Lectures on Elliptic Functions and Modular Forms in Conformal Field Theory

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Abstract

A concise review of the notions of elliptic functions, modular forms, and ϑ–functions is provided, devoting most of the paper to applications to Conformal Field Theory (CFT), introduced within the axiomatic framework of quantum field theory. Many features, believed to be peculiar to chiral 2D (= two dimensional) CFT, are shown to have a counterpart in any (even dimensional) globally conformal invariant quantum field theory. The treatment is based on a recently introduced higher dimensional extension of the concept of vertex algebra.

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1. Introduction

Arguably, the most attractive part of Conformal Field Theory (CFT) is that involving elliptic functions and modular forms. Modular inversion, the involutive S–transformation of the upper half–plane

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in SL(2, \mathbb{Z}); \quad \tau \to -\frac{1}{\tau} \quad (Im\, \tau > 0)\,, \qquad (1..1)$$

relates high and low temperature behaviour, thus providing the oldest and best studied [1] example of a duality transformation [37].

The aim of these lectures is threefold:

(1) To offer a brief introduction to the mathematical background, including a survey of the notions of elliptic functions, elliptic curve (and its moduli), modular forms and ϑ–functions. (The abundant footnotes are designed to provide some historical background.)

(2) To give a concise survey of axiomatic CFT in higher (even) dimensions with an emphasis on the vertex algebra approach developed in [51], [55].

[1] For a physicist oriented review of modular inversion – see [18].

(3) To give an argument indicating that *finite temperature correlation functions* in a globally conformal invariant (GCI) quantum field theory in any even number of space–time dimensions are (doubly periodic) *elliptic functions* and to study the modular properties of the corresponding temperature mean values of the conformal Hamiltonian.

Two-dimensional (2D) CFT models provide basic known examples in which the chiral energy average in a given superselection sector is a modular from of weight 2. In a rational CFT these energy mean values span a finite dimensional representation of $SL(2,\mathbb{Z})$. We demonstrate that modular transformation properties can also be used to derive high temperature asymptotics of thermal energy densities in a 4–dimensional CFT.

We include in the bibliography some selected texts on the mathematical background briefly annotated in our (half page long) "Guide to references" at the end of the lectures. (Concerning modular forms we follow the notation of Don Zagier in [22].) A detailed exposition of the authors' original results can be found in [55].

2. Elliptic functions and curves

The theory of elliptic functions has been a centre of attention of the 19th and the early 20th century mathematics (since the discovery of the double periodicity by N. H. Abel in 1826 until the work of Hecke [2] and Hurwitz's [3] book [30] in the 1920's – see [38] for an engaging historical survey). This is followed by a period of relative dormancy when E. Wigner ventured to say that it is "falling into oblivion" [4]. (Even today physics students rarely get to learn this chapter of mathematics during their undergraduate years.) The topic experiences a renaissance in the early 1970's, which continues to these days (see the guide to the literature until 1989 by D. Zagier in [22] pp. 288–291). The proceedings [22] of the 1989 Les Houches Conference on Number Theory and Physics provide an excellent shortcut into the subject and further references. The subject continues to be a focus of mathematical physicists' attention (for a recent application to noncommutative geometry – see [14] [13]).

2.1. Elliptic integrals and functions

If we did not know about trigonometric functions when first calculating the integral $z = \int\limits_0^x \frac{dt}{\sqrt{1-t^2}}$, we would have come out with a rather nasty multivalued function $z(x)$. Then an unprejudiced young man might

[2] Erich Hecke (1887–1947) was awarded his doctorate under David Hilbert (1862–1943) in 1910 in Göttingen for a dissertation on modular forms and their application to number theory.

[3] Adolf Hurwitz (1859-1919).

[4] E. Wigner, *The limits of science*, Proc. Amer. Phil. Soc. **94** (1950) 422; see also his collection of Scientific Essays, *Symmetries and Reflections* p. 219 (Eugene Paul Wigner, 1902–1995, Nobel Prize in physics, 1963).

have discovered that one should instead work with the inverse function $x(z) = \sin z$, which is a nice *single valued entire periodic function.* This is more or less what happened for elliptic integrals[5], say, for an integral of the type

$$z = \int\limits_x^\infty \frac{1}{\sqrt{4\xi^3 - g_2\xi - g_3}}\, d\xi\,, \qquad g_2^3 - 27g_3^2 \neq 0\,. \tag{2..1}$$

The inverse function $x = x(z)$ can be written in the Weierstrass' notation[6] as a manifestly *meromorphic* (*single valued*), *doubly periodic function* (see Exercise 2.2 (a)):

$$x(z) \,=\, \wp(z;\,\omega_1,\,\omega_2)\ (=\,\wp(z))\,=\,\frac{1}{z^2} + \sum_{\omega\,\in\,\Lambda\backslash\{0\}} \left(\frac{1}{(z+\omega)^2} - \frac{1}{\omega^2}\right), \tag{2..2}$$

where Λ is the 2–dimensional lattice of periods,

$$\Lambda = \left\{\omega = m\omega_1 + n\omega_2\ :\ m,n \in \mathbb{Z},\ Im\,\frac{\omega_1}{\omega_2} > 0\right\}\,,$$
$$g_2 = 60 \sum_{\omega\,\in\,\Lambda\backslash\{0\}} \omega^{-4}\,, \qquad g_3 = 140 \sum_{\omega\,\in\,\Lambda\backslash\{0\}} \omega^{-6}\,. \tag{2..3}$$

Indeed, knowing the final answer (2..2) it is easy to check that $x(z)$ satisfies a first order differential equation (Exercise 2.2 (b)):

$$y^2 \,=\, 4x^3 - g_2x - g_3 \quad \text{for} \quad x \,=\, \wp(z)\,, \quad y \,=\, \wp'(z)\,. \tag{2..4}$$

This is the counterpart of the equation $y^2 = 1 - x^2$ for $x(z) = \sin(z)$. The condition that the third degree polynomial y^2 (2..4) has no multiple zero can be expressed by the nonvanishing of the discriminant, proportional to $g_2^3 - 27g_3^2$ (in the case of coinciding roots, the integral (2..1) reduces to a trigonometric one).

Remark 2..1. More generally, elliptic integrals are integrals over *rational functions* $R(x,y)$, when y^2 is a *third or a fourth degree polynomial* in x with

[5] After nearly 200 years of study of elliptic integrals, starting with the 17th century work of John Wallis (1616–1703) and going through the entire 18th century with contributions from Leonard Euler (1707–1783) and Adrien–Marie Legendre (1752–1833), a 23–years old Norwegian, the pastor's son, Niels Henrik Abel (1802–1829) had the bright idea to look at the inverse function and prove that it is single valued, meromorphic and doubly periodic. As it often happens with 19th century discoveries, Carl Friedrich Gauss (1777–1855) also had developed this idea in his notebooks – back in 1798 – on the example of the lemniscate (see [47] Sects. 2.3 and 2.5).

[6] Karl Theodor Wilhelm Weierstrass (1815–1897); the $\wp$-function appeared in his Berlin lectures in 1862. Series of the type (2..2) were, in fact, introduced by another young deceased mathematician (one of the precious few appreciated by Gauss – whom he visited in Göttingen in 1844) Ferdinand Gotthold Eisenstein (1823–1852) – see [68].

different roots. A *fourth–degree curve*, $\widetilde{y}^2 = a_0\widetilde{x}^4 + a_1\widetilde{x}^3 + a_2\widetilde{x}^2 + a_3\widetilde{x} + a_4$ can be brought to the Weierstrass canonical form (2..2) by what may be called a "*Möbius* [7] *phase space transformation*": $\widetilde{x} = \dfrac{ax+b}{cx+d}$, $\widetilde{y} = \dfrac{A}{(cx+d)^2}\, y$, $ad - bc \neq 0 \neq A$ i. e., if y transforms as a derivative (with a possible dilation of the independent variable z in accord with the realization (2..2)). We have, in particular, to equate $\dfrac{a}{c}$ to one of the zeroes of the polynomial $\widetilde{y}^2(\widetilde{x})$, thus killing the coefficient of x^4 (see Exercise 2.3). An example of such type of integrals is the *Jacobi's* [8] "*sinus amplitudinus*",

$$x = \mathrm{sn}\left(z, k^2\right) , \quad z := \int_1^x \frac{d\xi}{\sqrt{(\xi^2-1)\,(1-k^2\xi^2)}} , \quad k^2 \neq 0, 1 \qquad (2..5)$$

which is proven to be a doubly periodic meromorphic function [9] with periods $4K$ and $2iK'$, where $K := \displaystyle\int_0^1 \frac{dx}{\sqrt{(1-x^2)\,(1-k^2x^2)}}$ $(= \dfrac{\pi}{2}\, F(\dfrac{1}{2}, \dfrac{1}{2}; 1; k^2)$, F being the hypergeometric function) and $K' := \displaystyle\int_1^{\frac{1}{k}} \frac{dx}{\sqrt{(x^2-1)\,(1-k^2x^2)}}$ (see, e.g. [47] Sects. 2.1 and 2.5; concerning the other Jacobi functions, $\mathrm{cn}(z,k^2) = \sqrt{1 - \mathrm{sn}(z,k^2)}$ and $\mathrm{dn}(z,k^2) = \sqrt{1 - k^2\mathrm{sn}(z,k^2)}$ see Sect. 2.16 of [47]).

We proceed to displaying some simple properties of elliptic functions, defined as *doubly periodic meromorphic functions on the complex plane.* Basic facts of complex analysis, such as Liouville's and Cauchy's [10] theorems, allow one to establish far reaching non–obvious results in the study of elliptic functions.

(1) Periodicity implies that an elliptic function $f(z)$ is determined by its values in a basic parallelogram, called a *fundamental domain:*

$$F = \{\alpha\,\omega_1 + \beta\,\omega_2\,;\ 0 \leqslant \alpha,\, \beta < 1\} \ .$$

[7] Augustus Ferdinand Möbius (1790–1868).

[8] Carl Gustav Jacob Jacobi (1804–1851) rediscovers in 1828 the elliptic functions (by inverting the elliptic integrals) and is the first to apply them to number theory. Jacobi, himself, says that the theory of elliptic functions was born when Euler presented to the Brelin Academy (in January 1752) the first series of papers, eventually proving the addition and multiplication theorems for elliptic integrals (see [69]). Bourbaki (in particular, Jean Dieudonné) have taken as a motto his words (from a letter to Legendre of 1830, deploring the worries of Fourier about applications): "le but unique de la science, c'est l'honneur de l'esprit humain."

[9] It is not difficult to show that the solution of the Newton equation of a length L and mass m *pendulum*, $m\frac{d^2\theta}{dt^2} + m\frac{G}{L}\sin\theta = 0$ (G being the Earth gravitational acceleration), is expressed in terms of the elliptic sinus (2..5) – see [47] Sect. 2.1 Example 4 and p. 77.

[10] Augustin–Louis Cauchy (1789–1857); Joseph Liouville (1809–1882).

(2) *If f is bounded in F, then it is a constant.* Indeed, periodicity would imply that f is bounded on the whole complex plane. The statement then follows from Liouville's theorem. Thus a non–constant elliptic function must have a pole in F.

(3) *The sum of the residues of the simple poles of f in F is zero.* This follows from Cauchy's theorem, since the integral over the boundary ∂F of F vanishes:

$$\oint_{\partial F} f(z)\, dz \,=\, 0\,,$$

as a consequence of the periodicity. (By shifting, if necessary, the boundary on opposite sides we can assume that f has no poles on ∂F.) It follows that f has at least 2 poles in F (counting multiplicities).

(4) *Let $\{a_i\}$ be the zeroes and poles of f in F and n_i be the multiplicity of a_i ($n_i > 0$ if a_i is a zero, $n_i < 0$ if a_i is a pole). Then applying (3) to $\dfrac{f'(z)}{f(z)}$ gives $\sum n_i = 0$.*

More properties of zeroes and poles of an elliptic function in a fundamental domain are contained in Theorem (1.1.2) of Cohen in [22], p. 213 (see also [47] Sect. 2.7). The above list allows to write down the general form of an elliptic function $f(z)$. If the singular part of $f(z)$ in F has the form:

$$\sum_{k\,=\,1}^{K} \sum_{s\,=\,1}^{S_k} N_{k,s}\, \frac{1}{(z - z_s)^k} \qquad (2..6)$$

for some $K, S_1, \dots, S_K \in \mathbb{N}$, $N, N_{s,k} \in \mathbb{C}$, $z_s \in F$ ($k = 1, \dots, K$, $s = 1, \dots, S_k$), then $f(z)$ can be represented in a finite sum:

$$f(z) = N \,+\, \sum_{k\,=\,1}^{K} \sum_{s\,=\,1}^{S_k} N_{k,s}\; p_k(z - z_s;\, \omega_1,\, \omega_2) \qquad (2..7)$$

where $p_k(z; \omega_1, \omega_2)$ are [11], roughly speaking, equal to:

$$p_k(z;\, \omega_1,\, \omega_2) \,:=\, \sum_{\omega\,\in\,\Lambda} \frac{1}{(z + \omega)^k}\,. \qquad (2..8)$$

The series (2..8) are absolutely convergent for $k \geqslant 3$ and $z \notin \Lambda$, and

$$p_{k+1}(z;\, \omega_1,\, \omega_2) \,=\, -\frac{1}{k}\,(\partial_z p_k)\,(z;\, \omega_1,\, \omega_2) \qquad (\,\partial_z \,:=\, \frac{\partial}{\partial z}\,)\,. \qquad (2..9)$$

[11] These are the "basic elliptic functions" of Eisenstein according to André Weil (1906–1998) who denotes them by E_k – see [68] Chapter III.

For $k = 1, 2$ one should specialize the order of summation or, alternatively, add regularizing terms. Such a regularization for the $k = 2$ case has been used in fact in the definition of the Weierstrass' $\wp$ function (2..2); for $k = 1$, the function

$$\mathfrak{Z}(z;\, \omega_1,\, \omega_2) \;=\; \frac{1}{z} + \sum_{\omega\,\in\,\Lambda\backslash\{0\}} \left(\frac{1}{z+\omega} - \frac{1}{\omega} + \frac{z}{\omega^2} \right) , \qquad (2..10)$$

is known as Weierstrass' $\mathfrak{Z}$ function. Note that the $\mathfrak{Z}$–function (2..10) is not elliptic (due to the above property (2)) but any linear combination $\sum\limits_{s\,=\,1}^{S} N_{1,s}$ $\mathfrak{Z}(z-\ z_s;\, \omega_1,\, \omega_2)$ with $\sum\limits_{s\,=\,1}^{S} N_{1,s} = 0$ will be elliptic. This follows from the translation property [39]

$$\mathfrak{Z}(z+\omega_1;\, \omega_1,\, \omega_2) \;=\; \mathfrak{Z}(z;\, \omega_1,\, \omega_2) - 8\pi^2 G_2(\omega_1,\, \omega_2)\, \omega_1 - 2\pi i\,, \quad (2..11)$$

$$\mathfrak{Z}(z+\omega_2;\, \omega_1,\, \omega_2) \;=\; \mathfrak{Z}(z;\, \omega_1,\, \omega_2) - 8\pi^2 G_2(\omega_1,\, \omega_2)\ , \qquad (2..12)$$

where

$$-8\pi^2 G_2(\omega_1,\, \omega_2) \;=\; \sum_{n\,\in\,\mathbb{Z}\backslash\{0\}} \frac{1}{(n\,\omega_2)^2} + \sum_{m\,\in\,\mathbb{Z}\backslash\{0\}} \left(\sum_{n\,\in\,\mathbb{Z}} \frac{1}{(m\,\omega_1 + n\,\omega_2)^2} \right) \qquad (2..13)$$

will be considered in more details in Sect. 3.2.

Exercise 2..1. Prove the absolute convergence of the series (2..13) using the *Euler's formulae*

$$\lim_{N\to\infty} \sum_{n\,=\,-N}^{N} \frac{1}{z+n} \;=\; \pi \operatorname{cotg} \pi z\,, \qquad \lim_{N\to\infty} \sum_{n\,=\,-N}^{N} \frac{(-1)^n}{z+n} \;=\; \frac{\pi}{\sin \pi z} \qquad (2..14)$$

(with a subsequent differentiation).

It is convenient to single out a class of elliptic functions $f(z;\omega_1,\omega_2)$, which are *homogeneous* in the sense that $\rho^k\ f(\rho z; \rho\omega_1, \rho\omega_2) = f(z;\omega_1,\omega_2)$ for $\rho \neq 0$ and some $k = 1, 2, \ldots$. The Weierstrass function (2..2) provides an example of a homogeneous function of degree 2. In the applications to GCI QFT a natural system of basic elliptic functions is

$$p_k^{\kappa\lambda}(z;\, \omega_1,\, \omega_2) \;=\; \lim_{M\to\infty} \sum_{m=-M}^{M} \lim_{N\to\infty} \sum_{n=-N}^{N} \frac{(-1)^{\kappa m + \lambda n}}{(z + m\,\omega_1 + n\,\omega_2)^k}, \quad \kappa, \lambda = 0, 1 \qquad (2..15)$$

(cp. with Eq. (2..14)), characterized by the (anti)periodicity condition

$$p_k^{\kappa\lambda}(z+\omega_1;\,\omega_1,\,\omega_2) = (-1)^\kappa\ p_k^{\kappa\lambda}(z;\,\omega_1,\,\omega_2) \quad \text{for} \quad k+\kappa+\lambda>0\,, \tag{2..16}$$

$$p_k^{\kappa\lambda}(z+\omega_2;\,\omega_1,\,\omega_2)\,(-1)^\lambda\ p_k^{\kappa\lambda}(z;\omega_1,\omega_2) \quad \text{for} \quad \kappa,\,\lambda=0,\,1\,. \tag{2..17}$$

The functions p_k are encountered in a family of examples, described in Sect. 4.4; $p_1^{\kappa\lambda}$ with $\kappa+\lambda>0$ appear in the study of Gibbs states of a chiral 2D Weyl field (Sect. 5.3); the thermal 2–point function of a free massless scalar field in 4–dimensions is presented as a difference of two p_1^{00} functions – see (6..13). The functions (2..15) are connected for different k by:

$$p_{k+1}^{\kappa\lambda}(z;\,\omega_1,\,\omega_2) \;=\; -\frac{1}{k}\left(\partial_z p_k^{\kappa\lambda}\right)(z;\,\omega_1,\,\omega_2) \tag{2..18}$$

and we will set

$$p_k(z;\,\omega_1,\,\omega_2) \;\equiv\; p_k^{00}(z;\,\omega_1,\,\omega_2)\,. \tag{2..19}$$

Exercise 2..2. (*a*) Prove that $\wp(z;\,\omega_1,\,\omega_2)$ (2..2) is doubly periodic in z with periods ω_1 and ω_2. *Hint*: prove that the derivative $-\frac{1}{2}\,\partial_z\wp$ is the elliptic function p_3 (2..8) so that $\wp(z+\omega)-\wp(z)$ is a constant for $\omega\in\Lambda$; show that the constant is zero by taking $z=-\frac{\omega}{2}$.

(*b*) Prove that $\wp(z;\,\omega_1,\,\omega_2)$ (2..2) satisfy the equation (2..4). *Hint*: prove that the difference between the two sides of Eq. (2..4) is an entire elliptic function vanishing at $z=0$.

(*c*) Prove the relations

$$p_k^{10}(z;\,\omega_1,\,\omega_2) \;=\; 2p_k(z;\,\omega_1,\,2\omega_2)-p_k(z;\,\omega_1,\,\omega_2) \tag{2..20}$$

$$p_k^{01}(z;\,\omega_1,\,\omega_2) \;=\; 2p_k(z;\,2\omega_1,\,\omega_2)-p_k(z;\,\omega_1,\,\omega_2)\ , \tag{2..21}$$

$$p_k^{11}(z;\,\omega_1,\,\omega_2) \;=\; 2p_k(z;\,\omega_1+\omega_2,\,2\omega_2)-p_k(z;\,\omega_1,\,\omega_2)\ , \tag{2..22}$$

$$p_1(z;\,\omega_1,\,\omega_2) \;=\; \mathfrak{z}(z;\,\omega_1,\,\omega_2)+8\pi^2G_2(\omega_1,\,\omega_2)\,z\,, \tag{2..23}$$

$$p_2(z;\,\omega_1,\,\omega_2) \;=\; \wp(z;\,\omega_1,\,\omega_2)-8\pi^2G_2(\omega_1,\,\omega_2)\ , \tag{2..24}$$

$$p_1(z+\omega_1;\,\omega_1,\,\omega_2) \;=\; p_1(z;\,\omega_1,\,\omega_2) - 8\pi^2G_2(\omega_1,\,\omega_2)\,(\omega_1-\omega_2)-2\pi i\,, \tag{2..25}$$

$$p_1(z+\omega_2;\,\omega_1,\,\omega_2) \;=\; p_1(z;\,\omega_1,\,\omega_2)\ . \tag{2..26}$$

Hint: to prove Eqs. (2..20)–(2..22) take even M and N in Eq. (2..15) and split appropriately the resulting sum; proving Eqs. (2..23)–(2..26) one can first show that the difference between the two sides of Eq. (2..23) is an entire, doubly periodic, odd function and therefore, it is zero (see also Appendix A).

