Constrained Generalized Supersymmetries

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Abstract

Generalized supersymmetries with bosonic tensorial central charges are investigated w.r.t. their division algebra properties. It is shown that complex and quaternionic supersymmetries admit compatible division-algebra induced constraints which are fully classified. In special cases constrained generalized supersymmetries present a dual formulation. Constrained generalized supersymmetries arise as the analytic continuation of the M-algebra to the Euclidean and as underlying superalgebras of certain classes of supersymmetric dynamical models.

1. Introduction

This talk is based on a series of papers, [1, 2, 3, 4], devoted to the investigation of generalized supersymmetries in connection with division algebras, as well as the application of these results in the broad context of the M-theory. The last two works, in particular, address the problem of classifying the division-algebra compatible constraints that can be imposed on generalized supersymmetries in presence of complex and quaternionic spinors. It is further shown that, in certain specified cases, such constrained generalized supersymmetries admit a dual description. Physical motivations for this mathematical analysis are based on M-theory and its related topics. It was indeed proven in [2] that a given constrained complex generalized supersymmetry is required in order to perform the analytic continuation of the Minkowskian M-algebra to the Euclidean. The present results are further applicable to the construction and classification of various classes of supersymmetric dynamical systems presenting bosonic tensorial central charges.

It is worth recalling that the problem of classifying supersymmetries has recently regained interest and found a lot of attention in the literature. We can cite, e.g., a series of papers where the notion of "spin algebra" has been introduced and investigated [5]. An even more updated reference concerns the classification of the so-defined "polyvector super-Poincaré algebras" [6].

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The reasons behind all this activity are clear. In the seventies the HŁS scheme [7] was a cornerstone providing the supersymmetric extension of the Coleman-Mandula no-go theorem. However, in the nineties, the generalized space-time supersymmetries going beyond the HŁS scheme (and admitting, in particular, a bosonic sector of the Poincaré or conformal superalgebra which could no longer be expressed as a tensor product $B_{geom} \oplus B_{int}$, where B_{geom} describes space-time Poincaré or conformal algebras, while the remaining generators spanning B_{int} are scalars) found widespread recognition [8, 9] in association with the dynamics of extended objects like branes (see [10, 11]). The eleven-dimensional M-algebra underlying the M-theory as a possible "Theory Of Everything" (TOE), admitting 32-real component spinors and maximal number ($= 528$) of saturated bosonic generators [8, 9] falls into this class of generalized supersymmetries. The mathematical ingredients that have to be used have been known by mathematicians since at least the sixties ([12], see also [13] and, for quite a convenient presentation for physicists, [14]). They include the division-algebra classification of Clifford algebras and fundamental spinors. It is quite rewarding that, by using these available tools, we can conveniently formulate and solve the problem of classifying generalized supersymmetries.

It is well-known that the Clifford algebra irreps [14] are put in correspondence with the $\mathbf{R}, \mathbf{C}, \mathbf{H}$ division algebras. An analogous scheme works for fundamental spinors (here and in the following, fundamental spinors are defined to be the spinors admitting, in a given space-time, the maximal division algebra structure compatible with the minimal number of *real* components). Both the eleven-dimensional M-algebra and the F-algebra in $(10+2)$ dimensions are based on real spinors. Their analytic continuation to the Euclidean, however, see [2] and [3], are based on complex spinors. The presence of both complex and quaternionic spinors allows introducing division-algebra compatible extra-constraint on the available generalized supersymmetries. The reason for that lies in the fact that in these two extra cases one has at disposal the division-algebra principal conjugation (which simply coincides, for real numbers, with the identity operator) to further play with. It is of particular importance to determine the biggest ("saturated") generalized supersymmetry compatible with the given division-algebra structure and constraint. For the sake of simplicity, in this work we are only concerned with "generalized supertranslations". This means in particular that the bosonic generators are all abelian. The construction of, e.g., Lorentz generators requires a bigger algebra than the ones here examined. One viable scheme to produce them consists in introducing a generalized superconformal algebra (which, in its turn, allows recovering a generalized superPoincaré algebra through an Inonü-Wigner type of contraction). Following [15], this can be easily achieved by taking two separate copies of "generalized supertranslations" and imposing the Jacobi identities on the whole set of generators to fully determine the associated superconformal algebra.

