularizing quantum field theories. In general quantization of field theories on non-commutative spaces has turned out to be much more difficult and with less attractive ultraviolet features than expected [17, 18], see however ref. [19], and ref. [20]. Recall also that non-commutativity is not the only suggested tool for constructing finite field theories. Indeed four-dimensional finite gauge theories have been constructed in ordinary space-time and not only those which are $\mathcal{N}=4$ and $\mathcal{N}=2$ supersymmetric, and most probably phenomenologically uninteresting, but also chiral $\mathcal{N}=1$ gauge theories [21] which already have been successful in predicting the top quark mass and have rich phenomenology that could be tested in future colliders [21, 22]. In the present work we have not adressed the finiteness of non-commutative quantum field theories, rather we have used non-commutativity to produce, via Fuzzy-CSDR, new particle models from particle models on $M^4 \times (S/R)_F$.

The Fuzzy-CSDR has different features from the ordinary CSDR leading therefore to new four-dimensional particle models. It may well be that Fuzzy-CSDR provides more realistic four-dimensional theories. Having in mind the construction of realistic models one can also combine the fuzzy and the ordinary CSDR scheme, for example considering $M^4 \times S'/R' \times (S/R)_F$.

A major difference between fuzzy and ordinary SCDR is that in the fuzzy case one always embeds S in the gauge group G instead of embedding just R in G. This is due to the fact that the differential calculus on the fuzzy coset space is based on $dimS$ derivations instead of the restricted $dimS - dimR$ used in the ordinary one. As a result the four-dimensional gauge group $H = C_G(R)$ appearing in the ordinary CSDR after the geometrical breaking and before the spontaneous symmetry breaking due to the four-dimensional Higgs fields does not appear in the Fuzzy-CSDR. In Fuzzy-CSDR the spontaneous symmetry breaking mechanism takes already place by solving the Fuzzy-CSDR constraints. The four-dimensional potential has the typical "maxican hat" shape, but it appears already spontaneously broken. Therefore in four dimensions appears only the physical Higgs field that survives after a spontaneous symmetry breaking. Correspondingly in the Yukawa sector of the theory we have the results of the spontaneous symmetry breaking, i.e. massive fermions and Yukawa interactions among fermions and the physical Higgs field. Having massive fermions in the final theory is a generic feature of CSDR when S is embedded in G [6]. We see that if one would like to describe the spontaneous symmetry breaking of the SM in the present framework, then one would be naturally led to large extra dimensions.

A fundamental difference between the ordinary CSDR and its fuzzy version is the fact that a non-abelian gauge group G is not really required in high dimensions. Indeed the presence of a $U(1)$ in the higher-dimensional theory is enough to obtain non-abelian gauge theories in four dimensions.

The final point that we would like to stress here is the question of the renormalizability of the gauge theory defined on $M_4 \times (S/R)_F$. First we notice that the theory exhibits certain features so similar to a higher-dimensional gauge theory defined on $M_4 \times S/R$ that naturally it could be considered as a higher-dimensional theory too. For instance the isometries of the spaces $M_4 \times S/R$ and $M_4 \times (S/R)_F$ are the same. It does not matter if the compact space is fuzzy or not. For example in the case of the fuzzy sphere, i.e. $M_4 \times S_F^2$, the isometries are $SO(3,1) \times SO(3)$ as in the case of the continuous space, $M_4 \times S^2$. Similarly the coupling of a gauge theory defined on $M_4 \times S/R$ and on $M_4 \times (S/R)_F$ are both dimensionful and have exactly the same dimensionality. On the other hand the first theory is clearly non-renormalizable, while the latter is renormalizable (in the sense that divergencies can be removed by a finite number of counterterms). So from this point of view one finds a partial justification of the old hopes for considering quantum field theories on non-commutative structures. If this observation can lead to finite theories too, it remains as an open question.

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