Corollary 2..1. *Every elliptic function $f(z)$ satisfying the periodicity conditions*

$$\begin{aligned} f(z+\omega_1;\,\omega_1,\,\omega_2) \;&=\; (-1)^\kappa\ f(z;\,\omega_1,\,\omega_2)\,, \\ f(z+\omega_2;\,\omega_1,\,\omega_2) \;&=\; (-1)^\lambda\ f(z;\omega_1,\omega_2)\,, \end{aligned} \tag{2..27}$$

for some $\kappa, \lambda = 0,\, 1$ admits unique (nontrivial) expansion

$$f(z) = N + \sum_{k=1}^{K} \sum_{s=1}^{S_k} N_{k,s}\; p_k^{\kappa\lambda}(z - z_s;\, \omega_1,\, \omega_2) \tag{2..28}$$

where $K, S_1, \dots, S_K \in \mathbb{N}$, $N, N_{s,k} \in \mathbb{C}$, $z_s \in F$ ($k = 1, \dots, K$, $s = 1, \dots, S_k$). In the case $\kappa = \lambda = 0$ the coefficients $N_{1,k}$ satisfy

$$\sum_{s=1}^{S_1} N_{1,s} \,=\, 0\,. \tag{2..29}$$

Exercise 2..3. Transform the forth degree equation $y^2 = (x - e_0)\,(x - e_1)\,(x - e_2)\,(x - e_3)$ (with different roots e_ν) into a third degree one $y^2 = 4\,(x - e'_1)\,(x - e'_2)\,(x - e'_3)$, using the Möbius transformation of Remark 2.1. (*Answer*: the transformation is $x \mapsto e_0 + (x - a)^{-1}$ and $y \mapsto \dfrac{A}{(x-a)^2}\, y$ with $A^2 = \dfrac{1}{4}\,(e_0 - e_1)\,(e_0 - e_2)\,(e_0 - e_3)$; then $e'_j = a - (e_0 - e_j)^{-1}$ ($j = 1, 2, 3$) where fixing $a = -\dfrac{1}{3} \displaystyle\sum_{j=1}^{3} (e_0 - e_j)^{-1}$ is equivalent to the condition $\displaystyle\sum_{j=1}^{3} e'_j = 0$ obeying the form (2..4).)

2.2. Elliptic curves

A nonsingular projective cubic curve with a distinguished "point at infinity" is called elliptic. An elliptic curve E over $\mathbb{C}$ (or, more generally, over any number field of characteristic different from 2 and 3) can be reduced to the Weierstrass form (in homogeneous coordinates X, Y, Z),

$$E\,:\, Y^2 Z = 4X^3 - g_2 X Z^2 - g_3 Z^3 \quad (g_2^3 - 27 g_3^2 \neq 0) \tag{2..30}$$

with the point at infinity, given by

$$e = (X : Y : Z) = (0 : 1 : 0)\,. \tag{2..31}$$

Let Λ be a (2–dimensional) period lattice (as in Eq. (2..3)). The uniformization map

$$z \;\mapsto\; \begin{cases} (\wp(z) : \wp'(z) : 1) & \text{for} \quad z \notin \Lambda \\ (0 : 1 : 0) & \text{for} \quad z \in \Lambda \end{cases} \tag{2..32}$$

($\wp(z) \equiv \wp(z;\omega_1,\omega_2)$, $\wp'(z) \equiv \partial_z \wp(z;\omega_1,\omega_2)$) from $\mathbb{C}$ to the projective complex plane provides an isomorphism between the torus $\mathbb{C}/\Lambda$ and the projective algebraic curve (2..30). It follows that E is a (commutative) algebraic group (as the quotient of the additive groups $\mathbb{C}$ and Λ). The addition theorem for Weierstrass functions,

$$\wp(z_1+z_2) \;=\; -\wp(z_1) - \wp(z_2) + \frac{1}{4}\left(\frac{\wp'(z_1)-\wp'(z_2)}{\wp(z_1)-\wp(z_2)}\right)^2$$
$$\left(\;\to\; -2\,\wp(z_1) + \frac{1}{4}\left(\frac{\wp''(z_1)}{\wp'(z_1)}\right)^2 \quad \text{for} \quad z_2 \to z_1\;\right) \qquad (2..33)$$

allows to express the group law in terms of the projective coordinates as follows.
The origin (or neutral element) of the group is the point at infinity e (2..31). If ($x = X/Z$, $y = Y/Z$) is a finite point of E (2..30) – i. e., a solution of the equation

$$y^2 = 4x^3 - g_2 x - g_3\,, \qquad (2..34)$$

then its *opposite* under the group law is the symmetric point $(x,\,-y)$ (which also satisfies (2..34)). If $P_1 = (x_1,\,y_1)$, $P_2 = (x_2,\,y_2)$ are non–opposite finite points of E, then their "sum" $P_3 = (x_3,\,y_3)$ is defined by

$$x_3 \;=\; -x_1 - x_2 + \frac{m^2}{4}\,, \quad y_3 \;=\; -y_1 - m\,(x_3 - x_1) \qquad \text{for}$$
$$m = \frac{y_1-y_2}{x_1-x_2} \quad \text{if} \quad P_1 \neq P_2\,; \quad m \;=\; \frac{12\,x_1^2 - g_2}{2y_1} \quad \text{if} \quad P_1 \;=\; P_2\,. \;(2..35)$$

The structure of rational points on an elliptic curve – a hot topic of modern mathematics – is reviewed in [58].

Exercise 2..4. Compute the sum $P+Q$ of points $P = (-\frac{11}{9},\,\frac{17}{27})$, $Q = (0,\,1)$ of an elliptic curve $y^2 = x^3 - x + 1$. (Answer: $(x,\,y) = \frac{1}{121}(159,\, -\frac{1861}{11})$).

Proposition 2..2. ([62] Proposition 4.1). *Two elliptic curves $E : y^2 = 4x^3 - g_2 x - g_3$ and $\widetilde{E} : \widetilde{y}^2 = 4\widetilde{x}^3 - \widetilde{g}_2\widetilde{x} - \widetilde{g}_3$ are isomorphic (as complex manifolds with a distinguished point) iff there exists $\rho \neq 0$, such that $\widetilde{g}_2 = \rho^4 g_2$, $\widetilde{g}_3 = \rho^6 g_3$; the isomorphism between them is then realized by the relation $\widetilde{x} = \rho^2 x$, $\widetilde{y} = \rho^3 y$.*

$\star$ The following text (between asterisks) is designed to widen the scope of a mathematically oriented reader and can be skipped in a first reading.

An elliptic curve, as well as, every algebraic (regular, projective) curve M can be fully characterized by its *function field* ([71]). This is the space $\mathbb{C}(M)$ of all meromorphic functions over M, i.e., functions

f such that in the vicinity of each point $p \in M$, f takes the form $(w - w(p))^d (a + (w - w(p))\, g(w))$ for some local coordinate w and an analytic function $g(w)$ around p, $d \in \mathbb{Z}$, and a *nonzero* constant a for $f \neq 0$. The number $ord_p f := d$ is then uniquely determined for nonzero f, depending only on f and p: it is called the *order of f at p*. Thus, the order is a function $ord_p : \mathbb{C}(M)\backslash\{0\} \to \mathbb{Z}$ satisfying the following natural properties

(ord_1) $ord_p(fg) = ord_p f + ord_p g$;

(ord_2) $ord_p(f+g) \geqslant \min\{ord_p f, ord_p g\}$ for $f \neq -g$;

(ord_3) $ord_p\, c = 0$ for $c \in \mathbb{C}\setminus\{0\}$

(it sometimes is convenient to set $ord_p 0 := \infty$). Functions $\nu : \mathbb{C}(M)\backslash\{0\} \to \mathbb{Z}$ satisfying (ord_1)–(ord_3) are called *discrete valuations* (on the field $\mathbb{C}(M)$). They are in one-to-one correspondence with the points $p \in M$: $p \mapsto ord_p$. Moreover, the *regular functions at p*, i.e. the functions taking finite (complex) values at p, are those for which $ord_p f \geqslant 0$; these functions form a ring R_p with a (unique) maximal ideal $\mathfrak{m}_p := \{f : ord_p f > 0\}$. Then the value $f(p)$ can be algebraically expressed as the coset $[f]_p$ of f in the quotient ring $R_p/\mathfrak{m}_p \cong \mathbb{C}$ (since the quotient by a maximal ideal is a field!).

On the other hand, the field $\mathbb{C}(M)$ of meromorphic functions on a (compact) projective curve can be algebraically characterized as a degree one transcendental extension of the field $\mathbb{C}$ of complex numbers: $\mathbb{C}(M)$ contains a non algebraic element over $\mathbb{C}$ and every two elements of $\mathbb{C}(M)$ are algebraically dependent (over $\mathbb{C}$, i.e., satisfy a polynomial equation with complex coefficients). Such fields are called *function fields*. The simplest example is the field $\mathbb{C}(z)$ of the complex rational functions of a single variable z. This is, in fact, the function field of the *Riemann*[12] *sphere* $\mathbb{P}^1$. Summarizing the above statements we have:

Theorem 2..3. ([71]) Chapt. VI *The nonsingular algebraic projective curves are in one–to–one correspondence with the degree one transcendental extensions of $\mathbb{C}$, naturally isomorphic to the fields of meromorphic functions over the curves.* $\star$

The function field of an elliptic curve $E := \mathbb{C}/\Lambda$ is generated by $\wp$ and $\wp'$ ([47], Sect. 2.13),

$$\mathbb{C}(E) \,=\, \mathbb{C}(\wp)[\wp'] \,=\, \mathbb{C}(\wp)\big[\sqrt{(\wp - e_1)(\wp - e_2)(\wp - e_3)}\,\big]\ , \qquad (2..36)$$

where e_1, e_2 and e_3 are the roots of the third order polynomial (2..34),

$$4\wp^3 - g_2\wp - g_3 \,=\, 4(\wp - e_1)(\wp - e_2)(\wp - e_3)\,(=\,(\wp')^2) \qquad (2..37)$$

(which should be different in order to have an elliptic curve). Thus, $\mathbb{C}(E)$ is a quadratic algebraic extension of the field $\mathbb{C}(\wp)$ of rational functions in $\wp$.

[12] Georg Friedrich Bernhard Riemann (1826–1866) introduced the "Riemann surfaces" in his Ph.D. thesis in Göttingen, supervised by Gauss (1851).

Exercise 2..5. Let (ω_1, ω_2) be a basis of Λ. Prove that the roots of $\wp'$ (2..37) in the basic cell $\{\lambda\,\omega_1 + \mu\,\omega_2 : 0 \leqslant \lambda, \mu < 1\}$ are (up to ordering) $\omega_1/2$, $\omega_2/2$, $(\omega_1+\omega_2)/2$, corresponding to $e_1 = \wp(\omega_1/2)$, $e_2 = \wp(\omega_2/2)$, $e_3 = \wp((\omega_1+\omega_2)/2)$. *Hint*: use the fact that $\wp'$ is an odd function of z as in Exercise 2.2. (a).

Exercise 2..6. Show that $\wp_j(z) = \sqrt{\wp(z) - e_j}$, $j = 1,2,3$ have single valued branches in the neighbourhood of the points $z_1 = \omega_1/2$, $z_2 = \omega_2/2$, $z_3 = (\omega_1+\omega_2)/2$, respectively. Prove that they have simple poles on the lattice Λ and may be standardized by fixing the residue at the origin as 1. Demonstrate that they belong to different quadratic extensions of the field $\mathbb{C}(E)$ corresponding to double covers of the torus E with primitive periods $(\omega_1, 2\,\omega_2)$, $(2\,\omega_1, \omega_2)$ and $(\omega_1+\omega_2, 2\,\omega_2)$, respectively (we shall also meet the corresponding index 2 sublattices in Sect. 2.4). Deduce that,

$$\begin{aligned}
\wp_1(z)\,(=\sqrt{\wp(z)-e_1}\,) &= p_1^{01}(z;\omega_1,\omega_2)\,, \\
\wp_2(z)\,(=\sqrt{\wp(z)-e_2}\,) &= p_1^{10}(z;\omega_1,\omega_2)\,, \\
\wp_3(z)\,(=\sqrt{\wp(z)-e_3}\,) &= p_1^{11}(z;\omega_1,\omega_2)\,,
\end{aligned} \tag{2..38}$$

where $p_1^{\kappa\lambda}$ are the functions (2..15) (see [47], Sect. 2.17).

Exercise 2..7. Find a relation between the sinus amplitudinus function $\mathrm{sn}(z,k^2)$ (2..5) and the functions $\wp_j$ of Exercise 2.6. *Answer*:

$$\wp_3(z)\,(\equiv p_1^{11}(z;\omega_1,\omega_2)\,) = \frac{\sqrt{e_2-e_3}}{\mathrm{sn}\,(z\sqrt{e_2-e_3}, k^2)}\,. \tag{2..39}$$

Exercise 2..8. Use the change of variables $x \mapsto e_3 + (e_2-e_3)/x^2$ to convert the indefinite integral $\displaystyle\int \big[4\ (x-e_1)\ (x-e_2)\ (x-e_3)\big]^{-\frac{1}{2}}\,dx$ into $(e_2-e_3)^{-\frac{1}{2}} \displaystyle\int \big[(1-x^2)\ (1-k^2x^2)\big]^{-\frac{1}{2}}\,dx$ for $k^2 = \dfrac{e_1-e_3}{e_2-e_3}$ (as in Exercise 2.3). Deduce as a consequence the relations:

$$\wp(z) \,=\, e_3 + \frac{e_2-e_3}{\{\mathrm{sn}\,(z\sqrt{e_2-e_3}, k^2)\}^2}\,. \tag{2..40}$$

Exercise 2..9. Prove that addition of half–periods and the reflection $z \mapsto -z$ are the only involutions of $E = \mathbb{C}/\Lambda$. Prove that the quotient space $E/(z \sim -z)$ is isomorphic to $\mathbb{P}^1$. Identify the quotient map $E \to E/(z \sim -z)$ as the Weierstrass function $\wp(z)$.

2.3. Modular invariance

Proposition 2.2 implies, in particular, that two lattices, Λ and $\rho\Lambda$, with the same ratio of the periods,

$$\tau := \frac{\omega_1}{\omega_2} \in \mathfrak{H} = \{\tau \in \mathbb{C}\,;\ Im\,\tau > 0\} \tag{2..41}$$

correspond to isomorphic elliptic curves. The isomorphism is given by multiplication with a non-zero complex number ρ:

$$\mathbb{C}/\Lambda \;\cong\; \mathbb{C}/\left(\rho\Lambda\right) \,:\, z \ (mod\,\Lambda) \;\mapsto\; \rho z \ (mod\,\rho\Lambda) \tag{2..42}$$
$$(x:y:1) \mapsto \left(\rho^2 x : \rho^3 y : 1\right) = \left(\wp\left(\rho z;\, \rho\omega_1,\, \rho\omega_2\right) : \partial_z \wp\left(\rho z;\, \rho\omega_1,\, \rho\omega_2\right) : 1\right).$$

On the other hand, the choice of basis (ω_1, ω_2) in a given lattice Λ is not unique. Any linear transformation of the form

$$\begin{aligned}(\omega_1,\, \omega_2) \;\mapsto\; (\omega_1',\, \omega_2') &:= (a\,\omega_1 + b\,\omega_2,\; c\,\omega_1 + d\,\omega_2)\,,\\ a, b, c, d \;\in\; \mathbb{Z}\,, &\quad ad - bc \;=\; \pm 1\end{aligned} \tag{2..43}$$

gives rise to a new basis $(\omega_1',\, \omega_2')$ in Λ which is as good as the original one. Had we been given a basis $(\omega_1,\, \omega_2)$ for which $Im\,(\omega_1/\omega_2) < 0$, we could impose (2..41) for $(\omega_1',\, \omega_2') = (\omega_2,\, \omega_1)$. Orientation preserving transformations (2..43) form the *modular group*

$$\Gamma(1) := SL(2,\mathbb{Z}) = \left\{\gamma = \begin{pmatrix} a & b\\ c & d\end{pmatrix} \,:\; a,\, b,\, c,\, d \,\in\, \mathbb{Z},\ \det\gamma \,=\, ad - bc \,=\, 1\right\}. \tag{2..44}$$

Thus, on one hand, we can define an elliptic curve, up to isomorphism, factorizing $\mathbb{C}$ by the lattice $\mathbb{Z}\tau + \mathbb{Z}$ with $\tau \in \mathfrak{H}$ and on the other, we can pass by a modular transformation $\gamma = \begin{pmatrix} a & b\\ c & d\end{pmatrix} \in \Gamma(1)$ to an equivalent basis $(a\tau + b, c\tau + d)$. Normalizing then the second period to 1 we obtain the classical action of $\Gamma(1)$ on $\mathfrak{H}$ (2..41) (mapping the upper half plane onto itself),

$$\tau \;\mapsto\; \frac{a\tau + b}{c\tau + d}\,. \tag{2..45}$$

This action obviously has a $\mathbb{Z}_2$ kernel $\{\pm 1\} \in \Gamma(1)$.
Note that the series (2..10) and (2..2), as well as (2..8) for $k \geqslant 3$, are absolutely convergent for $z \notin \Lambda$. This implies, in particular, their independence of the choice of basis,

$$\begin{aligned}\mathfrak{Z}(z;\omega_1,\omega_2) \;&=\; \mathfrak{Z}(z; a\omega_1 + b\omega_2, c\omega_1 + d\omega_2)\,,\\ \wp(z;\omega_1,\omega_2) \;&=\; \wp(z; a\omega_1 + b\omega_2, c\omega_1 + d\omega_2)\,,\\ p_k(z;\omega_1,\omega_2) &=\; p_k(z; a\omega_1 + b\omega_2, c\omega_1 + d\omega_2) \quad (k \geqslant 3).\end{aligned} \tag{2..46}$$

for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1)$. Using the homogeneity

$$\begin{aligned} \mathfrak{Z}(\rho z; \rho\omega_1, \rho\omega_2) &= \rho^{-1}\, \mathfrak{Z}(z;\omega_1,\omega_2)\,, \\ \wp(\rho z; \rho\omega_1, \rho\omega_2) &= \rho^{-2}\, \wp(z;\omega_1,\omega_2)\,, \\ p_k(\rho z; \rho\omega_1, \rho\omega_2) &= \rho^{-k}\, p_k(z;\omega_1,\omega_2)\,, \end{aligned} \tag{2..47}$$

($\rho \in \mathbb{C}\backslash\{0\}$) and setting

$$\begin{gathered} \mathfrak{Z}(z,\tau) := \mathfrak{Z}(z;\tau,1)\,, \quad \wp(z,\tau) := \wp(z;\tau,1)\,, \quad p_k(z,\tau) := p_k(z;\tau,1) \\ p_k^{\kappa\lambda}(z,\tau) := p_k^{\kappa\lambda}(z;\tau,1) \quad (\kappa,\lambda = 0,1), \quad p_k(z,\tau) \equiv p_k^{00}(z,\tau) \end{gathered} \tag{2..48}$$

(see (2..15)) we find as a result, the modular transformation laws

$$\begin{aligned} (c\tau+d)^{-1}\, \mathfrak{Z}\left(\frac{z}{c\tau+d}, \frac{a\tau+b}{c\tau+d}\right) &= \mathfrak{Z}(z,\tau)\,, \\ (c\tau+d)^{-2}\, \wp\left(\frac{z}{c\tau+d}, \frac{a\tau+b}{c\tau+d}\right) &= \wp(z,\tau)\,, \\ (c\tau+d)^{-k}\, p_k\left(\frac{z}{c\tau+d}, \frac{a\tau+b}{c\tau+d}\right) &= p_k(z,\tau) \quad (k \geqslant 3). \end{aligned} \tag{2..49}$$

The functions $p_1(z,\tau)$ and $p_2(z,\tau)$ obey *inhomogeneous* modular transformation laws since $G_2(\omega_1,\omega_2)$ transforms inhomogeneously (see Sect. 3.2). This is the price for preserving the periodicity property for $z \mapsto z+1$ according to (2..26). Nevertheless, all the functions $p_k^{\kappa\lambda}$ for $k \geqslant 1$ and $\kappa+\lambda > 0$ transform homogeneously among themselves:

$$(c\tau+d)^{-k}\, p_k^{[a\kappa+b\lambda]_2\,,\,[c\kappa+d\lambda]_2}\,\Big(\frac{z}{c\tau+d}, \frac{a\tau+b}{c\tau+d}\Big) = p_k^{\kappa\lambda}\,(z,\,\tau) \tag{2..50}$$

where $[\lambda]_2 = 0,1$ stands for the $\lambda\, mod\, 2$.

Exercise 2..10. Prove the relation (2..50) for $k \geqslant 3$ using the absolute convergence of the series in Eq. (2..15). (For $k = 1,2$ one should use the uniqueness property of the functions $p_k^{\kappa\lambda}$ given in Appendix A.)

Exercise 2..11. (a) Prove the representations

$$p_1\,(z,\tau) = \lim_{N\to\infty} \sum_{k=-N}^{N} \pi \operatorname{cotg} \pi(z + k\tau) = \tag{2..51}$$

$$= \pi \operatorname{cotg} \pi z + 4\pi \sum_{n=1}^{\infty} \frac{q^n}{1-q^n}\, \sin 2\pi n z \tag{2..52}$$

where $q := e^{2\pi i\tau}$ and the series (2..51) absolutely converges for all $z \notin \mathbb{Z}\tau+\mathbb{Z}$ and $\tau \in \mathfrak{H}$ while (2..52) absolutely converges for $|q| < |e^{-2\pi i z}| < 1$. *Hint*:

take the difference between the two sides of (2..51) and prove that it is an entire, odd, elliptic function using (2..25) and (2..26); to derive (2..52) from (2..51) use the expansion

$$\operatorname{cotg}\pi\,(z+k\tau) + \operatorname{cotg}\pi\,(z-k\tau) \;=\; -i\,\frac{1+e^{2\pi iz}q^k}{1-e^{2\pi iz}q^k} + i\,\frac{1+e^{-2\pi iz}q^k}{1-e^{-2\pi iz}q^k}$$
$$=\; 4\sum_{n=1}^{\infty} q^{nk}\,\sin 2\pi nz.$$

(b) Find similar representations for $p_2(z,\tau)$, $p_1^{11}(z,\tau)$ and $p_2^{11}(z,\tau)$.

2.4. Modular groups

As an abstract discrete group, $\Gamma\,(1)$ has two generators S and T satisfying the relations

$$S^2 \;=\; (ST)^3\;, \tag{2..53}$$
$$S^4 \;=\; 1\,; \tag{2..54}$$

their 2×2 matrix realization is

$$S \;=\; \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\,, \quad T \;=\; \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\,. \tag{2..55}$$

This can be established by the following Exercise.