2. Generalized supersymmetries: the M and F algebra examples

Let us introduce the notion of generalized supersymmetries by illustrating the specific examples of the M and F algebras.

For our purposes we should recall, at first, that three matrices, denoted as A, B, C, have to be introduced in association with the three conjugations (hermitian, complex and transposition) acting on Gamma matrices [16]. Since only two of the above matrices are independent we choose here, following [1], to work with A and C. A plays the role of the time-like Γ^0 matrix in the Minkowskian space-time and is used to introduce barred spinors. C, on the other hand, is the charge conjugation matrix. Up to an overall sign, in a generic (s,t) space-time, A and C are given by the products of all the time-like and, respectively, all the symmetric (or antisymmetric) Gamma-matrices[1]. The properties of A and C immediately follow from their explicit construction, see [16] and [1].

In a representation of the Clifford algebra realized by matrices with real entries, the conjugation acts as the identity, see [1]. In this case the space-like gamma matrices are symmetric, while the time-like gamma matrices are antisymmmetric, so that A can be identified with the charge conjugation matrix C_A.

For our purposes the importance of A and the charge conjugation matrix C lies on the fact that, in a D-dimensional space-time ($D = s + t$) spanned by $d \times d$ Gamma matrices, they allow to construct a basis for $d \times d$ (anti)hermitian and (anti)symmetric matrices, respectively. It is indeed easily proven that, in the real and the complex cases (the quaternionic case is different), the $\begin{pmatrix} D \\ k \end{pmatrix}$ antisymmetrized products of k Gamma matrices $A\Gamma^{[\mu_1 \cdots \mu_k]}$ are all hermitian or all antihermitian, depending on the value of $k \leq D$. Similarly, the antisymmetrized products $C\Gamma^{[\mu_1 \cdots \mu_k]}$ are all symmetric or all antisymmetric.

For what concerns the M-algebra, the 32-component real spinors of the $(10,1)$-spacetime admit anticommutators $\{Q_a, Q_b\}$ which are 32×32 symmetric real matrices with, at most, $32 + \frac{32 \times 31}{2} = 528$ components. Expanding the r.h.s. in terms of the antisymmetrized product of Gamma matrices, we get that it can be saturated by the so-called M-algebra

$$\{Q_a, Q_b\} \;=\; (A\Gamma_\mu)_{ab} P^\mu + \left(A\Gamma_{[\mu\nu]}\right)_{ab} Z^{[\mu\nu]} + \left(A\Gamma_{[\mu_1 \ldots \mu_5]}\right)_{ab} Z^{[\mu_1 \ldots \mu_5]}. \tag{1}$$

Indeed, the $k = 1, 2, 5$ sectors of the r.h.s. furnish $11 + 55 + 462 = 528$ overall components. Besides the translations P^μ, in the r.h.s. the anti-

[1]Depending on the given space-time (see [16] and [1]), there are at most two charge conjugations matrices, C_S, C_A, given by the product of all symmetric and all antisymmetric gamma matrices, respectively. In special space-time signatures they collapse into a single matrix C.

symmetric rank-2 and rank-5 abelian tensorial central charges, $Z^{[\mu\nu]}$ and $Z^{[\mu_1\cdots\mu_5]}$ respectively, appear.

The (1) saturated M-algebra admits a finite number of subalgebras which are consistent with the Lorentz properties of the Minkowskian eleven dimensions. There are 6 such subalgebras which are recovered by setting either one or two among the three sets of tensorial central charges P^μ, $Z^{[\mu\nu]}$, $Z^{[\mu_1\cdots\mu_5]}$ identically equal to zero (a completely degenerate subalgebra is further obtained by setting the whole r.h.s. identically equal to zero).

The fact that the fundamental spinors in a $(10,2)$-spacetime also admit 32 components is due to the existence of the Weyl projection. This implies that the saturated M-algebra admits a $(10,2)$ space-time presentation, the so-called F-algebra, in terms of $(10,2)$ Majorana-Weyl spinors $\tilde{Q}_{\tilde{a}}$, $\tilde{a} = 1,2,\ldots,32$.