Exercise 2..12. A subset $D \subset \mathfrak{H}$ is called a *fundamental domain* for $\Gamma(1)$ if each orbit $\Gamma(1)\tau$ of a $\tau \in \mathfrak{H}$ has at least one point in D, and if two points of D belong to the same orbit, they should belong to the boundary of D. Let

$$D = \left\{\tau \in \mathfrak{H} : -\frac{1}{2} \leqslant Re\,(\tau) \leqslant \frac{1}{2}\;,\; |\tau| \geqslant 1\right\}\;; \tag{2..56}$$

prove that D is a *fundamental domain* of $\Gamma(1)$. Moreover, prove that

1) $\tau \in \mathfrak{H}$ then there exists a $\gamma \in \Gamma(1)$, such that $\gamma\tau \in D$;

2) if $\tau \neq \tau'$ are two points in D such that $\tau' = \gamma\,\tau$ then either $Re(\tau) = \mp 1/2$ and $\tau' = \tau \pm 1$ or $|\tau| = 1$ and $\tau' = S\tau = -1/\tau$.

3) Let $P_1 := PSL(2,\mathbb{Z}) = SL\,(2,\,\mathbb{Z})\,/\mathbb{Z}_2$ be the (projective) modular group acting faithfully on $\mathfrak{H}$ and let $I(\tau) = \{\gamma \in P_1 : \gamma\tau = \tau\}$ be the stabilizer of τ in P_1. Then if $\tau \in D$, $I(\tau) = 1$ with the following three exceptions: $\tau = i$, then $I(\tau)$ is a 2–element subgroup of P_1 generated by S; if $\tau = \varrho := e^{\frac{2\pi i}{3}}$ then $I(\tau)$ is a 3–element subgroup of P_1 generated by ST; if $\tau = -\overline{\varrho} := e^{\frac{\pi i}{3}}$ then $I(\tau)$ is a 3–element subgroup of P_1 generated by TS.

(See Sect. 1.2 of Chapter VII of [59].) Derive, as a corollary, that S and T generate P_1.

Exercise 2..13. Verify that there are six images of the fundamental domain D (2..56) under the action of $\Gamma(1)$ incident with the vertex $e^{\frac{i\pi}{3}}$: they are obtained from D by applying the modular transformations 1, T, TS, TST, ST^{-1} and S. Note that all these domains are *triangles in the Lobachevsky's plane* [13] with two 60° ($=\pi/3$) angles and a zero angle vertex at the *oricycle*. They split into two orbits under the 3–element cyclic subgroup of P_1 generated by TS. Their union is the fundamental domain fo the index six subgroup $\Gamma(2)$ (defined in (2..60) below; cp. [47] Sect. 4.3).

Remark 2..2. $\Gamma(1)$ can be viewed, alternatively, as a homomorphic image of the braid group B_3 on three strands. Indeed, the group B_3 can be characterized in terms of the elementary braidings $b_i, i=1,2$, which interchange the end points i and $i+1$ and are subject to the *braid relation*

$$b_1\, b_2\, b_1 = b_2\, b_1\, b_2\,. \tag{2..57}$$

On the other hand, the group $\widetilde{\Gamma}$ with generators $\widetilde{S}$ and $\widetilde{T}$ satisfying only the relation (2..53) is isomorphic to the group B_3 since the mutually inverse maps

$$\widetilde{S} \mapsto b_1\, b_2\, b_1\,, \quad \widetilde{T} \mapsto b_1^{-1} \quad \text{and} \quad b_1 \mapsto \widetilde{T}^{-1}\,, \quad b_2 \mapsto \widetilde{T}\, \widetilde{S}\, \widetilde{T} \tag{2..58}$$

convert the relations (2..53) and (2..57) into one another. The element $\widetilde{S}^2$ is mapped to the generator of the (infinite) centre of B_3. Its image S^2 in $\Gamma(1)$ satisfies the additional constraint (2..54). It follows that B_3 appears as a central extension of $\Gamma(1)$.

We shall also need in what follows some *finite index subgroups* $\Gamma \subset \Gamma(1)$ (i. e. such that $\Gamma(1)/\Gamma$ has a finite number of cosets).

Let Λ' be a sublattice of Λ ($:= \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$) of a finite index N. This means that the factor group Λ/Λ' is a finite Abelian group of order $|\Lambda/\Lambda'| = N$. The set of such sublattices, $\{\Lambda' : |\Lambda/\Lambda'| = N\}$, is *finite* and the group $\Gamma(1)$ acts on it via $\Lambda' \mapsto \gamma(\Lambda')$ for $\gamma \in \Gamma(1)$ (here we assume that $\gamma\omega := (am+bn)\,\omega_1 + (cm+dn)\,\omega_2 \equiv (\omega_1,\omega_2)\,\gamma \begin{pmatrix} m \\ n \end{pmatrix}$ for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\omega = m\omega_1 + n\omega_2 \in \Lambda$).

[13] We thank Stanislaw Woronowicz for drawing our attention to this property. Nikolai Ivanovich Lobachevsky (1793–1856) publishes his work on the noneuclidean geometry in 1829/30. Jules–Henri Poincaré (1854–1912) proposes his interpretation of Lobachevsky's plane in 1882: it is the closed unit disk whose boundary is called *oricycle* with straight lines corresponding to either diameters of the disk or to circular arcs intersecting the oricycle under right angles. The upper half plane is mapped on the Poincaré disk by the complex conformal transformation $\tau \mapsto z = \dfrac{1+i\tau}{\tau+i}$.

Exercise 2..14. Find all index 2 sublattices Λ' of the lattice Λ from (2..3). *Answer*: $\Lambda_{01} := \mathbb{Z}\omega_1 + 2\mathbb{Z}\omega_2$, $\Lambda_{10} := 2\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ and $\Lambda_{11} := \mathbb{Z}(\omega_1 + \omega_2) + 2\mathbb{Z}\omega_2$. Prove that the stabilizer of Λ_{11}, denoted further by Γ_θ $(:= \{\gamma \in \Gamma(1) : \gamma(\Lambda_{11}) \subseteq \Lambda_{11}\})$, is

$$\Gamma_\theta = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1) : ac \text{ and } bd \text{ even} \right\}. \tag{2..59}$$

The group Γ_θ can be also characterized as the index 3 subgroup of $\Gamma(1)$ generated by S and T^2 (see [31] Sect. 13.4).

Other important finite index subgroups of $\Gamma(1)$ are the (normal) *principal congruence subgroups*

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1) \, : \, a \equiv 1 \; mod\, N \equiv d, \; b \equiv 0 \; mod\, N \equiv c \right\}, \tag{2..60}$$

(which justifies the notation $\Gamma(1)$ for $SL(2,\mathbb{Z})$) and the subgroups

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1) \; : \; c \equiv 0 \; mod\, N \right\}. \tag{2..61}$$

Proposition 2..4. ([62] Lemma 1.38). *Let $SL(2,\mathbb{Z}_N)$ be the (finite) group of 2×2 matrices γ whose elements belong to the finite ring $\mathbb{Z}_N \cong \mathbb{Z}/N\mathbb{Z}$ of integers mod N (and such that* $\det\gamma \equiv 1 \; mod\, N$*). If $f : \Gamma(1) \to SL(2,\mathbb{Z}_N)$ is defined by $f(\gamma) = \gamma \; mod\, N$, then the sequence*

$$1 \longrightarrow \Gamma(N) \longrightarrow \Gamma(1) \xrightarrow{f} SL(2,\mathbb{Z}_N) \longrightarrow 1 \tag{2..62}$$

is exact, i.e., the factor group $\Gamma(1)/\Gamma(N)$ is isomorphic to $SL(2,\mathbb{Z}_N)$.

We note that in the case $N = 2$ the factor group $SL(2,\mathbb{Z}_2)$ is isomorphic to the permutation group $\mathcal{S}_3$ with the identification

$$s_1 = f(T) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \;\; s_2 = f(TST) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \;\; (s_1^2 \equiv 1 \, mod\, 2 \equiv s_2^2). \tag{2..63}$$

In general, the number of elements of the factor group $SL(2, \mathbb{Z}_N)$ (i.e., the index of $\Gamma(N)$ in $\Gamma(1)$, by Proposition 2.3) is

$$\mu = N^3 \prod_{p|N} \left(1 - \frac{1}{p^2}\right) \tag{2..64}$$

(the product being taken over the primes p which divide N, [62] Sect. 1.6).

Remark 2..3. The (invariant) commutator subgroup of the braid group B_3 is the monodromy group M_3, which can, alternatively, be defined as the kernel of the group homomorphism of B_3 onto the 6–element symmetric group S_3 realized by the map $b_i \mapsto s_i$, $i = 1, 2$, where s_i are the elementary transpositions satisfying (2..57) and $s_i^2 = 1$. In other words, we have an exact sequence of groups and group homomorphisms,

$$1 \to M_3 \to B_3 \to S_3 \to 1, \quad \text{i. e.} \quad S_3 = B_3/M_3\,. \tag{2..65}$$

Exercise 2..15. Prove that the stabilizer of the sublattice $\Lambda_{01} = \mathbb{Z}\omega_1 + 2\mathbb{Z}\omega_2 \subseteq \Lambda$ is the subgroup $\Gamma_0\,(2)$. Thus $\Gamma_0\,(2)$ and Γ_θ are mutually conjugate subgroups of $\Gamma\,(1)$. Prove also that the action of $\Gamma\,(1)$ on the three element set $\{\Lambda_{10}, \Lambda_{01}, \Lambda_{11}\}$ of index 2 sublattices of Λ is equivalent to the above homomorphism $\Gamma\,(1) \xrightarrow{f} SL\,(2, \mathbb{Z}_2) \cong \mathcal{S}_3$. In fact, this action is given by the formula, $\Lambda_{\kappa\lambda} \to \Lambda_{[a\kappa+b\lambda]_2\,,\,[c\kappa+d\lambda]_2}$ for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1)$ (note that this is precisely the action of γ on the upper indices of $p_k^{\kappa\lambda}$ in (2..50)).

Exercise 2..16. Let Γ be a finite index subgroup of $\Gamma(1)$. Prove that there exists a nonzero power T^h (i.e., $h \neq 0$) belonging to Γ. (*Hint*: since there are finite number of right cosets ${}_{\Gamma}\backslash\Gamma(1)$ there exist $\gamma \in \Gamma$ and h_1, $h_2 \in \mathbb{Z}$, $h_1 \neq h_2$ such that $T^{h_1} = \gamma\, T^{h_2}$.)

3. Modular forms and ϑ–functions

3.1. Modular forms

Using the equivalence of proportional lattices we shall, from now on, normalize the periods as $(\omega_1, \omega_2) = (\tau, 1)$ with τ belonging to the upper half plane $\mathfrak{H}$ (2..41).

Let Γ be a subgroup of the modular group $\Gamma\,(1)$. An analytic function $G_k\,(\tau)$ defined on the upper half plane $\mathfrak{H}$ $(\ni \tau)$ is called a *modular form* of weight k and level Γ if

(i) it is Γ–covariant:

$$(c\tau + d)^{-k}\, G_k\!\left(\frac{a\tau + b}{c\tau + d}\right) = G_k(\tau) \quad \text{for} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma, \tag{3..1}$$

i.e., the expression $G_k(\tau)\,(d\tau)^{\frac{k}{2}}$ is Γ–invariant:

$$G_k(\gamma\tau)\,(d(\gamma\tau))^{\frac{k}{2}} = G_k(\tau)\,(d\tau)^{\frac{k}{2}} \quad \text{for} \quad \gamma\tau = \frac{a\tau + b}{c\tau + d} \tag{3..2}$$

(in view of the identity $d(\gamma\tau) = \dfrac{d\tau}{(c\tau + d)^2}$);

(ii) $G_k(\tau)$ admits a *Fourier*[14] *expansion* in non–negative powers of

$$q = e^{2\pi i \tau} \quad (\,|q| < 1\,). \tag{3..3}$$

The coefficients $g_2(\tau)$ and $g_3(\tau)$ (2..4) of the Weierstrass equation provide examples of modular forms of level $\Gamma(1)$ and weights 4 and 6, respectively.

Remark 3..1. The prefactor $j\,(\gamma,\tau) = (c\tau+d)^{-k}$ in (3..1), called "an automorphy factor", can be replaced by a general *cocycle*: $j\,(\gamma_1,\tau)\,j\,(\gamma_2,\gamma_1\tau) = j\,(\gamma_1\gamma_2,\tau)$ $(\gamma_1,\,\gamma_2\,\in\,\Gamma)$. If we stick to the prefactor $(c\tau+d)^{-k}$, then there are no non–zero modular forms of odd weights and level Γ provided $-1 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \in \Gamma$. Indeed, applying (3..1) to this element we find $G_k(\tau) = (-1)^k G_k(\tau)$, i. e. $G_k(\tau) = 0$ for odd k. For this reason we will mainly consider the case of even weights (for an example of a modular form of weight one – see Proposition 3.8).

Remark 3..2. If the modular group $\Gamma \subset \Gamma(1)$ contains a subgroup of type $\Gamma(N)$ (2..60) we shall also use *level* N for the minimal such N instead of "level Γ". In particular, a modular form of level $\Gamma(1)$ is commonly called a *level one form.*

Remarkably, the space of modular forms of a given weight and level is finite dimensional. This is based on the fact that every such modular form can be viewed as a *holomorphic section* of a line bundle over a *compact Riemann surface.* To explain this let us introduce the *extended* upper half plane

$$\mathfrak{H}^* := \mathfrak{H} \cup \mathbb{Q} \cup \{\infty\} \tag{3..4}$$

on which the modular group $\Gamma\,(1)$ acts so that $\mathbb{Q} \cup \{\infty\}$ is a single orbit. The set $\mathfrak{H}^*$ can be endowed with a Hausdorff[15] topology, extending that of $\mathfrak{H}$, in such a way that the quotient space[16] ${}_{\Gamma\,(1)}\backslash\mathfrak{H}^*$ is isomorphic, as a topological space (i.e., it is homeomorphic), to the Riemann sphere with a distinguished point, the orbit $\mathbb{Q} \cup \{\infty\}$. The points of the set $\mathbb{Q} \cup \{\infty\}$ are called *cusps* of the group[17] $\Gamma(1)$ as well as of any finite index subgroup Γ in $\Gamma(1)$. Then the quotient space ${}_{\Gamma}\backslash\mathfrak{H}^*$ is homeomorphic to a compact Riemann surface with distinguished points, the cusps' orbits (with respect to Γ). For more details on this constructions we refer the reader to [62].

[14] Jean Baptiste Joseph Fourier (1768–1830).
[15] Felix Hausdorff (1868–1942).
[16] Following the custom we will use the left coset notation for the discrete group action while $\mathfrak{H}$ can be viewed as a right coset, $\mathfrak{H} = SL\,(2,\mathbb{R})\,/SO\,(2)$, the maximal compact subgroup $SO\,(2)$ of $SL\,(2,\mathbb{R})$ being the stabilizer of the point i in the upper half–plane.
[17] The cusps τ of $\mathfrak{H}^*$ (with respect to some subgroup Γ of $\Gamma(1)$) are characterized by the property that they are left invariant by an element of Γ conjugate to T^n (2..55) for some $n \in \mathbb{Z}$.

Proposition 3..1. ([48] Chapt. 4) *Every modular form of weight* $2k$ *and level* Γ, *for a finite index subgroup* Γ *of* $\Gamma(1)$, *can be extended to a meromorphic section of the line bundle of* k*–differentials* $g(\tau)(d\tau)^k$ *over the compact Riemann surface* ${}_{\Gamma}\backslash\mathfrak{H}^*$. *The degree of the pole of the resulting meromorphic section at every cusp is not smaller than* $-k$ *and the degree of the pole at an image* $[\tau]_\Gamma \in {}_{\Gamma}\backslash\mathfrak{H}$ *of a point* $\tau \in \mathfrak{H}$ *is not smaller than* $-\llbracket k\left(1-\dfrac{1}{e_{\tau,\Gamma}}\right)\rrbracket$ *where* $e_{\tau,\Gamma}$ *is the order of the stabilizer of* τ *in* $\Gamma/\{\pm 1\}$ *and* $\llbracket a \rrbracket$ *stands for the integer part of the real number* a.

Note that for the points $\tau \in \mathfrak{H}$ having unit stabilizer in $\Gamma/\{\pm 1\}$ (i.e., $e_{\tau,\Gamma} = 1$) the corresponding holomorphic sections of Proposition 3.1 have no poles at $[\tau]_\Gamma$. This is because then the canonical projection $\mathfrak{H}^* \to {}_{\Gamma}\backslash\mathfrak{H}^*$ is local (analytic) diffeomorphism around $[\tau]_\Gamma$. On the other hand, if $e_{\tau,\Gamma} > 1$ then $[\tau]_\Gamma$ is a *ramification* point for the projection $\mathfrak{H}^* \to {}_{\Gamma}\backslash\mathfrak{H}^*$, so that a holomorphic (invariant) differential is projected, in general, to a meromorphic differential. For example, the weight 2 holomorphic differential $(dz)^2$ is invariant under the projection $z \mapsto w = z^2$ and it is projected to $(1/4)\,w^{-1}\,(dw)^2$ $(= (1/4)\,z^{-2}\,(2z)^2\,(dz)^2)$.
Exercise 2.12 implies that $e_{\tau,\Gamma}$ for any subgroup Γ of $\Gamma(1)$ is either 1 or 2, or 3. Let us set ν_ℓ to be the number of points $[\tau]_\Gamma \in {}_{\Gamma}\backslash\mathfrak{H}^*$ with $e_{\tau,\Gamma} = \ell$ for $\ell = 2, 3$ and let ν_∞ be the number of cusps' images in ${}_{\Gamma}\backslash\mathfrak{H}^*$.

Corollary 3..2. *For* $k = 0$ *Proposition 3.1 implies that every modular form of weight* 0 *is represented by holomorphic function over a compact Riemann surface and therefore, it is constant by Liouville's theorem.*

The Liouville theorem has a generalization to meromorphic sections of line bundles over compact complex surfaces – this is the Riemann–Roch[18] theorem ([48], Chapt. 1) stating that the vector space of such sections with fixed singularities is *finite dimensional.*

Theorem 3..3. (See [48] Theorem 2.22 and Theorem 4.9, and [62] Proposition 1.40.) *The vector space of modular forms of weight* $2k$ *and level* Γ *(a finite index subgroup of* $\Gamma(1)$*) has finite dimension*

$$d_{2k,\Gamma} = \begin{cases} 0 & \text{for } k < 0 \\ 1 & \text{for } k = 0 \\ (2k-1)\,(g_\Gamma - 1) + \nu_\infty k + & \text{for } k > 0, \\ \quad + \llbracket \dfrac{k\,\nu_2}{2} \rrbracket + \llbracket \dfrac{2k\,\nu_3}{3} \rrbracket & \end{cases} \tag{3..5}$$

where g_Γ *is the genus of the Riemann surface* ${}_{\Gamma}\backslash\mathfrak{H}^*$ *which can be calculated using the index* μ *of the subgroup* Γ *in* $\Gamma(1)$ *by the formula*

$$g_\Gamma = 1 + \frac{\mu}{12} - \frac{\nu_2}{4} - \frac{\nu_3}{3} - \frac{\nu_\infty}{2}\,. \tag{3..6}$$

[18] Gustav Roch (1839–1866).

Remark 3..3. In the case of level 1 modular forms: $g_{\Gamma(1)} = 0$, $\nu_\infty = 1$ and $e_{\tau,\Gamma(1)}$ takes nonunit values only at the images $[\tau]_{\Gamma(1)}$ of $\tau = i$ and $\tau = e^{\frac{2\pi i}{3}}$ which are 2 and 3, respectively, i.e., $\nu_2 = \nu_3 = 1$ (see Exercise 2.12). Then Eq. (3..5) takes for $k = 1, \ldots, 17, \ldots$ the form

$$d_{2k,\Gamma(1)} = 1 - k + [\![\frac{k}{2}]\!] + [\![\frac{2k}{3}]\!] = 1, 0, 1, 1, 1, 1, 2, 1, 2, 2, 2, 2, 3, 2, 3, 3, 3, 3, \ldots \tag{3..7}$$

(in the next subsection we will derive independently this formula in a more direct fashion, establishing on the way the recurrence relation $d_{2k+12,\Gamma(1)} = d_{2k,\Gamma(1)} + 1$). For the principal congruence subgroups $\Gamma(N)$ (2..60), we have, when $N > 1$ (see [62], Sect. 1.6): $\nu_2 = \nu_3 = 0$, $\nu_\infty = \dfrac{\mu}{N}$ and μ is given by Eq. (2..64). In particular, $g_{\Gamma(N)} = 0$ for $1 \leqslant N \leqslant 5$, $g_{\Gamma(6)} = 1$, $g_{\Gamma(7)} = 3$, $g_{\Gamma(8)} = 5$, $g_{\Gamma(9)} = 10$, $g_{\Gamma(10)} = 13$, $g_{\Gamma(11)} = 26$.