In the case of Weyl projected spinors the r.h.s. has to be reconstructed with the help of a projection operator which selects the upper left block in a 2×2 block decomposition. Specifically, if $\mathcal{M}$ is a matrix decomposed in 2×2 blocks as $\mathcal{M} = \begin{pmatrix} \mathcal{M}_1 & \mathcal{M}_2 \\ \mathcal{M}_3 & \mathcal{M}_4 \end{pmatrix}$, we can define

$$P(\mathcal{M}) \equiv \mathcal{M}_1. \tag{2}$$

The saturated M-algebra (1) can therefore be rewritten as

$$\left\{\tilde{Q}_{\tilde{a}}, \tilde{Q}_{\tilde{b}}\right\} = P\left(\tilde{A}\tilde{\Gamma}_{\tilde{\mu}\tilde{\nu}}\right)_{\tilde{a}\tilde{b}} \tilde{Z}^{[\tilde{\mu}\tilde{\nu}]} + P\left(\tilde{A}\tilde{\Gamma}_{[\tilde{\mu}_1\ldots\tilde{\mu}_6]}\right)_{\tilde{a}\tilde{b}} \tilde{Z}^{[\tilde{\mu}_1\cdots\tilde{\mu}_6]}, \tag{3}$$

where all tilde's are referred to the corresponding $(10,2)$ quantities. The matrices in the r.h.s. are symmetric in the exchange $\tilde{a}\leftrightarrow\tilde{b}$. This time the rank-2 and selfdual rank-6 antisymmetric abelian tensorial central charges, $\tilde{Z}^{[\tilde{\mu}\tilde{\nu}]}$ and respectively $\tilde{Z}^{[\tilde{\mu}_1\cdots\tilde{\mu}_6]}$, appear. Their total number of components is $66+462=528$, therefore proving the saturation of the r.h.s.. The saturated equation (3) is named the F-algebra.

3. Real, complex and quaternionic generalized supersymmetries.

For real n-component spinors Q_a, the most general supersymmetry algebra is represented by

$$\{Q_a, Q_b\} = \mathcal{Z}_{ab}, \tag{4}$$

where the matrix $\mathcal{Z}$ appearing in the r.h.s. is the most general $n\times n$ symmetric matrix with total number of $\frac{n(n+1)}{2}$ components. For any given space-time we can easily compute its associated decomposition of $\mathcal{Z}$ in terms of the antisymmetrized products of k-Gamma matrices, namely

$$\mathcal{Z}_{ab} = \sum_k (A\Gamma_{[\mu_1\ldots\mu_k]})_{ab} Z^{[\mu_1\ldots\mu_k]}, \tag{5}$$

where the values k entering the sum in the r.h.s. are restricted by the symmetry requirement for the $a \leftrightarrow b$ exchange and are specific for the given spacetime. The coefficients $Z^{[\mu_1\cdots\mu_k]}$ are the rank-k abelian tensorial central charges.

When the fundamental spinors are complex or quaternionic they can be organized in complex (for the **C** and **H** cases) and quaternionic (for the **H** case) multiplets, whose entries are respectively complex numbers or quaternions.

The real generalized supersymmetry algebra (4) can now be replaced by the most general complex or quaternionic supersymmetry algebras, given by the anticommutators among the fundamental spinors Q_a and their conjugate ${Q^*}_{\dot{a}}$ (where the conjugation refers to the principal conjugation in the given division algebra, see ([1])). We have in this case

$$\{Q_a, Q_b\} = \mathcal{P}_{ab} \qquad , \qquad \{{Q^*}_{\dot{a}}, {Q^*}_{\dot{b}}\} = {\mathcal{P}^*}_{\dot{a}\dot{b}}, \tag{6}$$

together with

$$\{Q_a, {Q^*}_{\dot{b}}\} \quad = \quad \mathcal{R}_{a\dot{b}}, \tag{7}$$

where the matrix $\mathcal{P}_{ab}$ (${\mathcal{P}^*}_{\dot{a}\dot{b}}$ is its conjugate and does not contain new degrees of freedom) is symmetric, while $\mathcal{R}_{a\dot{b}}$ is hermitian.