3.2. Eisenstein series. The discriminant cusp form

We proceed to describing the modular forms of level one. Let $\mathcal{M}_k$ be the space of all such modular forms. As a consequence of Corollary 3.2 and Theorem 3.3, $\mathcal{M}_0$ is 1–dimensional (it consists of constant functions) and $\mathcal{M}_1 = \mathcal{M}_{2k+1} = \{0\}$.
Examples of non–trivial modular forms are given by the *Eisenstein series*

$$\begin{aligned} G_{2k}(\tau) &= \frac{(2k-1)!}{2(2\pi i)^{2k}} \sum_{(m,n)}{}' (m\tau + n)^{-2k} := \\ &= \frac{(2k-1)!}{(2\pi i)^{2k}} \left\{ \sum_{n=1}^{\infty} \frac{1}{n^{2k}} + \sum_{m=1}^{\infty} \sum_{n\in\mathbb{Z}} (m\tau + n)^{-2k} \right\} . \end{aligned} \tag{3..8}$$

Note that for $2k \geqslant 4$ we have $G_{2k}(\tau) = G_{2k}(\tau, 1)$:

$$G_{2k}(\omega_1, \omega_2) := \frac{(2k-1)!}{2(2\pi i)^{2k}} \sum_{\omega\in\Lambda\backslash\{0\}} \omega^{-2k} \quad (\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2) \tag{3..9}$$

where the series is absolutely convergent and therefore, it does not depend on the basis (ω_1, ω_2):

$$G_{2k}(a\omega_1 + b\omega_2, c\omega_1 + d\omega_2) = G_{2k}(\omega_1, \omega_2) \quad \text{for} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1) . \tag{3..10}$$

It follows that, $G_{2k}(\tau)$ satisfies for $2k \geqslant 4$ the conditions (i) for modular forms since we have

$$G_{2k}(\rho\omega_1, \rho\omega_2) = \rho^{-2k} G_{2k}(\omega_1, \omega_2) . \tag{3..11}$$

For $2k = 2$ the series (3..8) is only conditionally convergent and it is not modular invariant (the sum depends on the choice of lattice basis, see below). To verify the second condition one can use the *Lipschitz formula* (see, e. g., Zagier in [22] Appendix)

$$\frac{(k-1)!}{(-2\pi i)^k} \sum_{n\in\mathbb{Z}} \frac{1}{(z+n)^k} \; = \; \sum_{l=1}^{\infty} l^{k-1} e^{2\pi i l z} \tag{3..12}$$

and deduce the Fourier expansion of G_{2k} (for $k \geqslant 1$):

$$G_{2k}(\tau) = -\frac{B_{2k}}{4k} + \sum_{n=1}^{\infty} \frac{n^{2k-1}}{1-q^n} q^n = \frac{1}{2}\,\zeta(1-2k) + \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n \tag{3..13}$$

where $\sigma_l(n) \; = \; \sum\limits_{r|n} r^l$ (sum over all positive divisors r of n), B_l are the *Bernoulli numbers* [19] which are generated by the Planck [20] distribution function:

$$\frac{x}{e^x-1} = \sum_{l=0}^{\infty} B_l \, \frac{x^l}{l!}; \; B_0 = 1, \; B_1 = -\frac{1}{2}, \; B_2 = \frac{1}{6}, \; B_3 = \cdots = B_{2k+1} = 0,$$

$$B_4 = B_8 = -\frac{1}{30}\,, \; B_6 = \frac{1}{42}\,, \; B_{10} = \frac{5}{66}\,, \; B_{12} = -\frac{691}{2730}\,, \; B_{14} = \frac{7}{6}\,, \; ... \tag{3..14}$$

$\zeta(s)$ is the Riemann ζ–function [21]. Remarkably, for $n \geqslant 1$, all Fourier coefficients of G_{2k} are positive integers.
Thus, for $k \geqslant 2$ the functions G_{2k} are modular forms of weight $2k$ (satisfying (3..1)). For $k = 1$, however, we have instead

$$(c\tau+d)^{-2} G_2(\frac{a\tau+b}{c\tau+d}) \; = \; G_2(\tau) + \frac{i}{4\pi}\frac{c}{c\tau+d} \tag{3..15}$$

so that only $G_2^*(\tau)d\tau$ is modular invariant where G_2^* is the non–holomorphic

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[19] Jacob Bernoulli (1654–1705) is the first in the great family of Basel mathematicians (see [3], pp. 131–138 for a brief but lively account). The Bernoulli numbers are contained in his treatise Ars Conjectandi on the theory of probability, published posthumously in 1713.

[20] Max Planck (1858–1947) proposed his law of the spectral distribution of the black–body radiation in the fall of 1900 (Nobel Prize in Physics, 1918) – see M. J. Klein in [29].

[21] The functional equation $\Gamma(\frac{s}{2})\pi^{-\frac{s}{2}}\zeta(s) = \Gamma(\frac{1-s}{2})\pi^{\frac{s-1}{2}}\zeta(1-s)$, which allows the analytic continuation of $\zeta(s) = \sum\limits_{n=1}^{\infty} n^{-s}$ as a meromorphic function (with a pole at $s = 1$) to the entire complex plane s, was proven by Riemann in 1859 (see also Cartier's lecture in [22]).

function

$$G_2^*(\tau) := -\frac{1}{8\pi^2} \lim_{\varepsilon \searrow 0} \left(\sum_{(m,n) \neq (0,0)} (m\tau + n)^{-2} |m\tau + n|^{-\varepsilon} \right)$$
$$= G_2(\tau) + \frac{1}{8\pi\tau_2}, \tag{3..16}$$

$\tau_2 = Im\, \tau$. (As the Eisenstein series (3..10) is divergent for $k = 1$, Eq. (3..16) can be taken as an alternative definition of $G_2(\tau)$ which can be shown to agree with (2..13).) In fact, there is no non–zero (level 1, holomorphic) modular form of weight 2 as a consequence of Theorem 3.3. There exist, on the other hand, level two forms of weight 2. We shall use in applications to CFT the fact that

$$F_2(\tau) := 2G_2(\tau) - G_2(\frac{\tau+1}{2}) \tag{3..17}$$

is a modular form of weight 2 and level Γ_θ (2..59).

Exercise 3..1. Prove that the functions $p_k^{\kappa\lambda}(z,\tau)$, (2..15), (2..48), ($k = 1, 2, \ldots,\ \kappa, \lambda = 0, 1$) have the Laurent [22] expansions

$$p_k^{\kappa\lambda}(z,\tau) = \frac{1}{z^k} + (-1)^k \sum_{n=1}^{\infty} \binom{2n-1}{k-1} \frac{2(2\pi i)^{2n}}{(2n-1)!} G_{2n}^{\kappa\lambda}(\tau)\, z^{2n-k}, \tag{3..18}$$

where $G_{2k}^{00}(\tau)$ coincides with the above introduced $G_{2k}(\tau)$ for $k = 1, 2, \ldots$, $G_2^{11}(\tau)$ coincides with $F_2(\tau)$ (3..17) and $G_{2k}^{\kappa\lambda}(\tau)$ has the following absolutely convergent, Eisenstein series representation:

$$G_{2k}^{\kappa\lambda}(\tau) := \frac{(2k-1)!}{2(2\pi i)^{2k}} \sum_{(m,n) \in \mathbb{Z}\times\mathbb{Z} \backslash \{(0,0)\}} (-1)^{\kappa m + \lambda n} (m\tau + n)^{-2k} \tag{3..19}$$

for $k \geqslant 2$. Using this prove that all $G_{2k}^{11}(\tau)$ (including $F_2(\tau) \equiv G_2^{11}(\tau)$) are modular forms of weight $2k$ and level Γ_θ for every $k = 1, 2, \ldots$. (See Sect. III.7 of [68] where the case $\kappa = \lambda = 0$ is considered.)

If there are $d_k > 1$ modular forms of weight k and a fixed level, then one can form $d_k - 1$ linearly independent linear combinations S_k of them, which have no constant term in their Fourier expansion. Such forms, characterized by the condition $S_k \to 0$ for $q \to 0$, are called *cusp forms* [23]. We denote by $\mathcal{S}_k$ the subspace of cusp forms. The first nonzero cusp form of level one appears for weight 12 and, as we shall see, its properties allow to determine the general structure of level one modular forms.

[22] P.A. Laurent (1813–1854) introduces his series in 1843.

[23] The notation S_k for the cusp forms comes from the German word Spitzenform. (The term "parabolic form" is used in the Russian literature.)

Proposition 3..4. *The 24th power of the Dedekind[24] η–function*

$$\Delta(\tau) \,=\, [\eta(\tau)]^{24} \,=\, q \prod_{n=1}^{\infty} (1-q^n)^{24} \tag{3..20}$$

is a cusp form of weight 12.

Proof. As $\Delta(\tau)$ clearly vanish for $q=0$ we have just to show that it is a modular form of degree 12. To this end we compute the logarithmic derivative

$$\frac{\Delta'(\tau)}{\Delta(\tau)} \,=\, 2\pi i \left(1 - 24 \sum_{n=1}^{\infty} \frac{nq^n}{1-q^n}\right) \,=\, -48\pi i G_2(\tau). \tag{3..21}$$

It then follows from (3..15) that

$$\frac{d}{d\tau}\left(\log \Delta\left(\frac{a\tau+b}{c\tau+d}\right)\right) \,=\, \frac{d}{d\tau}\left(\log\left[(c\tau+d)^{12}\Delta(\tau)\right]\right) \tag{3..22}$$

and hence, noting that $\Delta(\tau)$ (3..20) is T–invariant (i.e., periodic of period 1 in τ), we conclude that it is indeed a modular form of weight 12. $\square$

Theorem 3..5. *The only non–zero dimensions $d_k = \dim \mathcal{M}_k$ are given (recursively) by*

$$d_0 = d_4 = d_6 = d_8 = d_{10} = 1\,; \quad d_{12+2k} = d_{2k} + 1\,, \quad k = 0, 1, 2, \ldots. \tag{3..23}$$

In particular, $d_k = 0$ for $k < 1$ and $\dim \mathcal{S}_k = 0$ for $k < 12$.

Proof. 1. If there were a modular form f of weight $-m$ ($m > 0$) then the function $f^{12}\triangle^m$ would have had weight 0 and a Fourier expansion with no constant term, which would contradict the Liouville theorem (cf. Theorem 3.3).

2. There is no cusp form of weight smaller than 12. Had there been one, say $S_k(\tau)$, with $k < 12$, then $S_k(\tau)/\Delta(\tau)$ would be a modular form of weight $k-12 < 0$ in contradiction with the above argument. (Here we use the fact that S_k/Δ is holomorphic in $\mathfrak{H}$ since the product formula (3..20) shows that $1/\Delta$ has no poles in the upper half plane.)

The theorem follows by combining these results with Proposition 3.3 and the argument that there is no level 1 modular form of weight 2. $\square$

Remark 3..4. In fact, the linear span of all level 1 modular forms of an arbitrary weight is the free commutative algebra generated by $G_4(\tau)$ and $G_6(\tau)$, i.e., it is the polynomial algebra $\mathbb{C}\,[G_4, G_6]$ (D. Zagier [22]).

[24] Richard Dedekind (1831–1916) also introduced (in 1877) the absolute invariant j (3..27) – as well as the modern concepts of a ring and an ideal.

Corollary 3..6. *The Dedekind η–function (3..20) is proportional to the discriminant of the right hand side of (2..34):*

$$\Delta(\tau) = (2\pi)^{-12}\left[g_2^3(\tau) - 27 g_3^2(\tau)\right] \qquad (3..24)$$
$$= [20\,G_4(\tau)]^3 - 3\,(7\,G_6(\tau))^2 \qquad (\in \mathcal{S}_{12}).$$

Proof. The difference $(20\,G_4)^3 - 3\,(7\,G_6)^2$ also belongs to $\mathcal{S}_{12}$ since, due to (3..14),

$$\left(-20\frac{B_4}{8}\right)^3 - 3\left(-7\frac{B_6}{12}\right)^2 = \frac{1}{(12)^3} - \frac{3}{(72)^2} = 0\,.$$

Noting further that $\dim \mathcal{S}_{12} = 1$ (Theorem 3.5) and comparing the coefficient to q in (3..25) ($1 = 60\left(20\frac{B_4}{8}\right)^2 + 21\left(\frac{7B_6}{6}\right)$), we verify the relation $\Delta(\tau) = (20\,G_4)^3 - 3\,(7\,G_6)^2$. The first equation (3..25) then follows from the relations

$$20\,G_4(\tau) = (2\pi)^{-4} g_2(\tau)\,; \quad 7\,G_6(\tau) = -\frac{3}{(2\pi)^6}\, g_3(\tau)\,. \qquad (3..25)$$

□

Remark 3..5. $\mathcal{S}_k$ is a Hilbert space, equipped with the *Peterson scalar product*

$$(f,g) = \iint\limits_{B/\Gamma(1)} \tau_2^k \overline{f}(\tau) g(\tau) d\mu \quad \text{where} \quad \tau = \tau_1 + i\tau_2\,, \quad d\mu = \frac{d\tau_1 d\tau_2}{\tau_2^2}\,, \qquad (3..26)$$

$d\mu$ being the $SL(2,\mathbb{R})$–invariant measure on $\mathfrak{H}$.

Remark 3..6. Comparing the constant term in the expansion (3..13) of G_{2k} with the first few dimensions d_{2k} we notice that $\mathcal{S}_{2k} = \{0\}$ for exactly those values of k for which $-B_{2k}/(4k)$ is the reciprocal of an integer (namely for $2k = 2$, 4, 6, 8, 10, and 14). The curious reader will find a brief discussion of this (nonaccidental) fact in Sect. 1B of Zagier's lectures in [22]. The existence of the discriminant form $\Delta(\tau)$ (3..20) – whose zeros are precisely the cusps of $\Gamma(1)$ – allows to define the modular invariant function [25]

$$j(\tau) = \frac{[240\,G_4(\tau)]^4}{\Delta(\tau)} = q^{-1} + 744 + 196\,884\,q + 21\,493\,760\,q^2 + \ldots \qquad (3..27)$$

[25] Gauss was apparently aware of the j-function before 1800; Charles Hermite (1822–1901) used it in solving the quintic equation in 1859; Dedekind gave a nice definition in 1877; Klein studied the function around 1880. (The authors thank John McKay for drawing their attention to the early work of Gauss and Hermite.)

that is analytic in the upper half plane $\mathfrak{H}$ but grows exponentially for $\tau \to i\infty$. The following proposition shows that j is, in some sense, the unique function with these properties.

Proposition 3..7. *If $\Phi(\tau)$ is any modular invariant analytic function in $\mathfrak{H}$ that grows at most exponentially for $Im\,\tau \to \infty$ then $\Phi(\tau)$ is a polynomial in $j(\tau)$.*

Proof. The function $f(\tau) = \Phi(\tau)\,[\Delta(\tau)]^m$ transforms as a modular form of weight $12m$ and if m is large enough, it is bounded at infinity, hence $f(\tau) \in \mathcal{M}_{12m}$. It then follows from Theorem 3.5 and Remark 3.4 that f is a homogeneous polynomial of degree m in G_4^3 and Δ. Therefore, $\Phi = f/\Delta^m$ is a polynomial of degree not exceeding m in j. $\square$

In fact, j can be viewed as a (complex valued) function on the set of 2–dimensional Euclidean lattices invariant under rotation and rescaling – see the thought provoking discussion in Sect. 6 of [46].

Remark 3..7. The function $j(\tau) - 744 = q^{-1} + 196\,884\,q + \ldots$ is called *Hauptmodul* of $\Gamma(1)$ (see [24], Sect. 2).

Remark 3..8. As observed by McKay in 1978 (see [24] for a review and references)

$$[j(\tau)]^{\frac{1}{3}} \;=\; q^{-\frac{1}{3}}\left(1 + 248\,q + 4124\,q^2 + \ldots\right) \qquad (3..28)$$

is the character of the level 1 affine Kac–Moody algebra $(\hat{E}_8)_1$ (see also Sect. 5 below).

3.3. ϑ–functions

Each meromorphic (and hence each elliptic) function can be presented as a ratio of two entire functions. According to property (2) of Sect. 2.1 these cannot be doubly periodic but, as we shall see, they may satisfy a twisted periodicity condition.
We shall construct a family of entire analytic functions which allow for a multiplicative cocycle defining the twisted periodicity condition. A classical example of this type is provided by the *Riemann ϑ–function*[26]:

$$\vartheta(z,\tau) \;=\; \sum_{n\in\mathbb{Z}} q^{\frac{1}{2}n^2} e^{2\pi i n z} \;=\; 1+2\sum_{n=1}^{\infty} q^{\frac{1}{2}n^2}\cos 2\pi n z\,, \quad q^{\frac{1}{2}} = e^{i\pi\tau}, \quad (3..29)$$

[26] ϑ functions appear before Riemann in Bernoulli's *Ars Conjectandi* (1713), in the number theoretic studies of Euler (1773) and Gauss (1801), in the study of the heat equation of Fourier (1826), and, most importantly, in Jacobi's *Fundamenta Nova* (1829).

which belongs to the family of four Jacobi ϑ–functions[27]

$$\vartheta_{\mu\nu}(z,\tau) \;=\; e^{i\pi\frac{\mu}{2}(\frac{\mu}{2}\tau-2z+\nu)}\,\vartheta(z-\frac{\mu\tau}{2}-\frac{\nu}{2},\tau), \quad \mu,\nu=0,1 \tag{3..30}$$

($\vartheta_{00}=\vartheta$). They satisfy the "twisted periodicity" conditions

$$\vartheta_{\mu\nu}(z+1,\tau) = \;(-1)^{\mu}\,\vartheta_{\mu\nu}(z,\tau), \tag{3..31}$$
$$\vartheta_{\mu\nu}(z+\tau,\tau) = \;(-1)^{\nu}q^{-\frac{1}{2}}e^{-2\pi i z}\,\vartheta_{\mu\nu}(z,\tau). \tag{3..32}$$

Note, in particular, that ϑ_{11} is the only odd in z among the four ϑ–functions and it can be written in the form

$$\vartheta_{11}(z,\tau) \;=\; 2\sum_{n=0}^{\infty}\,(-1)^n\,q^{\frac{1}{2}(n+\frac{1}{2})^2}\,\sin{(2n+1)\,\pi z} \tag{3..33}$$

while the others are even and can be written as follows (together with (3..29))

$$\vartheta_{01}(z,\tau) = \;1+2\sum_{n=1}^{\infty}\,(-1)^n\,q^{\frac{1}{2}n^2}\,\cos 2\pi n z\,,$$
$$\vartheta_{10}(z,\tau) = \;2\sum_{n=1}^{\infty}\,q^{\frac{1}{2}(n-\frac{1}{2})^2}\,\cos{(2n-1)\,\pi z}\,. \tag{3..34}$$

Thus ϑ_{11} has an obvious zero for $z=0$ and hence vanishes (due to the twisted periodicity) for all $z=m\tau+n$. In fact, this is the full set of zeros in z of ϑ_{11} (which one can prove applying the Cauchy theorem to the logarithmic derivative of ϑ_{11}). Using (3..30) we can then also find the zeros of all four Jacobi ϑ–functions. This allows to deduce the following infinite product expression for $\vartheta_{\mu\nu}$:

$$\vartheta_{\substack{00\\1}}(z,\tau) = \;\prod_{n=1}^{\infty}(1-q^n)\big(1\pm 2q^{n-\frac{1}{2}}\,\cos 2\pi z+q^{2n-1}\big)\;,$$
$$\vartheta_{10}(z,\tau) = \;2q^{\frac{1}{8}}\cos 2\pi z\prod_{n=1}^{\infty}(1-q^n)\big(1+2q^n\,\cos 2\pi z+q^{2n}\big)\;, \tag{3..35}$$
$$\vartheta_{11}(z,\tau) = \;2q^{\frac{1}{8}}\sin 2\pi z\prod_{n=1}^{\infty}(1-q^n)\big(1-2q^n\,\cos 2\pi z+q^{2n}\big)\;.$$

[27] A more common notation for the Jacobi ϑ–functions is $\vartheta_{11}=\vartheta_1$, $\vartheta_{10}=\vartheta_2$, $\vartheta_{00}=\vartheta_3$, and $\vartheta_{01}=\vartheta_4$. Many authors also write $q=e^{i\pi\tau}$ instead of $q=e^{2\pi i\tau}$; with our choice the exponent of q will coincide with the conformal dimension – see Sects. 4-7 below. The function $\vartheta_{11}=\vartheta_1$ plays an important role both in the study of the elliptic Calogero–Sutherland model [41] and in the study of thermal correlation functions (Sect. 4.4 below).

One is naturally led to the above definition by considering (as in [50]) the action on ϑ of the *Heisenberg–Weyl* [28] *group* $U(1) \times \mathbb{R}^2$ that appears as a central extension of the 2–dimensional Abelian group $\mathbb{R}^2$. It is generated by the two 1–parameter subgroups U_a and V_b acting on (say, entire analytic) functions $f(z)$ as:

$$(U_a f)(z) = e^{\pi i (a^2 \tau + 2az)}\, f(z + a\tau)\,, \quad (V_b f)(z) = f(z+b)\,. \tag{3..36}$$

A simple calculation gives $U_{a_1+a_2} = U_{a_1} U_{a_2}$,

$$e^{2\pi i ab} U_a V_b = V_b U_a\,. \tag{3..37}$$

The function $\vartheta(z,\tau)$ (3..29) is invariant under the discrete subgroup $\mathbb{Z}^2 = \{(a,b) \,:\, a,b \in \mathbb{Z}\}$ which is commutative since $e^{2\pi i ab} = 1$ for integer ab. If $a = \dfrac{\mu}{2}$, $b = \dfrac{\nu}{2}$ then the action of $V_b U_a$ gives rise to the four functions (3..30)

$$\vartheta_{\mu\nu}(z,\tau) \;=\; \Big(V_{-\frac{\nu}{2}} U_{-\frac{\mu}{2}} \vartheta \Big)(z,\tau)\,. \tag{3..38}$$

Similarly, for $a,b \in (1/l)\,\mathbb{Z}$ we obtain l^2 ϑ–functions (that are encountered in CFT applications).
The functions $\vartheta_{\mu\nu}(z,\tau)$ (3..30) are solutions of the Schrödinger equation [29]

$$i \frac{\partial}{\partial \tau} \vartheta_{\mu\nu}(z,\tau) \;=\; \frac{1}{4\pi} \frac{\partial^2}{\partial z^2}\, \vartheta_{\mu\nu}(z,\tau) \tag{3..39}$$

and so are, in fact, all functions of the type $V_b U_a \vartheta$.
The following fact is basic in the general theory of ϑ–functions:

Proposition 3..8. (See D. Zagier [22] Sect. 1C p. 245). *Given an r–dimensional lattice Λ_r in which the length squared $Q(x)$ of any vector $x \in \Lambda_r$ is an integer, the multiplicities of these lengths are the Fourier coefficients of a modular form*

$$\Theta_Q(\tau) = \sum_{x \in \Lambda_r} q^{Q(x)} \tag{3..40}$$

of weight $r/2$. More precisely, there exists a positive integer N and a character χ of $\Gamma_0\,(N)$ such that

$$\Theta_Q\,(\gamma\tau) = \chi\,(d)\,(c\tau + d)^{\frac{r}{2}}\, \Theta_Q\,(\tau) \quad \textit{for} \;\; \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0\,(N)\,. \tag{3..41}$$

[28] Werner Heisenberg (1901–1976), Nobel Prize in physics 1932; Hermann Weyl (1885–1955).
[29] Eq. (3..39) was known to J. Fourier (as "the heat equation"), long before Erwin Schrödinger (1887–1961, Nobel Prize in Physics 1933) was born.