4. Constrained generalized supersymmetries and their duality relations.

Let us investigate and classify now the set of consistent constraints that can be imposed on the complex generalized supersymmetries (the quaternionic case was investigated in [3]).

The saturated complex generalized supersymmetries (i.e. the ones admitting as bosonic r.h.s. both the most general symmetric matrix $\mathcal{P}$ entering (6) and the most general hermitian matrix $\mathcal{R}$ entering (7)) contain the same number of bosonic degrees of freedom as the corresponding saturated generalized supersymmetries realized with *real* spinors [3]. In this respect the big advantage of the introduction of the complex formalism, whenever this is indeed possible, consists in the implementation of some constraint that cannot be otherwise imposed within the real framework.

The bosonic r.h.s. can be expressed in terms of the rank-k totally antisymmetric tensors denoted as M_k, see the previous section. It is clear that any restriction on the saturated bosonic generators which allows all possible combinations of the rank-k antisymmetric tensors entering the r.h.s. is in principle allowed by Lorentz-covariant requirement. On the other hand, few particular combinations of the rank-k antisymmetric tensors have more compelling reasons to appear than just arising as a hand-imposed restriction on the saturated bosonic r.h.s. They can indeed be present due to a division-algebra constraint based on an underlying symmetry. It is expected that restrictions of this type offer a protecting mechanism towards the arising of anomalous terms, in application to the supersymmetries realized by certain classes of dynamical systems. This is an important reason to analyze and classify these constraints. Their whole class is presented in the table below. It consists of all possible combinations of restrictions on the $\mathcal{P}$, $\mathcal{R}$ matrices of (6) and (7) (e.g. whether both of them are present or just one of them, if a reality or an imaginary condition is applied). The entries in the table below specify the number of bosonic components (in the real counting) associated with the given constrained supersymmetry realized by n-component complex spinors. The columns represent the restrictions on $\mathcal{R}$, the rows the restrictions on $\mathcal{P}$ (an imaginary condition on $\mathcal{P}$ is equivalent to the reality condition and therefore is not reported in the table below). We have

$\mathcal{P}\backslash\mathcal{R}$	1) $Full$	2) $Real$	3) $Imag.$	4) $Abs.$
$a)$ $Full$	$2n^2+n$	$\frac{3}{2}(n^2+n)$	$\frac{1}{2}(3n^2+n)$	n^2+n
$b)$ $Real$	$\frac{1}{2}(3n^2+n)$	n^2+n	n^2	$\frac{1}{2}(n^2+n)$
$c)$ $Abs.$	n^2	$\frac{1}{2}(n^2+n)$	$\frac{1}{2}(n^2-n)$	0

(8)

Some comments are in order. The above list of constraints is not necessarily implemented for any given supersymmetric dynamical system. One should check, e.g., that the above restrictions are indeed compatible with the equations of motion. On a purely algebraic basis, however, they are admissible restrictions which require a careful investigation.

One can notice that certain numbers appear twice as entries in the above table. This is related with the fact that the same constrained superalgebra can admit a different, but equivalent, presentation. We refer to these equivalent presentations as "dual formulation" of the constrained supersymmetries. Dual formulations are expected in correspondence of the constraints

$$\begin{array}{rcl} a3 & \leftrightarrow & b1, \\ a4 & \leftrightarrow & b2, \\ b3 & \leftrightarrow & c1, \\ b4 & \leftrightarrow & c2. \end{array} \tag{9}$$

It is worth stressing that in application to dynamical systems, which need more data than just superalgebraic data, one should explicitly verify whether the above related constraints indeed lead to equivalent theories.

The inequivalent constrained generalized supersymmetries can be listed as follows

I	$(a1)$	$2n^2+n,$	$k=3,$	$l=1$
II	$(a2)$	$\frac{3}{2}(n^2+n),$	$k=3,$	$l=0$
III	$(a3\,\&\,b1)$	$\frac{1}{2}(3n^2+n),$	$k=2,$	$l=1$
IV	$(a4\,\&\,b2)$	$n^2+n,$	$k=2,$	$l=0$
V	$(b3\,\&\,c1)$	$n^2,$	$k=1,$	$l=1$
VI	$(b4\,\&\,c2)$	$\frac{1}{2}(n^2+n),$	$k=1,$	$l=0$
VII	$(c3)$	$\frac{1}{2}(n^2-n),$	$k=0,$	$l=1$

(10)

The integral numbers k, l have the following meaning. For the given constrained supersymmetry the bosonic r.h.s. can be presented in the following form

$$Z = kX + lY, \quad k = 0,1,2,3, \quad l = 0,1, \tag{11}$$

where X and Y denote the bosonic sectors associated with the VI and respectively VII constrained supersymmetry.

In association with the maximal Clifford algebras in D-dimensional spacetimes (with no dependence on their signature), the X and Y bosonic sectors are given by the following set of rank-k antisymmetric tensors

	X	Y
$D=3$	M_1	M_0
$D=5$	M_2	M_0+M_1
$D=7$	M_0+M_3	M_1+M_2
$D=9$	$M_0+M_1+M_4$	M_2+M_3
$D=11$	$M_1+M_2+M_5$	$M_0+M_3+M_4$
$D=13$	$M_2+M_3+M_6$	$M_0+M_1+M_4+M_5$

(12)

Formula (11) specifies the admissible class of division-algebra related, constrained bosonic sectors.

This analysis concludes the investigation of constrained complex generalized supersymmetries for maximal Clifford algebras (i.e. the Clifford algebras associated to the maximal number of Clifford algebras of given size that can be constructed). The extension of these results to the case of non-maximal Clifford algebras (obtained form the previous ones through a dimensional-reduction procedure) can be easily recovered, see [4] .

An example of application of a constrained generalized supersymmetry was given in [2], where the analytical continuation of the M-algebra to the 11-dimensional Euclidean space was made possible by the introduction of a holomorphic complex generalized supersymmetry.

5. Conclusions.

In this paper we discussed the classification of (real, complex and quaternionic) generalized supersymmetries. The notion of constrained (complex and quaternionic) generalized supersymmetry was given. Constrained complex generalized supersymmetries have been explicitly classified. It was further shown that, in some given cases, they can be presented in a dual formulation, according to the choice of decomposition of the bosonic degrees of freedom in the two matrices $\mathcal{P}$ and $\mathcal{R}$ entering formulas (6) and (7).

Physical implications of these mathematical structures are quite obvious. The classification of generalized supersymmetries allow to understand the web of interrelated dualities of different classes of theories which can be either analitically continued (let's say, to the Euclidean) or recovered through dimensional reduction.

As an example, we can cite that the analytic continuation of the M algebra was proven in [2] to correspond to an eleven-dimensional complex holomorphic supersymmetry. It was further shown in [3] that the same algebra also admits a 12-dimensional Euclidean presentation in terms of Weyl-projected spinors. These two examples of Euclidean supersymmetries can find application in the functional integral formulation of higher-dimensional supersymmetric models.

There is an interesting class of models which nicely fits [4] in the framework here described and is currently under intense investigation. It is the class of superparticle models, introduced at first in [17] and later studied in [18], whose bosonic coordinates correspond to tensorial central charges. It was shown in [19] that a 4-dimensional theory of this kind leads to a tower of massless higher spin states, concretely implementing a Fronsdal's proposal [20] of introducing bosonic tensorial coordinates to describe massless higher spin theories (admitting helicity states greater than two). This is an active area of investigation, the main motivation being the investigation the tensionless limit of superstring theory, corresponding to a tower of higher helicity massless particles (see e.g. [21]).

In a somehow "orthogonal" direction, a class of theories which can be investigated in the present framework is the class of supersymmetric extensions of Chern-Simon supergravities in higher dimensions, requiring as a basic ingredient a Lie superalgebra admitting a Casimir of appropriate order, see e.g. [22].

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