For $Q(x) = 1/2\, xAx$ where A is an even symmetric $r \times r$ matrix, the level N is the smallest positive integer such that NA^{-1} is again even.

Here is an $r = 2$ example:

$$\begin{aligned} &Q = x_1^2 + x_2^2\,, \quad \Theta_Q \;=\; 1 + 4q + 4q^2 + 4q^4 + 8q^5 + 4q^8 + \ldots\,, \\ &A \;=\; \left(\begin{array}{cc} 2 & 0 \\ 0 & 2 \end{array}\right), \quad N = 4\,, \quad \chi(d) \;=\; (-1)^{\frac{d-1}{2}}\,. \end{aligned} \tag{3..42}$$

Remark 3..9. The modular form $G_4(\tau)$ can be expressed in terms of the Jacobi ϑ–functions as:

$$\begin{aligned} 240\, G_4(\tau) \;&=\; \frac{1}{2}\left\{\vartheta_{00}^8(0,\tau) + \vartheta_{10}^8(0,\tau) + \vartheta_{01}^8(0,\tau)\right\} \\ &=\; 1 + 240\,\frac{q}{1-q} + \ldots\,. \end{aligned} \tag{3..43}$$

4. Quantum field theory and conformal invariance (a synopsis)

For the benefit of mathematician readers we shall give a brief summary of the general properties of quantum fields (see [63], [4], [15] for more details and proofs), and of the role of the conformal group. (Our review will be necessarily one-sided: such central concepts of real world quantum field theory (QFT) as perturbative expansions, Feynman graphs, and Feynman path integral won't be even mentioned.)

4.1. Minkowski space axioms. Analyticity in tube domains

Quantum fields generate – in the sense of [63] – an operator algebra in a vacuum state space $\mathcal{V}$. A closely related approach, [27], [1], starts with an abstract C^*-algebra - the algebra of local observables – and constructs different state spaces as Hilbert space representations of this algebra, defining the different superselection sectors of the theory. (Important recent progress relating Haag's algebraic approach to 2D CFT – see [36] – is beyond the scope of the present notes.)

In the Wightman approach the fields are described as operator valued distributions over *Minkowski*[30] *space–time* M. It is a D-dimensional real affine space equipped with a Poincaré invariant interval, which assumes, in Cartesian coordinates, the form

$$x_{12}^2 = \boldsymbol{x}_{12}^2 - (x_{12}^0)^2\,, \qquad x_{12} = x_1 - x_2\,, \qquad \boldsymbol{x}^2 = \sum_{i=1}^{D-1} x_i^2. \tag{4..1}$$

[30] Hermann Minkowski (1864–1909) introduces the 4–dimensional space-time (in 1908 in Göttingen), thus completing the creation of the special theory of relativity of Hedrik Antoon Lorentz (1853-1928, Nobel Prize in physics, 1902), Henri Poincaré, and Albert Einstein (1979-1955, Nobel Prize in physics, 1921).

The *state space* $\mathcal{V}$ is a *pre-Hilbert space* carrying a (reducible) *unitary positive energy representation* $U(a,\widetilde{\Lambda})$ of the (quantum mechanical) *Poincaré group* $Spin(D-1,1) \ltimes \mathbb{R}^D$. This means that the joint spectrum of the (hermitian, commuting) translation generators $P_0, \boldsymbol{P}$ in (the Hilbert space completion of) $\mathcal{V}$ belongs to the positive light-cone V_+ (*spectral condition*):

$$V_+ := \left\{ P \in \mathbb{R}^D \,:\, P_0 \geq |\boldsymbol{P}| \equiv \sqrt{\boldsymbol{P}^2} \right\} \qquad (U(a,\mathbb{I}) \,=\, e^{-iaP}). \tag{4..2}$$

(Boldface letters, $\boldsymbol{x}$, $\boldsymbol{P}$, denote throughout $(D-1)$–vectors.) Furthermore, $\mathcal{V}$ is assumed to have a *1-dimensional translation invariant subspace* spanned by the *vacuum vector* $|0\rangle$ (which is, as a consequence, also Lorentz invariant):

$$|0\rangle \in \mathcal{V}, \quad P^\mu |0\rangle = 0 \, (= (U(a,\widetilde{\Lambda}) - 1)|0\rangle \,), \quad \langle 0|0\rangle = 1. \tag{4..3}$$

The field algebra is generated by a finite number of (finite component) spin-tensor fields $\phi(x)$. Each ϕ is an *operator valued distribution on* $\mathcal{V}$: the smeared field $\phi(f)$ for any Schwartz[31] test function $f(x)$ is defined on $\mathcal{V}$ and leaves it invariant. The fields ϕ obey the *relativistic covariance condition*:

$$U(a,\widetilde{\Lambda})\,\phi(x)\,U^{-1}(a,\widetilde{\Lambda}) \,=\, S(\widetilde{\Lambda}^{-1})\,\phi(\Lambda x + a). \tag{4..4}$$

Here $S(\widetilde{\Lambda})$ is a finite dimensional representation of the spinorial (quantum mechanical) Lorentz group $Spin(D-1,1)$ of $2^{d_0} \times 2^{d_0}$ matrices; for even D, the case of interest here, the exponent d_0 coincides with the canonical dimension of a free massless scalar field,

$$d_0 = \frac{D-2}{2} \tag{4..5}$$

(in general, the spinorial representation has dimension $2^{[\![\frac{D-1}{2}]\!]}$ where $[\![\rho]\!]$ stands for the integer part of the positive real ρ); $\Lambda \in SO(D-1,1)$ is the (proper) Lorentz transformation corresponding to the matrices $\pm\widetilde{\Lambda}$ ($-\mathbb{I}$ belonging to the centre of the group $Spin(D-1,1)$). We assume that $S(-\mathbb{I})$ is a multiple (with a sign factor) of the identity operator:

$$S(-\mathbb{I})\phi(x) = \varepsilon_\phi \phi(x), \quad \varepsilon_\phi = \pm 1. \tag{4..6}$$

The sign ε_ϕ is related to the *valuedness* of $S(\widetilde{\Lambda})$: $\varepsilon_\phi = 1$ for single valued (tensor) representations of $SO(D-1,1)$; $\varepsilon_\phi = -1$, for double valued (spinor) representations. The field ϕ and its hermitian conjugate ϕ^* (which is assumed to belong to the field algebra whenever ϕ does) satisfy the *locality condition*

$$\phi(x_1)\,\phi^*(x_2) - \varepsilon_\phi\,\phi^*(x_2)\,\phi(x_1) = 0 \quad \text{for} \quad x_{12}^2 > 0\,, \tag{4..7}$$

[31] Laurent Schwartz (1915-2002).

which reflects the independence of the "operations" $\phi(x_1)$ and $\phi^*(x_2)$ at space-like separated points. Finally, we assume that the *vacuum* is a *cyclic vector* of the field algebra. In other words, (smeared) vector valued monomials $\phi_1(x_1)\phi_2(x_2)\dots\phi_n(x_n)|0\rangle$ span $\mathcal{V}$.

A QFT is fully characterized by its *correlation* (or Wightman) *functions*

$$w_n\left(x_{12},\dots,x_{n-1n}\right) \;=\; \langle 0|\,\phi_1(x_1)\dots\phi_n(x_n)\,|0\rangle \tag{4..8}$$

which are, in fact, tempered distributions in $M^{\times(D-1)}$, only depending (due to translation invariance) on the independent coordinate differences $x_{a\,a+1}$, $a = 1,\dots,n-1$.

The spectral condition allows to view the above vector valued monomials and Wightman distributions as boundary values of analytic functions.

Proposition 4..1.

(a) *The vector valued distribution $\phi\,(x)\,|0\rangle$ is the boundary value (for $y \to 0$, $y^0 > |\boldsymbol{y}|$) of a vector-valued function analytic in the forward tube domain $\mathfrak{T}_+$ where*

$$\mathfrak{T}_\pm \;:=\; \left\{ z = x + iy \in M + iM \,:\, \pm y^0 > |\boldsymbol{y}| \right\}. \tag{4..9}$$

(b) *The Wightman distribution (4..8) is a boundary value of an analytic function $w_n\left(z_{12},\dots,z_{n-1n}\right)$ holomorphic in the product of backward tubes $\mathfrak{T}_-^{\times(n-1)}$ and polynomially bounded on its boundary*

$$\left|w_n\left(z_{12},\,\dots,z_{n-1n}\right)\right| \leqslant A\left(1 + \sum_{a=1}^{n-1}\left|z_{aa+1}\right|^2\right)\left(\min_a \left|y_{a\,a+1}^2\right|\right)^{-l}. \tag{4..10}$$

The *proof* [63], [4] uses the spectral condition and standard properties of Laplace transform of tempered distributions.

Each of the tube domains $\mathfrak{T}_\varepsilon$, $\varepsilon = +,-$ is clearly invariant under Poincaré transformations and uniform dilations $z \mapsto \varrho z$ ($\varrho > 0$). A straightforward calculation shows that it is also invariant under the *Weyl inversion* w,

$$z \;\mapsto\; wz \;=\; \frac{I_s z}{z^2}\,, \qquad I_s\left(z^0,\,\boldsymbol{z}\right) \;:=\; \left(z^0,\,-\boldsymbol{z}\right). \tag{4..11}$$

It follows that $\mathfrak{T}_\varepsilon$ is actually *conformally invariant* – as w and the (real) translations generate the full $\binom{D+2}{2}$ parameter *conformal group* $\mathcal{C}$. Moreover, each $\mathfrak{T}_\varepsilon$ is a homogeneous space of $\mathcal{C}$ [67] [32] (see also Sect. 4.2 below). In fact, $\mathfrak{T}_\varepsilon$ is a *coadjoint orbit* of $\mathcal{C}$ equipped with a *conformally invariant symplectic form* proportional to $dx^\mu \wedge d\dfrac{y_\mu}{y^2}$ (see [64] Sect. 3.3).

[32] This has been known earlier – e.g. to the late Vladimir Glaser (1924–1984) who communicated it to I. Todorov back in 1962.

Remark 4..1. Note that the n–point tubes of Proposition 4.1 (b), $\{(x_1+iy_1, \dots, x_n+iy_n) : x_{a\,a+1} + iy_{a\,a+1} \in \mathfrak{T}_\varepsilon\}$, are not conformally invariant for $n>1$.

From now on we shall consider conformally invariant QFT models. Apart from the free Maxwell [33] (photon) field and the massless neutrino, real world fields are not conformally invariant. The interest in unrealistic higher symmetry models comes from the fact that the only (mathematically) existing so far QFT in four space time dimensions – after three quarters of a century of vigorous efforts – are the free field theories. Conformal QFT has the additional advantage to provide (at least, conjecturably) the short distance behaviour of more realistic (massive) theories (for a discussion of this point – see [66])

4.2. Conformal compactification of space–time. The conformal Lie algebra

The quantum mechanical conformal group $\mathcal{C}$ of D-dimensional space–time can be defined as (a finite covering of) the group of real rational coordinate transformations $g : x \to x'(x)$ (with singularities) of Minkowski space M for which

$$dx'^2 = \omega^{-2}(x,g)\,dx^2\,, \quad dx^2 = \boldsymbol{dx}^2 - (dx^0)^2\,, \quad \boldsymbol{dx}^{\,2} = \sum_{i=1}^{D}\left(dx^i\right)^2, \quad (4..12)$$

where $\omega(x,g)$ is found below to be a polynomial in x^μ of degree not exceeding 2. An extension of the classical Liouville theorem says that, for $D>2$, $\mathcal{C}$ is locally isomorphic to the $\binom{D+2}{2}$ parameter (connected) group $\mathcal{C} = Spin(D,2)$, a double cover of the of pseudo–rotation $SO_0\,(D,2)$ of $\mathbb{R}^{D,2}$. In fact, the action of $\mathcal{C}$ on M having singularities can be extended to entirely regular action on a compactification of M called *conformal compactification* or just *compactified Minkowski space* $\overline{M}$. A classical manifestly covariant description of compactified Minkowski space is provided by the projective quadric in $\mathbb{R}^{D,2}$, introduced by Dirac [34] [17] (it generalizes to Lorentzian metric and to higher dimension of a construction of Klein [35]):

$$\overline{M} = Q\,/\,\mathbb{R}^*,\ Q = \left\{\vec{\xi} \in \mathbb{R}^{D,2}\backslash\{0\} : \vec{\xi}^{\,2} := \boldsymbol{\xi}^{\,2} + \xi_D^2 - \xi_0^2 - \xi_{-1}^2\,(= \xi^a \eta_{ab} \xi^b) = 0\right\}. \quad (4..13)$$

[33] James Clerk Maxwell (1831–1879) wrote his *Treatise on Electricity and Magnetism* in 1873.

[34] Paul Adrian Maurice Dirac (1902–1984), Nobel Prize in physics 1933, known for his equation and for the prediction of antiparticles, speaks (in Varenna – [29]) of his great appreciation of projective geometry since his student years at Bristol.

[35] Felix Klein (1849–1925), a believer in a preestablished harmony between physics and mathematics, has outlined this construction without formulae in his famous 1872 Erlanger program.

The *conformal Lie algebra* $\mathfrak{c}$, generated in the projective picture, by the infinitesimal pseudo–rotations $X_{ab} = \xi_b \dfrac{\partial}{\partial \xi^a} - \xi_b \dfrac{\partial}{\partial \xi^a}$ is characterized by the commutation relations

$$[X_{ab}, X_{cd}] \,=\, \eta_{ac} X_{bd} \,-\, \eta_{bc} X_{ad} \,+\, \eta_{bd} X_{ac} \,-\, \eta_{ad} X_{bc} \tag{4..14}$$

for $a, b, c, d \,=\, -1, 0, \ldots, D$ ($\eta_{11} \,=\, \cdots \,=\, \eta_{DD} \,=\, 1 \,=\, -\eta_{00} \,=\, -\eta_{-1-1}$, $\eta_{ab} = 0$ for $a \neq b$). Then the embedding of M in $\overline{M}$ is given by

$$x \mapsto \{\lambda \vec{\xi}_x\} \in \overline{M}_{\mathbb{C}}, \quad \vec{\xi}_x \,=\, x^\mu \vec{e}_\mu + \frac{1 + x^2}{2}\, \vec{e}_{-1} + \frac{1 - x^2}{2}\, \vec{e}_D \quad \text{or,}$$
$$x^\mu \,=\, \frac{\xi^\mu}{\kappa}, \quad \kappa = \xi^D + \xi^{-1}, \tag{4..15}$$

where $\left\{\vec{e}_a : -1 \leqslant a \leqslant D\right\}$ is an orthonormal basis in $\mathbb{R}^{D,2}$, so that the conformal structure on M, or the isotropy relation, is encoded on $\overline{M}$ by the ($SO(D,2)$–invariant relation of) orthogonality of the rays because of the simple formula:

$$x_{12}^2 \,=\, -2\, \vec{\xi}_{x_1} \cdot \vec{\xi}_{x_2} \,=\, \left(\vec{\xi}_{x_1} - \vec{\xi}_{x_2}\right)^2. \tag{4..16}$$

Since the vectors $\vec{\xi}_x$ of the map (4..15) can be characterized by the condition

$$\kappa := \vec{\xi}_\infty \cdot \vec{\xi}_x \,=\, 1\,, \tag{4..17}$$

where $\vec{\xi}_\infty \,=\, (-1, 0, \mathbf{0}, 1)$, we conclude that the complement set $K_\infty :=$ $\overline{M} \backslash M$, the set of points at "infinity", is the $(D-1)$–cone with tip $\{\lambda \vec{\xi}_\infty\}$:

$$K_\infty \,=\, \left\{\{\lambda \vec{\xi}\} \in \overline{M} : \; \vec{\xi}_\infty \cdot \vec{\xi} \; (= \kappa = \xi^D + \xi^{-1} = \xi_D - \xi_{-1}) = 0\right\}. \tag{4..18}$$

Note also that the Weyl inversion (4..11) is a proper conformal transformation given by a rotation of angle π in the $(-1, 0)$ plane: $w\,(\xi_{-1}, \xi_0, \boldsymbol{\xi}, \xi_D)$ $= (-\xi_{-1}, -\xi_0, \boldsymbol{\xi}, \xi_D)$.

Remark 4..2. One can, sure, identify the circle and the $(D-1)$–sphere in the definition (4..13) of the quadric Q, as well. Indeed, the quotient space $Q/\mathbb{R}_+$ can be defined by the equations

$$Q/\mathbb{R}_+ \,=\, \left\{\, \vec{\xi} \in \mathbb{R}^{D,2} \backslash \{0\} : \xi_0^2 + \xi_{-1}^2 = 1 = \boldsymbol{\xi}^{\,2} + \xi_D^2 \,\right\} \,\cong\, \mathbb{S}^1 \times \mathbb{S}^{D-1}. \tag{4..19}$$

Going from $Q/\mathbb{R}_+$ to $\overline{M} = Q/\mathbb{R}^*$ amounts to dividing $Q/\mathbb{R}_+$ by $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$, i.e., by identifying $\vec{\xi}$ and $-\vec{\xi}$ in the product of the circle and the sphere.

Thus we conclude that $\overline{M}$ is diffeomorphic to $\mathbb{S}^1 \times \mathbb{S}^{D-1}/\mathbb{Z}_2$. It also follows that $\overline{M}$ is a *nonorientable* manifold for odd $D > 1$.

The (linear) $SO(D,2)$ action $g\vec{\xi} = (g^a_b \xi^b)$ on Q induces the nonlinear transformations (4..12) by $g\vec{\xi}_x \sim \vec{\xi}_{g(x)}$ where the proportionality coefficient turns out to be equal to the (square root of the) conformal factor, $\omega(x,g)$:

$$g\vec{\xi}_x \,=\, \omega(x,g)\,\vec{\xi}_{g(x)}, \tag{4..20}$$

since the relation (4..17) together with $\vec{\xi}_\infty \cdot \vec{\xi}_{g(x)} = 1$ and Eq. (4..16) imply

$$(g(x_1) - g(x_2))^2 \,=\, \frac{x_{12}^2}{\omega(x_1,g)\,\omega(x_2,g)} \tag{4..21}$$

(proving, in particular, the conformal property of the $SO(D,2)$–action). Note that the *Weyl subgroup*, the Poincaré group with dilations, is characterized by the condition that $\omega(g,x)$ does not depend on x and $\omega(g,x) = 1$ iff g is a Poincaré transformation.

There is a natural basis of the conformal Lie algebra $\mathfrak{c}$ generating the simplest transformations in Minkowski space:

- *Poincaré translations* $e^{i\,a\cdot P}\,(x)\ (\equiv e^{i\,a^\mu P_\mu}\,(x)\,)\ = x + a$ (for $x, a \in M$),
- *Lorentz transformations* $e^{tX_{\mu\nu}}$, $0 \leqslant \mu < \nu \leqslant D-1$ $(X_{\nu\mu} = -X_{\mu\nu})$,
- *dilations* $x \mapsto \rho x$, $\rho > 0$,
- *special conformal transformations* $e^{i\,a\cdot K}\,(x) = \dfrac{x + x^2\,a}{1 + 2\,a\cdot x + a^2\,x^2}$.

The generators iP_μ $(\mapsto \dfrac{\partial}{\partial x^\mu})$, iK_μ and the dilations are expressed in terms of X_{ab} as:

$$\begin{aligned} &i P_\mu = -X_{-1\mu} - X_{\mu D}\,, \qquad i K_\mu = -X_{-1\mu} + X_{\mu D}\,, \\ &\rho^{X_{-1D}}\,(x) = \rho\, x \quad (\rho > 0) \end{aligned} \tag{4..22}$$

(see Appen. B); the Lorentz generators $X_{\mu\nu}$ correspond to $0 \leqslant \mu, \nu \leqslant D-1$.

There is a remarkable complex variable parametrization of $\overline{M}$ given by:

$$\begin{aligned} \overline{M} \,=\, &\Big\{ z_\alpha = e^{2\pi i \zeta}\, u_\alpha : \zeta \in \mathbb{R},\, u^2 := \boldsymbol{u}^2 + u_D^2 = 1,\, u \in \mathbb{R}^D \Big\} \cong \\ &\cong \mathbb{S}^1 \times \mathbb{S}^{D-1}\big/\mathbb{Z}_2, \end{aligned} \tag{4..23}$$

where z_α can be extended to the whole complex Euclidean space $\mathbb{E}_{\mathbb{C}}$ $(\cong \mathbb{C}^D)$ thus defining a chart in the *complexification* $\overline{M}_{\mathbb{C}}$ of compactified Minkowski space; they are connected to the (complex) Minkowski coordi nates $\mathcal{z} = x + iy \in M_{\mathbb{C}}$ $(:= M + iM)$ by the rational conformal transformation ([64], [54], [51]):

$$\boldsymbol{z} = \frac{\boldsymbol{\mathcal{z}}}{\omega(\mathcal{z})}\ , \qquad z_D = \frac{1-\mathcal{z}^2}{2\,\omega(\mathcal{z})}\ , \quad \omega(\mathcal{z}) = \frac{1+\mathcal{z}^2}{2} - i\mathcal{z}^0\ . \tag{4..24}$$

Proposition 4..2. *The rational complex coordinate transformation $g_c : M_{\mathbb{C}}\ (\ni \mathcal{z}) \to E_{\mathbb{C}}\ (\ni z)$, defined by (4..24), is a complex conformal map (with singularities) between the complex Minkowski and Euclidean spaces, such that*

$$z_{12}^2 = \frac{\mathcal{z}_{12}^2}{\omega\left(\mathcal{z}_1\right)\omega\left(\mathcal{z}_2\right)}\ , \quad dz^2 \left(= d\boldsymbol{z}^2 + dz_D^2\right) = \frac{d\mathcal{z}^2}{\omega\left(\mathcal{z}\right)^2}\ . \tag{4..25}$$

The transformation g_c is regular in the tube domain $\mathfrak{T}_+ = \left\{ \mathcal{z} = x + i\,y : y^0 > |\boldsymbol{y}| \right\}$ and on the real Minkowski space M. The image T_+ of $\mathfrak{T}_+$ under g_c

$$T_+ := \left\{ z \in \mathbb{C}^D : \left| z^2 \right| < 1,\ z \cdot \overline{z} = \left| z^1 \right|^2 + \cdots + \left| z^D \right|^2 < \frac{1}{2}\left(1 + \left| z^2 \right|^2\right) \right\} \tag{4..26}$$

is a precompact submanifold of $E_{\mathbb{C}}$. The closure $\overline{M}$ of the precompact image of the real Minkowski space M in $E_{\mathbb{C}}$ has the form (4..23).

The statement is verified by a direct computation (see [54], [51]).

The transformation (4..24) generalizes the Cayley[36] transformation

$$\mathcal{z} \mapsto z = \frac{1+i\mathcal{z}}{1-i\mathcal{z}} \qquad (\mathcal{z}, z \in \mathbb{C}) \tag{4..27}$$

arising in the description of the chiral (1-dimensional light-ray) projection of the 2D CFT (see Sect. 5). In the $D = 4$ dimensional case (4..24) can be viewed as the *Cayley compactification map* $u(2) \to U(2)$ in the space of (complex) quaternions $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{H}$ (see [67] [64]):

$$\begin{aligned} &M \ni x \mapsto i\widetilde{x} := ix^0\mathbb{I} + \boldsymbol{Q}\cdot\boldsymbol{x} \in u(2), \\ &\overline{M} \ni z \mapsto \not{z} := z^\alpha Q_\alpha \equiv z^4\mathbb{I} + \boldsymbol{Q}\cdot\boldsymbol{z} \in U(2)\,, \\ &i\widetilde{x} \mapsto \not{z} = \frac{1+i\widetilde{x}}{1-i\widetilde{x}}\ , \end{aligned} \tag{4..28}$$

where Q_k $(k = 1, 2, 3)$ are the *quaternion units* (expressed in terms of the Pauli matrices, see Sect. 6.2.). Another point of view on the transformation (4..24) is developed in [51] (see Appendix A): to each pair of mutually

[36] Arthur Cayley (1821–1895) has introduced, in 1843, the notion of n–dimensional space and is a pioneer of the theory of invariants.

nonisotropic points, say q_0, $q_\infty \in \overline{M}_{\mathbb{C}}$, one assigns an affine chart of $\overline{M}_{\mathbb{C}}$ with a distinguished centre q_0 and a centre q_∞ of the infinite light cone. This is provided by the fact that the stability subgroup in $\mathcal{C}$ of a point, say q_∞, is isomorphic (conjugate) to the Weyl group (the group of affine conformal transformations). In the above 1–dimensional case $q_0 = i$, $q_\infty = -i$, while in a general dimension D, q_0 and q_∞ are mutually conjugate points in the forward and backward tubes, respectively. (Remarkably, only when $q_\infty \in T_\pm$ the corresponding affine chart entirely cover the real compact space $\overline{M}$; this has no analog for signatures different from the Lorentz type $(D-1,1)$ or $(1,D-1)$.)

Note that all pairs of mutually nonisotropic points of $\overline{M}$ (or $\overline{M}_{\mathbb{C}}$) form a single orbit under the action of the (complex) conformal group ([53], Proposition 2.1). In particular, the transformation g_c of Proposition 4.2 can be considered as an element of $\mathcal{C}_{\mathbb{C}}$ such that $g_c(p_{0,\infty}) = q_{0,\infty}$. Thus the stabilizer of the pair p_0, p_∞, which is the Lorentz group with dilations, is conjugate to the stabilizer of q_0, q_∞. Since q_0 and q_∞ are complex conjugate to one another, it turns out that their stabilizer in $\mathcal{C}_{\mathbb{C}}$ is $*$–invariant. Moreover, its real part coincides with the *maximal compact subgroup* $\mathcal{K}$ of $\mathcal{C}$,

$$\mathcal{K} \,=\, Spin(D) \times U(1)/\mathbb{Z}_2\,, \tag{4..29}$$

generated by $X_{\alpha\beta}$ $(\alpha,\beta = 1,\,\dots,D)$ and the *conformal hamiltonian*,

$$H \,:=\, i\,X_{-10} \,\equiv\, \frac{1}{2}\,(P_0 + K_0)\,. \tag{4..30}$$

As noted by Segal[37] [60] H is positive whenever P_0 is (since $K_0 = wP_0w^{-}1$ with w defined in (4..11)). The factor $Spin(D)$ of $\mathcal{K}$ acts on the coordinates z by (Euclidean) rotations while the $U(1)$ subgroup multiplies z by a phase factor. Thus, $\mathcal{K}$ appears as the stability group of the origin $z = 0$ (i.e., q_0) in the *real* conformal group $\mathcal{C}$. Noting further the transitivity of the action of $\mathcal{C}$ on either T_+ or T_- we conclude that the forward tube is isomorphic to the coset space

$$T_+ \,\cong\, \mathcal{C}/\mathcal{K}\,. \tag{4..31}$$

We will need the complex Lie algebra generators T_α and C_α for $\alpha = 1,\,\dots,\,D$ of *z–translations* $e^{w\cdot T}(z) = z + w$ and *special conformal transformations* $e^{w\cdot C}(z) = \dfrac{z + z^2\,w}{1 + 2\,w\cdot z + w^2\,z^2}$ $(w,\,z \in \mathbb{C}^D)$ which are conjugate by g_c to the analogous generators $-iP_\mu$ and $-iK_\mu$. This new basis of generators (T_α, C_α, H and $X_{\alpha\beta}$) is expressed in terms of X_{ab} as:

$$T_\alpha = iX_{0\alpha} - X_{-1\alpha},\;\; C_\alpha = -iX_{0\alpha} - X_{-1\alpha} \;\text{ for } \alpha = 1,\dots,D \tag{4..32}$$

($[T_\alpha, C_\beta] = 2\,(\delta_{\alpha\beta}H - X_{\alpha\beta})$, $[H, C_\alpha] = -C_\alpha$, $[H, T_\alpha] = T_\alpha$, see Appendix B). The generators T_α, C_α for $\alpha = 1,\dots,D$, together with the above introduced $X_{\alpha\beta}$ $(\alpha,\beta = 1,\dots,D)$ and H span an *Euclidean real form*

[37] Irving Ezra Segal (1918–1998).

($\cong$ *spin* $(D+1,1)$) of the complex conformal algebra. The generators T_α, $X_{\alpha\beta}$ and H span the subalgebra $\mathfrak{c}_\infty$ of the complexified conformal Lie algebra $\mathfrak{c}_{\mathbb{C}}$, the *stabilizer of the central point at infinity* in the z–chart which is isomorphic to the complex Lie algebra of Euclidean transformations with dilations. The stabilizer of $z=0$ in $\mathfrak{c}_{\mathbb{C}}$ is its conjugate $\mathfrak{c}_0$,

$$\mathfrak{c}_0 \,:=\, Span_{\mathbb{C}}\,\{C_\alpha,\, X_{\alpha\beta},\, H\}\;. \tag{4..33}$$

Remark 4..3. The z–coordinates are expressed in terms of ξ_a as $z_\alpha = \xi_\alpha/(i\xi_0 - \xi_{-1})$ (in contrast with (4..15) the denominator $i\xi_0 - \xi_{-1}$ never vanishes for real $\vec{\xi}$) . We could have introduced a length scale R replacing the numerator ξ_α by $R\xi_\alpha$. We shall make use of the parameter R in Sect. 7, where it is viewed as "the radius of the Universe" (in the sense of Irving Segal [61]) and the thermodynamic ($R\to\infty$) limit is studied.

4.3. The concept of GCI QFT. Vertex algebras, strong locality, rationality

We proceed with a brief survey of the axiomatic QFT with GCI. The assumptions of the GCI QFT are the Wightman axioms [63], briefly sketched in Sect. 4.1, and the condition of GCI for the correlation functions [53]. The latter means that the Wightman functions [38] $\langle 0|\,\phi^M_{A_1}(x_1)\ldots\phi^M_{A_n}(x_n)|0\rangle$ are invariant (in the sense of distributions) under the substitution

$$\phi^M_A(x) \;\mapsto\; \left[\pi^M_x(g)^{-1}\right]^B_A\,\phi^M_B(g(x)) \tag{4..34}$$

for every conformal transformation $g\in\mathcal{C}$, outside its singularities. The matrix valued function $\pi^M_x(g)$ is called (Minkowski) cocycle and is characterized by the properties

$$\pi^M_x(g_1g_2) = \pi^M_{g_2(x)}(g_1)\,\pi^M_x(g_2)\;, \qquad \pi^M_x\left(e^{ia\cdot P}\right)^B_A \,=\, \delta^B_A\,. \tag{4..35}$$

The transformation law (4..34) extends the Poincaré covariance (4..4) for $S(\widetilde{\Lambda}) \equiv \pi^M_0(\widetilde{\Lambda})$ to the case of (nonlinear) conformal transformations. An example of such a transformation law is given by the *electromagnetic* field that is transforming as a 2–form

$$F^M_{\mu\nu}(x)\;dx^\mu\wedge dx^\nu \;=\; F^M_{\mu\nu}(g(x))\;dg(x)^\mu\wedge dg(x)^\nu\,. \tag{4..36}$$

As proven in [53], Theorem 3.1, GCI is equivalent to the rationality of the (analytically continued) Wightman functions. Moreover ([51] Theorem 9.1), the product of fields acting on the vacuum, $\phi^M_{A_1}(x_1)\ldots\phi^M_{A_n}(x_n)|0\rangle$, are boundary values of analytic functions $\phi_{A_1}(z_1)\,\ldots\phi_{A_n}(z_n)|0\rangle$ defined

[38] The superscript "M" will further mean that the corresponding objects are considered over the Minkowski space (chart in $\overline{M}$).

for all sets of mutually nonisotropic points $z_1, \dots, z_n \in \mathfrak{T}_+$ and the limit is obtained for $Im\, z_{kk+1} \in \mathfrak{T}_-$. This makes possible to consider the n–point vacuum correlation functions of the theory as *meromorphic sections* of the nth tensor power (over $\overline{M}_{\mathbb{C}}^{\times n}$) of a bundle. This bundle is defined over the complex compactified Minkowski space $\overline{M}_{\mathbb{C}}$ by the cocycle (4..35) and is called the *field bundle*. It is then naturally endowed with an action of the conformal group $\mathcal{C}_{\mathbb{C}}$ via (bundle) automorphisms.

Remark 4..4. Trivializing the bundle over every affine chart on $\overline{M}_{\mathbb{C}}$, for instance, in the z–coordinates (4..24), by the action of the corresponding Abelian group of affine translations, $t_w(z)$ $(\equiv e^{w \cdot T}(z)) = z + w$, the action of $\mathcal{C}_{\mathbb{C}}$ will take the form

$$(z = \{z^\alpha\}, \phi = \{\phi_A\}) \underset{g}{\longmapsto} \big(g\,(z)\,, \pi_z(g)\,\phi = \{\pi_z(g)_A^B \phi_B\}\big) \in \mathbb{C}^D \times F \quad (4..37)$$

where F is the standard fibre and $\pi_z\,(g)$ is the z–picture cocycle. This provides the general scheme for the passage from the GCI QFT over Minkowski space to the theory over a complex affine chart which contains the forward tube $\mathfrak{T}_+$ (4..10) – see [51] Sect. 9. The fibre F is the space of (classical) *field values* and the coordinates ϕ_A correspond to the collection of local fields in the theory.

The (analytic) *z-picture* of a GCI QFT is obtained by transformation of Minkowski space fields ϕ_A^M to the z–coordinates (4..24) as (operator valued) sections of the field bundle:

$$\phi_A\,(z) = \pi^M_{g_c^{-1}(z)}\big(g_c\big)_A^B\ \phi_B^M\big(g_c^{-1}(z)\big) \quad (z = g_c(x)) \qquad (4..38)$$

where g_c is the transformation (4..24) viewed as an element of $\mathcal{C}_{\mathbb{C}}$. Different normalization conventions in the x and z picture require an additional numerical factor in (4..38). For instance, the canonical commutation relations yield an extra factor 2π in (4..38) for a free massless scalar field φ, the standard conventions for the 2–point function being

$$\begin{aligned} &\langle 0|\, \varphi^M(x_1)\varphi^M(x_2)\,|0\rangle = \frac{1}{4\pi^2\left(x_{12}^2 + i0x_{12}^0\right)} \quad \text{while} \\ &\langle 0|\, \varphi(z_1)\varphi(z_2)\,|0\rangle = \frac{1}{z_{12}^2}\,. \end{aligned} \qquad (4..39)$$

The resulting theory is equivalent to the theory of vertex algebras ([5], [32], [6], [19]) extended to higher dimensions (see [51]). We proceed to sum up the properties of z-picture fields and of the more general *vertex operator fields* arising in their *operator product expansion* (OPE).

1) The *state space* $\mathcal{V}$ of the theory is a (*pre-Hilbert*) *inner product space* carrying a (reducible) unitary *vacuum representation* $U(g)$ of the conformal group $\mathcal{C}$, for which:

1a) the corresponding representation of the complex Lie algebra $\mathcal{c}_{\mathbb{C}}$ is such that the spectrum of the $U(1)$ generator H (4..30) belongs to $\left\{ 0, \frac{1}{2}, 1, \frac{3}{2}, .. \right\}$ and has a finite degeneracy:

$$\mathcal{V} = \bigoplus_{\rho = 0, \frac{1}{2}, 1, \ldots} \mathcal{V}_\rho \, , \ (H - \rho)\, \mathcal{V}_\rho = 0 \, , \quad \dim \mathcal{V}_\rho < \infty \, , \tag{4..40}$$

each $\mathcal{V}_\rho$ carrying a fully reducible representation of *Spin* (D) (generated by $X_{\alpha\beta}$). Moreover, the central element $-\mathbb{I}$ of the subgroup *Spin* (D) is represented on $\mathcal{V}_\rho$ by $(-1)^{2\rho}$.

1b) The *lowest energy space* $\mathcal{V}_0$ is 1-dimensional: it is spanned by the (normalized) *vacuum vector* $|0\rangle$, which is invariant under the full conformal group $\mathcal{C}$.

As a consequence (see [51], Sect. 7) the Lie subalgebra $\mathcal{c}_0$ (4..33) of $\mathcal{c}_{\mathbb{C}}$ has locally finite action on $\mathcal{V}$, i.e., every $v \in \mathcal{V}$ belongs to a finite dimensional subrepresentation of $\mathcal{c}_0$. Moreover, the action of $\mathcal{c}_0$ can be integrated to an action of the complex Euclidean group with dilations $\pi_0\,(g)$ and the function

$$\pi_z\,(g) \ := \ \pi_0 \left(t^{-1}_{g(z)}\, g\, t_z \right) \tag{4..41}$$

is rational in z with values in $End_{\mathbb{C}}\, \mathcal{V}$ (the space endomorphisms of $\mathcal{V}$) and satisfies the cocycle property

$$\pi_z\,(g_1 g_2) \ = \ \pi_{g_2(z)}\,(g_1)\, \pi_z\,(g_2) \quad \text{iff} \quad g_1 g_2\,(z)\, , \, g_2\,(z) \ \in \ \mathbb{C}^D \, . \tag{4..42}$$

2) The fields $\phi\,(z) \equiv \{\phi_a\,(z)\}$ ($\psi\,(z) \equiv \{\psi_b\,(z)\}$, etc.) are represented by infinite power series of the type

$$\phi\,(z) \ = \ \sum_{n \in \mathbb{Z}} \sum_{m \geqslant 0} \left(z^{\,2} \right)^n \, \phi_{\{n,m\}}\,(z) \ , \qquad z^{\,2} \ := \ \sum_{\alpha = 1}^{D} z_\alpha^2 \, , \tag{4..43}$$

$\phi_{\{n,m\}}\,(z)$ being (in general, multicomponent) operator valued polynomial in z which is *homogeneous of degree* m *and harmonic*,

$$\phi_{\{n,m\}}\,(\lambda z) = \lambda^m \, \phi_{\{n,m\}}\,(z) \ , \quad \Delta\, \phi_{\{n,m\}}\,(z) = 0 \, , \quad \Delta = \sum_{\alpha=1}^{D} \frac{\partial^2}{\partial z_\alpha^2} \tag{4..44}$$

and

$$\phi_{\{-n,m\}}\,(z)\ v \ = \ 0 \tag{4..45}$$

for all $m = 0, 1, \ldots$ if $n > N_v \in \mathbb{Z}$.

3) *Strong locality*: The fields ϕ, ψ, ... are assumed to have $\mathbb{Z}_2$–parities p_ϕ, p_ψ, ... such that

$$\rho_{12}^N \left\{\phi_a(z_1)\,\psi_b(z_2) - (-1)^{p_\phi p_\psi}\,\psi_b(z_2)\,\phi_a(z_1)\right\} = 0 \quad (\rho_{12} := z_{12}^2)\,, \tag{4..46}$$

for sufficiently large N.

In Minkowski space, the strong locality condition is implied by the Huygens'[39] principle and the Wightman positivity. Recall that the Huygens' principle is a stronger form of the locality condition (4..7) stating that the left hand side vanishes for all nonisotropic separations ($x_{12}^2 \neq 0$). This is a consequence of GCI [53] since, as we already mentioned, the mutually nonizotropic pairs of points form a single orbit for the (connected) conformal group $\mathcal{C}$. Let us also note that the assumption that the field algebra is $\mathbb{Z}_2$–graded, which underlies 3), fixes the commutation relations among different fields thus excluding the so called "Klein transformations" (whose role is discussed e. g. in [63]).

Strong locality implies an analogue of the *Reeh–Schlieder theorem*, the separating property of the vacuum, namely

Proposition 4..3.

(a) ([55], Proposition 3.2 (a).) *The series $\phi_a(z)|0\rangle$ does not contain negative powers of z^2.*

(b) ([51], Theorem 3.1.) *Every local field component $\phi_a(z)$ is uniquely determined by the vector $v_a = \phi_a(0)|0\rangle$.*

(c) ([51], Theorem 4.1 and Proposition 3.2.) *For every vector $v \in \mathcal{V}$ there exists unique local filed $Y(v, z)$ such that $Y(v,0)|0\rangle = v$. Moreover, we have*

$$Y(v,z)|0\rangle = e^{z\cdot T}\,v\,, \quad z\cdot T = z^1\,T_1 + \cdots + z^D\,T_D\,. \tag{4..47}$$

The part (c) of the above proposition is the higher dimensional analogue of the *state field correspondence*.

4) *Covariance*:

$$[T_\alpha, Y(v,z)] = \frac{\partial}{\partial z^\alpha}\,Y(v,z)\,, \tag{4..48}$$

$$[H, Y(v,z)] = z\cdot\frac{\partial}{\partial z}\,Y(v,\,z) + Y(Hv,z)\,, \tag{4..49}$$

$$\begin{aligned}[X_{\alpha\beta}, Y(v,z)] = {} & z^\alpha\,\frac{\partial}{\partial z^\beta}\,Y(v,z) - z^\beta\,\frac{\partial}{\partial z^\alpha}\,Y(v,\,z) \\ & + Y(X_{\alpha\beta}v,z)\,,\end{aligned} \tag{4..50}$$

$$\begin{aligned}[C_\alpha, Y(v,z)] = {} & \big(z^2\,\frac{\partial}{\partial z^\alpha} - 2\,z^\alpha\,z\cdot\frac{\partial}{\partial z}\big)\,Y(v,z) - 2\,z^\alpha\,Y(Hv,z) \\ & + 2\,z^\beta\,Y(X_{\beta\alpha}v,z) + Y(C_\alpha v,\,z)\,.\end{aligned} \tag{4..51}$$

[39] The Dutch physicist, mathematician, and astronomer Christian Huygens (1629–1695) is the originator of the wave theory of light.

If $v \in \mathcal{V}$ is a minimal energy state in an irreducible representation of $\mathfrak{c}$ then $C_\alpha v = 0$ ($\alpha = 1, \dots, D$) (as C_α plays role of a lowering operator: $HC_\alpha v = C_\alpha (H-1)\, v$); such vectors are called **quasiprimary**. Their linear span decomposes into irreducible representations of the maximal compact subgroup $\mathcal{K}$, each of them characterized by weights $\left(d;\, j_1, \dots, j_{\frac{D}{2}}\right)$. We assume that our basic fields ϕ_a, ψ_b, ..., correspond to such quasiprimary vectors so that the transformation laws (4..48)–(4..51) give rise to $\mathcal{K}$–induced representations of the conformal group $\mathcal{C}$.

If $v \in \mathcal{V}$ is an eigenvector of H with eigenvalue d_v then Eq. (4..49) implies that the field $Y\,(v, z)$ has dimension d_v:

$$[H\,,\,Y\,(v,\,z)\,] \,=\, z\cdot\frac{\partial}{\partial z}\,Y\,(v,\,z) + d_v\,Y\,(v,\,z) \qquad (H\,v \,=\, d_v v)\,. \tag{4..52}$$

It also follows from the correlation between the dimension and the spin in the property 1a) and from the spin and statistics theorem that the Z_2–parity p_v of v is related to its dimension by $p_v \equiv 2d_v \; mod\, 2$; therefore

$$\rho_{12}^{\mu(v_1,v_2)}\Big\{Y(v_1,z_1)\,Y(v_2,z_2) - (-1)^{4\,d_{v_1} d_{v_2}}\,Y(v_2,z_2)\,Y(v_1,z_1)\Big\} = 0, \tag{4..53}$$

where $\mu\,(v_1, v_2)$ depends on the spin and dimensions of v_1 and v_2 while the cocycle for $Y(v,z)$ satisfies $\pi_z(e^{2\pi X_{\alpha\beta}}) = (-1)^{2d_v}$. (For a description of the spinor representation of $Spin(D,2)$ for any D – see Appendix C.)

5) *Conjugation*:

$$\left\langle\, v_1 \,\middle|\, Y\left(v^+,\,z\right) v_2 \,\right\rangle \,=\, \left\langle\, Y\left(\pi_{z^*}(I)^{-1}\,v,\,z^*\right) v_1 \,\middle|\, v_2 \,\right\rangle \tag{4..54}$$

for every $v, v_1, v_2 \in \mathcal{V}$, where

$$z^* \,:=\, \frac{\overline{z}}{\overline{z}^{\,2}} \tag{4..55}$$

is the z–picture conjugation (leaving invariant the real space (4..13)) and I is the element of $\mathcal{C}_{\mathbb{C}}$ representing the complex inversion

$$I\,(z) \,:=\, \frac{R_D\,(z)}{z^{\,2}}\,, \quad R_\mu\left(z^1,\,\dots,\,z^D\right) \,:=\, \left(z^1,\,\dots,\,-z^\mu,\,\dots,\,z^D\right) \tag{4..56}$$

(I^2 is a central element of $\mathcal{C}$ while I does not belong to the real conformal group).

The above properties also imply the *Borcherds' OPE relation*. The equality

$$Y(v_1,z_1)\,Y(v_2,z_2) \text{ “=” } Y(Y(v_1,z_{12})\,v_2,z_2)\,, \tag{4..57}$$

is satisfied after applying some transformations to the formal power series on both sides which are not defined on the corresponding series' spaces (see [51], Theorem 4.3). Moreover, the vector valued function

$$Y(v,z_1)\,Y(v_2,z_2)|0\rangle = Y(Y(v_1,z_{12})\,v_2,z_2)|0\rangle$$

is analytic with respect to the Hilbert norm topology for $|z_2^2| < |z_1^2| < 1$ and sufficiently small ρ_{12}.

Proposition 4..4. *Under the above assumptions the vacuum correlation functions are (Euclidean invariant, homogeneous) rational functions of z_α.*

Proof. If we take vertex operators $Y(v_k, z) =: \phi_k(z)$, $k = 1, \dots, n$, having fixed dimensions d_k then the strong locality (4..53) implies that for large enough $N \in \mathbb{N}$ the product

$$F_{1\dots n}(z_1, \dots, z_n) := \Big(\prod_{1 \leqslant i < j \leqslant n} \rho_{ij} \Big)^N \langle 0|\, \phi_1(z_1) \dots \phi_n(z_n) |0\rangle \qquad (4..58)$$

$\rho_{ij} = z_{ij}^2 \equiv (z_i - z_j)^2$, is $\mathbb{Z}_2$ symmetric under any permutation of the factors within the vacuum expectation value. Energy positivity, on the other hand, implies that $\langle 0|\, \phi_1(z_1) \dots \phi_n(z_n) |0\rangle$, and hence $F_{1\dots n}(z_1, \dots, z_n)$ do not contain negative powers of z_n^2. It then follows from the symmetry and the homogeneity of $F_{1\dots n}$ that it is a polynomial in all z_i^μ. Thus the (Wightman) correlation functions are rational functions of the coordinate differences. (See for more detail [51], [55]; an equivalent Minkowski space argument based on the support properties of the Fourier transform of (the x-space counterpart of) (4..58) is given in [53].) $\square$

6) *The concept of stress–energy tensor.* The importance of assuming the existence of a stress–energy tensor T along with the Wightman axioms in a conformal field theory has been recognized long ago [45] (see also [44]). It is simpler to introduce T in a GCI theory extended on compactified Minkowski space. It is a *rank two conserved symmetric traceless tensor* which will be written in the analytic picture in the form:

$$T(z; v) := T_{\alpha\beta}(z)\, v^\alpha v^\beta\,, \quad \big(\frac{\partial}{\partial v} \cdot \frac{\partial}{\partial v}\big)\, T(z; v) = 0\,. \qquad (4..59)$$

Its scale dimension is equal to the space-time dimension D. The *conservation law* reads:

$$\frac{\partial}{\partial z_\alpha}\, T_{\alpha\beta}(z) = 0 \quad \Longleftrightarrow \quad \big(\frac{\partial}{\partial z} \cdot \frac{\partial}{\partial v}\big)\, T(z; v) = 0\,. \qquad (4..60)$$

Finally, the conformal Lie algebra generators should appear among its modes (see (4..88)); in particular, the generators of the Lie algebra $u(1) \times spin(D)$ of the maximal compact subgroup $\mathcal{K}$ of $\mathcal{C}$ can be expressed by (finite) linear combinations of the zero modes of T.

4.4. Real compact picture fields. Gibbs states and the KMS condition

The conjugation law (4..54) simplifies in the *real compact picture* in which

$$\phi(\zeta, u) := e^{2\pi i d \zeta}\, \phi(e^{2\pi i \zeta} u) \qquad (4..61)$$

where d is the conformal dimension of ϕ. The commutation relation (4..52) of the z–picture fields with the conformal energy operator implies then

$$e^{2\pi i t H}\phi(\zeta,u)=\phi(\zeta+t,u)\,e^{2\pi i t H}, \tag{4..62}$$

i.e., H appears as the translation generator in ζ in this realization. (While the z–picture fields correspond to the complex Euclidean invariant line element dz^2 (4..25), the compact picture fields correspond to the real $\mathcal{K}$-invariant line element $dz^2/z^2 = dx^2/|\omega|^2$.) Since all dimensions of GCI fields should be integer or half odd integer depending on their spin the corresponding compact picture fields are periodic or antiperiodic, respectively,

$$\phi(\zeta+1,u)=(-1)^{2d}\phi(\zeta,u)\,. \tag{4..63}$$

The (anti)periodicity property (4..63) implies that ϕ has a Fourier series expansion

$$\phi(\zeta,u)\;=\sum_{\nu\,\in\,d+\mathbb{Z}}\sum_{m\,=\,0}^{\infty}\phi_{\nu m}(u)\,e^{-2\pi i\nu\zeta} \tag{4..64}$$

where $\phi_{\nu m}(u)$ are operator valued homogeneous harmonic polynomials of degree m restricted to the unit sphere (we leave it to the reader to find the connection between the expansions (4..44) and (4..64)). Combined with (4..62), this gives

$$[H,\phi_{\nu m}(u)]=-\nu\phi_{\nu m}(u)\;(\Leftrightarrow\; q^H\phi_{\nu m}(u)\,q^{-H}=q^{-\nu}\phi_{\nu m}(u),|q|<1). \tag{4..65}$$

For a scalar field (of integer dimension) the hermiticity condition (4..54) reads:

$$\phi_{\nu m}(u)^*=\phi_{-\nu m}(u)\,. \tag{4..66}$$

We assume the existence of *Gibbs* [40] *temperature states* – i. e., the existence of all traces of the type

$$tr_{\mathcal{V}}(Aq^H)\quad\text{for}\quad q=e^{2\pi i\tau}\,,\quad Im\,\tau>0\quad(\text{i. e. }|q|<1) \tag{4..67}$$

where A is any polynomial in the (local, GCI) fields (including $A=1$) and $\mathcal{V}$ is the space of finite energy states (dense in the Hilbert state space). We then define the temperature mean of A by the standard relation:

$$\langle\, A\,\rangle_q\;=\;\frac{1}{Z(\tau)}\,tr_{\mathcal{V}}(Aq^H)\,,\quad Z(\tau)\;=\;tr_{\mathcal{V}}(q^H)\quad(2\pi\,Im\,\tau\;=\;\frac{h\nu}{kT}) \tag{4..68}$$

thus identifying the imaginary part of τ with the Planck's energy quantum divided by the absolute temperature. The parameter τ of the upper half plane $\mathfrak{H}$ thus appears in two guises: as a moduli labeling complex structures

[40] Josiah Williard Gibbs (1839–1903) has published his last work "Elementary Principles in Statistical Mechanics developed with special reference to the Rational Foundations of Thermodynamics" in 1902.

on a torus (that are inequivalent on $\Gamma(1)\backslash\mathfrak{H}$) and as (inverse) absolute temperature.

For free fields the partition function $Z(\tau)$ can, in fact, be computed given the dimensions $d_b(n)$ and $d_f(n)$ of 1-particle bosonic and fermionic states of energy n and $n-1/2$, respectively. The result is:

$$Z(\tau) \;=\; \prod_{n\,=\,1}^{\infty} \frac{\bigl(1+q^{n-\frac{1}{2}}\bigr)^{d_f(n)}}{\bigl(1-q^{n}\bigr)^{d_b(n)}}. \tag{4..69}$$

The mean thermal energy $\bigl\langle H \bigr\rangle_q$ is given by the logarithmic derivative of $Z(\tau)$,

$$\bigl\langle H \bigr\rangle_q \;=\; \frac{1}{2\pi i Z(\tau)} \frac{dZ(\tau)}{d\tau} \;=\; q\,\frac{d}{dq}\,\ln Z \tag{4..70}$$

so that for the generalized free field models we have

$$\bigl\langle H \bigr\rangle_q \;=\; \sum_{n\,=\,1}^{\infty} \frac{n\,d_b(n)\,q^n}{1-q^n} \;+\; \sum_{n\,=\,1}^{\infty} \frac{\bigl(n-\frac{1}{2}\bigr)\,d_f(n)\,q^{n-\frac{1}{2}}}{1+q^{n-\frac{1}{2}}}\,. \tag{4..71}$$

It has been established that the thermal (Gibbs) correlation functions are finite linear combinations of a fixed set of elliptic functions in each of the conformal time differences ζ_{kk+1}; the coefficients that are (depending on u_k) q–series whose convergence is conjectured (see [55] Theorem 3.5, where we have been motivated by an intuitive argument of Zhu [72]).

Theorem 4..5. (see [55] Theorem 3.5 and Corollary 3.6) *If the finite temperature correlation functions of a set of Bose fields* $\{\phi_a\}$*,*

$$\bigl\langle\, \phi_1(\zeta_1,u_1)\ldots\phi_n(\zeta_n,u_n)\,\bigr\rangle_q \;:=\; \frac{1}{Z(\tau)}\,tr_{\mathcal{V}}\,\bigl\{\phi_1(\zeta_1,u_1)\ldots\phi_n(\zeta_n,u_n)\,q^H\bigr\}\;, \tag{4..72}$$

are meromorphic and symmetric (as meromorphic functions) with respect to permutations of the factors ϕ_a*, then the KMS condition ([28] [10])*

$$\bigl\langle \phi\,(\zeta_2,u_2)\ldots\phi\,(\zeta_n,u_n)\,\phi\,(\zeta_1+\tau,u_1)\bigr\rangle_q = \bigl\langle \phi\,(\zeta_1,u_1)\ldots\phi\,(\zeta_n,u_n)\bigr\rangle_q \tag{4..73}$$

implies that the functions (4..72) are elliptic with respect to the $(n-1)$ *independent differences* $\zeta_{i\,i+1}=\zeta_i-\zeta_{i+1}$ $(i=1,\ldots,n-1)$ *of periods* 1 *and* τ*.*

We shall verify that the Gibbs correlation functions of (generalized) free fields are indeed symmetric elliptic functions of $\zeta_{i\,i+1}$ thus confirming the above conjecture.

It follows from the Huygens principle, established in [53], that all singularities of correlation functions are poles for isotropic intervals

$$z_{ab}^2 \;=\; 2\,e^{2\pi i(\zeta_a+\zeta_b)}(\cos 2\pi\zeta_{ab} - \cos 2\pi\alpha_{ab})\,(=0) \tag{4..74}$$

where $2\pi\alpha_{ab}$ is the angle between the unit Euclidean D–vector u_a and u_b :

$$u_a \cdot u_b = \cos 2\pi\alpha_{ab}. \tag{4..75}$$

Noting the relation

$$\cos 2\pi\zeta - \cos 2\pi\alpha \; = \; -2 \sin \pi(\zeta+\alpha) \sin \pi(\zeta-\alpha) \tag{4..76}$$

we deduce that the singularities of temperature means are poles in ζ_{ab} for

$$\zeta_{ab} \pm \alpha_{ab} = n \in \mathbb{Z}\,. \tag{4..77}$$

The decomposition formula

$$\frac{1}{\sin \pi\zeta_+ \sin \pi\zeta_-} \; = \; \frac{1}{\sin 2\pi\alpha} (\operatorname{cotg} \pi\zeta_+ - \operatorname{cotg} \pi\zeta_-) \quad \text{for} \quad \zeta_\pm \; = \; \zeta \pm \alpha \tag{4..78}$$

allows to separate the two poles arising in the vacuum correlation functions. The Euler expansion for $\operatorname{cotg} \pi\zeta$ and its fermionic counterpart for $\frac{1}{\sin \pi\zeta}$ gives

$$\operatorname{cotg} \pi\zeta = \frac{1}{\pi} \lim_{N\to\infty} \sum_{n=-N}^{N} (\zeta+n)^{-1}\,, \quad \frac{1}{\sin \pi\zeta} = \frac{1}{\pi} \lim_{N\to\infty} \sum_{n=-N}^{N} \frac{(-1)^n}{\zeta+n}\,. \tag{4..79}$$

In the finite temperature 2–point correlators the corresponding Eisenstein-Weierstrass type series (on the lattice $\mathbb{Z}\tau + \mathbb{Z}$) are expressed as linear combinations of derivatives of both sides of (4..79) proportional to the functions (2..15),

$$p_k^{\kappa\lambda}(\zeta,\tau) = \lim_{M\to\infty} \lim_{N\to\infty} \sum_{m=-M}^{M} \sum_{n=-N}^{N} \frac{(-1)^{\kappa m + \lambda n}}{(\zeta + m\tau + n)^k}\,, \tag{4..80}$$

$\kappa, \lambda = 0, 1,\ k = 1, 2, \dots$ ($\zeta = \zeta_+, \zeta_-$) introduced in Sect. 2.
For a generalized free field $\phi(\zeta, u)$ ($= \{\phi_A(\zeta, u)\}$) of (half)integer dimension d, characterized by its 2–point vacuum function, the Gibbs correlation functions can be also expressed in terms of the 2–point Wightman function. The result looks very simple ([55], Theorem 4.1):

$$\begin{aligned} \left\langle\, \phi(\zeta_1, u_1)\, \overset{*}{\phi}(\zeta_2, u_2)\, \right\rangle_q \; &= \; \sum_{k=-\infty}^{\infty} (-1)^{2d\,k}\, W(\zeta_{12} + k\tau; u_1, u_2)\,, \\ W(\zeta_{12}; u_1, u_2) \; &:= \; \langle 0|\, \phi(\zeta_1, u_1)\, \overset{*}{\phi}(\zeta_2, u_2) |0\rangle\;. \end{aligned} \tag{4..81}$$

To derive (4..81) one combines the KMS condition with the canonical (anti)commutation relations. Using the fact the canonical (anti)commutator is a c–number (i.e., proportional to the unit operator), equal to its vacuum expectation value (and that $\left\langle 1 \right\rangle_q = 1$), we find the following relation for the thermal mean value of products of ϕ–modes:

$$\begin{aligned} \left\langle \phi_{\nu_1 m_1}(u_1)\, \overset{*}{\phi}_{\nu_2 m_2}(u_2) \right\rangle_q - (-1)^{2d} \left\langle \overset{*}{\phi}_{\nu_2 m_2}(u_2)\, \phi_{\nu_1 m_1}(u_1) \right\rangle_q & \\ = \delta_{-\nu_1,\nu_2} \left\langle 0 \right| (\phi_{\nu_1 m_1}(u_1)\, \overset{*}{\phi}_{-\nu_1 m_2}(u_2) & \\ -(-1)^{2d} \overset{*}{\phi}_{-\nu_1 m_2}(u_2)\, \phi_{\nu_1 m_1}(u_1)) \left| 0 \right\rangle ; & \end{aligned} \tag{4..82}$$

on the other hand, the KMS condition together with (4..65) gives

$$\begin{aligned} \left\langle \overset{*}{\phi}_{\nu_2 m_2}(u_2)\, \phi_{\nu_1 m_1}(u_1) \right\rangle_q &= \left\langle \phi_{\nu_1 m_1}(u_1)\, q^H\, \overset{*}{\phi}_{\nu_2 m_2}(u_2)\, q^{-H} \right\rangle_q \\ &= q^{-\nu_2} \left\langle \phi_{\nu_1 m_1}(u_1)\, \overset{*}{\phi}_{\nu_2 m_2}(u_2) \right\rangle_q , \end{aligned} \tag{4..83}$$

and therefore,

$$\begin{aligned} \left\langle \phi_{\nu_1 m_1}(u_1)\, \overset{*}{\phi}_{\nu_2 m_2}(u_2) \right\rangle_q = \frac{\delta_{-\nu_1,\nu_2}}{1-(-1)^{2d} q^{\nu_1}} \left\langle 0 \right| (\phi_{\nu_1 m_1}(u_1)\, \overset{*}{\phi}_{-\nu_1 m_2}(u_2) \left| 0 \right\rangle & \\ + \frac{\delta_{-\nu_1,\nu_2} q^{\nu_2}}{1-(-1)^{2d} q^{\nu_2}} \left\langle 0 \right| (\overset{*}{\phi}_{\nu_2 m_2}(u_2)\, \phi_{-\nu_2 m_1}(u_1) \left| 0 \right\rangle & \end{aligned} \tag{4..84}$$

where the first term is nonzero for $\nu_1 = -\nu_2 \geqslant 0$ ($|q^{\nu_1}| < 1$) while the second is nonzero for $\nu_2 = -\nu_1 \geqslant 0$ ($|q^{\nu_2}| < 1$). Further, we expand the prefactors $\dfrac{1}{1-(-1)^{2d} q^{\nu_{1,2}}}$ in the right hand side of (4..84) in $|q| < 1$, as in Exercise 2.11, and take the corresponding sum (4..64) over the modes which gives (4..81) since, for instance, the term $\left((-1)^{2d} q^{\nu_1}\right)^k$ multiplying the vacuum expectation in the fist term (4..84) gives (after summing in ν_1) the expression $(-1)^{2dk}\, W(\zeta_{12} + k\tau;\, u_1,\, u_2)$ $(k = 0, 1, \dots)$.
To illustrate the conclusions of Theorem 4.5 in the case of a two point bosonic thermal correlation function let us choose as a basis of "bosonic" elliptic functions those of the generalized free scalar fields of dimension $k = 1, 2, \dots$:

$$P_k(\zeta_{12}; u_1, u_2; \tau) := \sum_{n=-\infty}^{\infty} \frac{\pi^{2k}}{\sin^k \pi\, (\zeta_+ + n\tau)\, \sin^k \pi\, (\zeta_- + n\tau)} \tag{4..85}$$

($\zeta_\pm = \zeta_{12} \pm \alpha$, $\cos 2\pi\alpha = u_1 \cdot u_2$). Then a corollary of Theorem 4.5 states that for any two GCI bosonic fields ϕ and ψ which obey the Huygens' principle (4..46) with some $N \in \mathbb{N}$ the Gibbs two point function can be presented as

$$\left\langle \phi(\zeta_1, u_1)\, \psi(\zeta_2, u_2) \right\rangle_q = \sum_{k=1}^{N} F_k(u_1, u_2; \tau)\, P_k\, (\zeta_{12}; u_1, u_2; \tau) \tag{4..86}$$

whose coefficients $F_k(u_1, u_2; \tau)$ are in general q-series. We will see in Sect. 6 that these coefficients carry an additional physical information that may recover quantities like the mean thermal energy.

Remark 4..5. The compact picture stress-energy tensor has a mode expansion of the form

$$T(\zeta, u; v) \,=\, e^{2D\pi i\zeta}\, T(e^{2\pi i\zeta}u; v) \,=\, \sum_{n=-\infty}^{\infty} \sum_{m=0}^{\infty} T_{nm}(u; v)\, e^{-2\pi i n\zeta} \quad (4..87)$$

where $T_{nm}(u; v)$ are operator valued, separately homogeneous and harmonic polynomials in u and v of degrees m and 2, respectively. Among all such polynomials there is exactly one, up to proportionality, which is $SO(D)$–invariant: $(u\cdot v)^2 - \frac{1}{D}\, u^2 v^2$. Then the conformal hamiltonian corresponds to the operator coefficient to this polynomial in the zero–mode part,

$$T_{02}(u; v) \,=\, N_0\, H \left((u\cdot v)^2 - \frac{1}{D}\, u^2 v^2\right) \,+\, \sum_{\sigma} T_{02;\sigma}\, h_{\sigma}^{(02)}(u; v)\,, \quad (4..88)$$

where $N_0 = \dfrac{D}{D-1}$, if the volume of the unite $(D-1)$–sphere is normalized to one, and $h_{\sigma}^{(02)}(u; v)$ is a basis in the $SO(D)$–nonscalar space. It follows from the $\mathcal{K}$–invariance of the thermal expectation values that

$$\left\langle T\,(\zeta, u; v)\right\rangle_q \,=\, N_0\, \left\langle H\right\rangle_q \left((u\cdot v)^2 - \frac{1}{D}\, u^2 v^2\right)\,. \quad (4..89)$$

5. Chiral fields in two dimensions

The simplest, long known example of a quantum field theory with elliptic correlation functions is provided by 2–dimensional ($2D$) (Euclidean) CFT on a torus or equivalently, by finite temperature $2D$ CFT on compactified Minkowski space $\overline{M}$ (for a rigorous discussion – see [72]). We adopt the latter point of view since it is the one that extends to higher dimensions.

The variables $\zeta_{\pm}$ (4..78) can be viewed for $D = 2$ as global coordinates on (the universal covering of) $\overline{M}$. Conserved currents and higher rank tensors (including the stress–energy tensor) split into chiral components depending on one of these variables. (This is simpler to derive in Minkowski space coordinates – see [23].) The vertex algebra corresponding to the full 2D theory then usually splits into tensor product of two copies of a vertex algebra over the real line satisfying the postulates of Sect. 4.3 with $D = 1$. A GCI chiral field $\phi(z)$ in this case is a formal Laurent series in a single complex variable z (one of the compactified light ray variables z_+ or z_- whose physical values belong to the circle $\mathbb{S}^1$), and the strong locality condition (4..46) for a pair of hermitian conjugate fields of dimension d assumes the form

$$z_{12}^{2d}\, \left\{ \overset{*}{\phi}\,(z_1)\, \phi\,(z_2) - (-1)^{2d}\, \phi\,(z_2)\, \overset{*}{\phi}\,(z_1) \right\} \,=\, 0\,. \quad (5..1)$$

5.1. $U(1)$ current, stress energy tensor, and the free Weyl field

A conformal $U(1)$ current $j_\mu(x)$ in 2D behaves as the gradient of a (dimensionless) free massless scalar field. Hence both its divergence and its curl vanish implying its splitting into chiral components:

$$\partial_\mu j^\mu = 0 = \partial_0 j_1 - \partial_1 j_0 \qquad \Longrightarrow \qquad (\partial_0 \pm \partial_1)(j^1 \pm j^0) = 0 \,. \tag{5..2}$$

Similarly, the symmetry and tracelessness of the conserved stress–energy tensor imply

$$(\partial_0 \pm \partial_1)(T_0^0 \pm T_0^1) = 0 \,. \tag{5..3}$$

We leave it to the reader to verify, on the other hand, that for z given by (4..23) with $D = 2$, $\boldsymbol{z} = x$ and the inverse transformation given, in general, by

$$\boldsymbol{x} = \frac{2\,\boldsymbol{z}}{1 + z^2 + 2z_D}\,, \quad -i\,x^0 = \frac{1 - z^2}{1 + z^2 + 2z_D} \tag{5..4}$$

setting in the $D = 2$ case $z_2 \pm i z_1 = e^{2\pi i \zeta_\pm}$ we find

$$x^0 \pm x^1 = \tan \pi\zeta_\pm \quad (\text{for } D = 2, \quad u = (\sin 2\pi\alpha, \cos 2\pi\alpha)). \tag{5..5}$$

Thus the "left movers" compact picture current and stress–energy tensor are functions of a single chiral variable ζ_-:

$$j(\zeta_-) := \frac{1}{2}(j^0 + j^1), \quad \mathcal{T}(\zeta_-) := \frac{1}{2}(\mathcal{T}_0^0 + \mathcal{T}_0^1) = \frac{1}{4}(\mathcal{T}_0^0 + \mathcal{T}_0^1 - \mathcal{T}_1^0 - \mathcal{T}_1^1). \tag{5..6}$$

The same is valid for the Weyl components of a free $D = 2$ Dirac field $\Psi(x)$ (and its conjugate $\overset{*}{\Psi}$). Introducing real off–diagonal $2D$ γ–matrices

$$\gamma^0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$\Psi(x) = \begin{pmatrix} \Psi_\uparrow \\ \Psi_\downarrow \end{pmatrix}, \quad \gamma^\mu \partial_\mu \Psi = 0 = (\partial_0 + \partial_1)\Psi_\uparrow\,, \quad \overset{*}{\Psi}(x) := \left(\Psi_\uparrow, \overset{*}{\Psi}_\downarrow\right) \tag{5..7}$$

we find

$$j(\zeta_-) \;=\; \frac{i}{2}\widetilde{\Psi}\,(\gamma^0 + \gamma^1)\,\Psi \;=\; \frac{1}{2}\overset{*}{\Psi}\,(1 + \gamma_0\gamma^1)\,\Psi \;=\; \overset{*}{\Psi}_\uparrow(\zeta_-)\Psi_\uparrow(\zeta_-) \tag{5..8}$$

($\widetilde{\Psi}(x) := \overset{*}{\Psi}(x)\gamma_0$).

Omitting from now on the arrow sign on the left mover's Weyl field $\Psi(\zeta)$ ($\equiv \Psi_\uparrow(\zeta)$) we obtain the chiral field $\Psi(\zeta)$ as a 1–component complex field which, together with its conjugate $\overset{*}{\Psi}$, can be written in the compact picture form

$$\overset{(*)}{\Psi}(\zeta) \;=\; \sum_n \overset{(*)}{\Psi}_{n-\frac{1}{2}} e^{i\pi(1-2n)\zeta}\,, \quad (\Psi_{n-\frac{1}{2}})^* \;=\; \overset{*}{\Psi}_{\frac{1}{2}-n} \tag{5..9}$$

(here ζ plays the role of a (compactified, chiral) light cone variable, say ζ_- of (4..78)). The Ψ–modes obey the *canonical anticommutation relations* and their index labels the energy they carry,

$$\Big[\Psi_{m-\frac{1}{2}}, \overset{*}{\Psi}_{\frac{1}{2}-n}\Big]_+ = \delta_{mn}, \qquad \Big[\overset{(*)}{\Psi}_{m-\frac{1}{2}}, \overset{(*)}{\Psi}_{\frac{1}{2}-n}\Big]_+ = 0\,,$$
$$\Big[L_0, \Psi_{\frac{1}{2}-n}\Big] = (n-\frac{1}{2})\Psi_{\frac{1}{2}-n}, \tag{5..10}$$

where L_0 stands for the Virasoro energy operator,

$$L_0 = \sum_{n=1}^{\infty}(n-\frac{1}{2})\,(\overset{*}{\Psi}_{\frac{1}{2}-n}\Psi_{n-\frac{1}{2}} + \Psi_{\frac{1}{2}-n}\overset{*}{\Psi}_{n-\frac{1}{2}})\,, \tag{5..11}$$

the counterpart of H in $D=1$ (in fact, the full 2D conformal hamiltonian is a sum of chiral energy operators, $H = L_0 + \overline{L}_0$, see, e.g., [16] or [23]). Energy positivity implies that the negative frequency modes annihilate the vacuum:

$$\overset{(*)}{\Psi}_{n-\frac{1}{2}}|0\rangle = 0 \quad \text{for} \quad n = 1, 2, \dots\,. \tag{5..12}$$

Then the vacuum 2–point correlation function is

$$\langle 0|\,\Psi(\zeta_1)\,\overset{*}{\Psi}(\zeta_2)\,|0\rangle \;=\; \frac{1}{2i\sin\pi\zeta_{12}}\,. \tag{5..13}$$

If we formally replace the normal product sum in (5..11) by the divergent sum over all integer n,

$$\begin{aligned}\widetilde{L}_0 \;&=\; \sum_{n\in\mathbb{Z}}(n-\frac{1}{2})\overset{*}{\Psi}_{\frac{1}{2}-n}\Psi_{n-\frac{1}{2}} = \sum_{n\in\mathbb{Z}}(n-\frac{1}{2})\Psi_{\frac{1}{2}-n}\overset{*}{\Psi}_{n-\frac{1}{2}}\\ &= L_0 - \frac{1}{2}\sum_{n=1}^{\infty}(2n-1),\end{aligned} \tag{5..14}$$

the last infinite term being understood by ζ–function regularization: $\sum\limits_{n=1}^{\infty}(2n-1)$ "=" $\sum\limits_{n=1}^{\infty} n - \sum\limits_{n=1}^{\infty} 2n$ "=" $-\zeta(-1)$, we will obtain [41]

$$\widetilde{L}_0 \;=\; L_0 + \frac{1}{2}\,\zeta(-1) \;=\; L_0 - \frac{1}{24} \quad (\text{as } \zeta(1-2k) = -\frac{B_{2k}}{2k})\,. \tag{5..15}$$

[41] Note that the passage from L_0 to $\widetilde{L}_0$ can be interpreted as the result of the non Möbius transformation $z \mapsto \zeta$ under which L_0 acquires a Schwarz derivative term: $\widetilde{L}_0 = L_0 + \frac{1}{12}\{z, 2\pi i\zeta\}$ where $\{z, w\} := \frac{z'''(w)}{z'(w)} - \frac{3}{2}\left(\frac{z''(w)}{z'(w)}\right)^2$. (Hermann Amandus Schwarz, 1843–1921 introduces his derivative in 1872.)

The calculations (4..82)–(4..84) in this case give

$$\left\langle \Psi_{m-\frac{1}{2}}\, q^{n-\frac{1}{2}}\, \overset{*}{\Psi}_{\frac{1}{2}-n} \right\rangle_q = \left\langle \overset{*}{\Psi}_{\frac{1}{2}-n} \Psi_{m-\frac{1}{2}} \right\rangle_q = \delta_{mn} - \left\langle \Psi_{m-\frac{1}{2}} \overset{*}{\Psi}_{\frac{1}{2}-n} \right\rangle_q ,$$

$$\left\langle \Psi_{m-\frac{1}{2}} \overset{*}{\Psi}_{\frac{1}{2}-n} \right\rangle_q = \frac{\delta_{mn}}{1+q^{m-\frac{1}{2}}} . \qquad (5..16)$$

Inserting (5..16) into the Gibbs 2-point function of the local Fermi field (5..9) we find (see Exercises 2.11 and 3.1, and $F_2(\tau) = 2G_2(\tau) - G_2\left(\frac{\tau+1}{2}\right)$ (3..17))

$$\left\langle \Psi(\zeta_1) \overset{*}{\Psi}(\zeta_2) \right\rangle_q = \frac{1}{2i \sin \pi\zeta_{12}} + 2i \sum_{n=1}^{\infty} \frac{q^{n-\frac{1}{2}}}{1+q^{n-\frac{1}{2}}} \sin(2n-1)\pi\zeta_{12} =$$

$$= \frac{1}{2\pi i}\, p_1^{11}(\zeta_{12}, \tau) \qquad (5..17)$$

(see Exercise 2.11 and the derivation of (4..81)). The temperature mean value of the chiral energy (5..11) is computed by Eq. (4..71) with $d_b(n) = 0$ and $d_f(n) = 1 + 1 = 2$ (for the two charges):

$$\left\langle \widetilde{L}_0 \right\rangle_q = -\frac{1}{24} + 2 \sum_{n=1}^{\infty} \left(n - \frac{1}{2}\right) \frac{q^{n-\frac{1}{2}}}{1+q^{n-\frac{1}{2}}} = F_2(\tau)\, (= 2\, G_2^{11}(\tau)) . \qquad (5..18)$$

Exercise 5..1. Verify the relation (5..18) for $F_2(\tau)$ defined by Eq. (3..17). (*Hint*: use the relation $q\left(\frac{\tau+1}{2}\right) = -q^{\frac{1}{2}}$ for $q^{\frac{1}{2}} \equiv q(\tau)^{\frac{1}{2}} = e^{i\pi\tau}$ and cancel the terms with even n in the expansion of $\frac{1}{2}\, G_2\left(\frac{\tau+1}{2}\right)$ with the expansion of $G_2(\tau)$.)

According to Sect. 3.2 (see Exercise 3.1), the *free energy* F_2 is a modular form of weight two and level Γ_θ.

For $H := \widetilde{L}_0$ we can integrate and exponentiate (4..70) (5..18) with the result

$$Z(\tau) = q^{-\frac{1}{24}} \prod_{n=1}^{\infty} (1 + q^{n-\frac{1}{2}})^2 . \qquad (5..19)$$

It follows from the Γ_θ invariance of the 1–form $F_2(\tau)d\tau$ that the partition function (5..19) is Γ_θ invariant. (It is, sure, not a modular form since it is not analytic for $q = 0$.) This invariance property of the partition function is peculiar for space–time dimension $D = 2$ since the leading term of the energy mean value has weight D and only for $D = 2$ the fact, that $G_D(\tau)$ is a modular form of weight D, implies that the 1–form $G_D(\tau)d\tau$ is invariant.

Remark 5..1. Minkowski space of dimension $D = 8n + 2, \quad n = 0, 1, \ldots$ is singled out by the property of admitting *Majorana-Weyl spinors* (i. e. $2^{\frac{D-2}{2}}$-component real semispinors). Thus, the Majorana-Weyl chiral field is a hermitian field $\Psi(\zeta)$ obeying (5..9) with $\Psi = \overset{*}{\Psi}$, and similar results can be obtained in this case, too. In particular, the free energy is $F_I(\tau) \equiv F_2(\tau)/2$ and the partition function is $Z_I(\tau) := Z(\tau)^{\frac{1}{2}}$, the partition function for the *Neveu-Schwarz sector* of the *chiral Ising model* (hence the index I on F and Z). Note that F_I and Z_I are transformed under the modular transformation $S\,T$ to the free energy and partition function of the *Ramond sector*

$$\begin{aligned} F_I^R(\tau) \;&:=\; (ST\,F_I)(\tau) \;=\; \frac{1}{\tau^2} F_I\left(1 - \frac{1}{\tau}\right) \;=\; G_2(\tau) - 2G_2(2\tau), \\ Z_I^R(\tau) \;&=\; q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 + q^n). \end{aligned} \tag{5..20}$$

The Neveu-Schwarz 0–point energy E_{NS} is the q–independent term $-1/48$ in F_I and it is uniquely determined by the modular covariance of F_I which thus selects Weyl symmetrization (accompanied by ζ–function regularization) rather than the normal ordering. The 0–point energy E_R in the Ramond sector differs from E_{NS} by the minimal conformal weight (eigenvalue of L_0) in that sector, Δ_R. It is calculated from (3..13) and (5..20) with the result

$$\begin{aligned} E_R \;&=\; -\Big(-\frac{B_2}{4}\Big) \;=\; \frac{1}{24} = E_{NS} + \Delta_R \quad \Longrightarrow \\ \Delta_R \;&=\; \frac{1}{24} + \frac{1}{48} \;=\; \frac{1}{16}\,. \end{aligned} \tag{5..21}$$

the magnetization field of the 2–dimensional Ising model has (left, right) conformal weight $(1/16, 1/16)$ – i. e., dimension $1/8$ and "spin" 0.

The knowledge of the energy eigenvalues does not suffice to label the states of the complex Weyl field. We also need the *charge operator* which appears as the zero mode of a composite field, the current

$$J(\zeta) = \frac{1}{2}\,[\overset{*}{\Psi}(\zeta), \Psi(\zeta)] \;= :\!\overset{*}{\Psi}(\zeta)\,\Psi(\zeta)\!: = \sum_n J_n e^{-2\pi i n\zeta} \tag{5..22}$$

where

$$\begin{aligned} J_0 &= \sum_{n=1}^{\infty} (\Psi^+_{\frac{1}{2}-n} \Psi_{n-\frac{1}{2}} - \Psi_{\frac{1}{2}-n} \Psi^+_{n-\frac{1}{2}})\,, \\ J_n &= \sum_{\rho \in \mathbb{Z}+\frac{1}{2}} \Psi^+_{-\rho} \Psi_{n+\rho} \quad \text{for} \quad n \neq 0\,. \end{aligned} \tag{5..23}$$

The current modes are characterized by their commutation relations

$$\left[J_n, \overset{*}{\Psi}(\zeta)\right] = e^{2\pi in\zeta}\overset{*}{\Psi}(\zeta)\,, \qquad [J_n, \Psi(\zeta)] = -e^{2\pi in\zeta}\Psi(\zeta)\,,$$
$$[J_n, J_m] = n\,\delta_{n,-m}\,. \tag{5..24}$$

The conformal energy (5..14) can be reexpressed in terms of the current modes – providing a special case of the so called Sugawara formula:

$$\begin{aligned}\widetilde{L}_0 &= \frac{1}{2}\sum_n J_{-n}J_n = L_0 + \frac{1}{2}\zeta(-1)\,,\\ L_0 &= \frac{1}{2}J_0^2 + \sum_{n=1}^{\infty} J_{-n}J_n, \quad \frac{1}{2}\zeta(-1) = -\frac{1}{24}\,.\end{aligned} \tag{5..25}$$

While the energy is coupled in the partition function with the (complexified) inverse temperature τ we shall express the charge distribution by a parameter μ called the *chemical potential* introducing the generalized partition function of Neveu–Schwarz sector

$$Z_{NS}(\tau,\,\mu) = tr_{\mathcal{V}}\left(q^{\widetilde{L}_0}q_\mu^{J_0}\right)\,, \quad q_\mu = e^{2\pi i\mu}\,. \tag{5..26}$$

Remark 5..2. The complex Weyl field model can be viewed as "the square of the chiral Ising model" of Remark 5.1. If we split Ψ into its real and imaginary parts $\sqrt{2}\Psi(\zeta) = \Psi^1(\zeta) - i\Psi^2(\zeta)$, then we shall have (setting $\langle\Psi^1(\zeta_1)\,\Psi^2(\zeta_2)\rangle_0 = 0$)

$$\langle\Psi(\zeta_1)\,\overset{*}{\Psi}(\zeta_2)\rangle_0 \;=\; \langle\Psi^1(\zeta_1)\,\Psi^1(\zeta_2)\rangle_0 \;=\; \langle\Psi^2(\zeta_1)\,\Psi^2(\zeta_2)\rangle_0\,, \tag{5..27}$$

the energy of the charged field (5..14) being twice the energy of the Ising model.

Exercise 5..2. Set $\langle A\rangle_{q,\mu} := \dfrac{1}{Z(\tau,\mu)}tr_{\mathcal{V}}\left(Aq^{\widetilde{L}_0}q^{J_0}\right)$; use the generalized KMS condition $\langle AB_{\tau,\mu}\rangle_{q\mu} = \langle BA\rangle_{q\mu}$ for $q^{\widetilde{L}_0}q_\mu^{J_0}\,B = B_{\tau,\mu}q^{\widetilde{L}_0}q^{J_0}$ to prove

$$\langle\Psi_{n-\frac{1}{2}}\Psi^+_{\frac{1}{2}-m}\rangle_{q,\mu} = \frac{\delta_{mn}}{1+q_\mu q^{n-\frac{1}{2}}}\,,\ \langle\Psi^+_{n-\frac{1}{2}}\Psi_{\frac{1}{2}-m}\rangle_{q,\mu} = \frac{\delta_{mn}}{1+q_\mu^{-1}q^{m-\frac{1}{2}}}\,, \tag{5..28}$$

$$2\pi i\big\langle\Psi(\zeta_1)\overset{*}{\Psi}(\zeta_2)\big\rangle_{q,\mu} = p_1^{11}(\zeta_{12},\tau,\mu) := \frac{\pi}{\sin\pi\zeta_{12}} + \sum_m \frac{e^{i\pi m(2\mu+\kappa)}}{\sin\pi(\zeta_{12}+m\tau)} \tag{5..29}$$

(cf. Appendix A).

Inserting (5..28) into the mean value of $\widetilde{L}_0$ (5..14), we find the following generalization of (5..18):

$$\langle\widetilde{L}_0\rangle_{q,\mu} = -\frac{1}{24} + \sum_{n=1}^{\infty}\left(n-\frac{1}{2}\right)\left(\frac{q_\mu^{-1}q^{n-\frac{1}{2}}}{1+q_\mu^{-1}q^{n-\frac{1}{2}}} + \frac{q_\mu q^{n-\frac{1}{2}}}{1+q_\mu q^{n-\frac{1}{2}}}\right). \qquad (5..30)$$

Integrating and exponentating the relation

$$\langle\widetilde{L}_0\rangle_{q,\mu} = \frac{1}{2\pi i}\frac{\partial}{\partial\tau}\log Z(\tau,\,\mu) = \frac{1}{Z}q\frac{\partial}{\partial q}Z \qquad (5..31)$$

which generalizes (4..70) we find

$$Z_{NS}(\tau,\,\mu) = q^{-\frac{1}{24}}\prod_{n=1}^{\infty}\left(1+q_\mu^{-1}q^{n-\frac{1}{2}}\right)\left(1+q_\mu q^{n-\frac{1}{2}}\right). \qquad (5..32)$$

Exercise 5..3. Use (5..25) and the KMS condition to derive the expression

$$Z_{NS}(\tau,\,\mu) = \frac{\vartheta(\mu,\,\tau)}{\eta(\tau)} = q^{-\frac{1}{24}}\prod_{m=1}^{\infty}(1-q^m)^{-1}\sum_n q^{\frac{1}{2}n^2}\,q_\mu^n \qquad (5..33)$$

($\vartheta(=\vartheta_{00})$) being the Riemann theta function (3..29); η, the Dedekind η–function of (3..20)).

Comparison between (5..32) and (5..33) yields a nontrivial identity called the *Jacobi triple product formula*.

We can repeat the study of modular properties of the chiral Ising model (Remark 5.1) for the model at hand of a complex Weyl field with the following results:

$$\begin{aligned} Z_{NS}(\tau+1,\,\mu) &= e^{-\frac{i\pi}{12}}Z_{NS}\left(\tau,\,\mu+\frac{1}{2}\right), \\ Z_{NS}\left(-\frac{1}{\tau},\,\frac{1}{2}\right) &= Z_R(\tau,\,0)\,; \end{aligned} \qquad (5..34)$$

here

$$Z_R(\tau,\,\mu) = q^{\frac{1}{12}}\left(q_\mu^{-\frac{1}{2}}+q_\mu^{\frac{1}{2}}\right)\prod_{n=1}^{\infty}\left(1+q_\mu^{-1}q^n\right)(1+q_\mu q^n) \qquad (5..35)$$

where Z_R is the partition function of the Ramond sector characterized by charge and conformal weight of its lowest energy state

$$e_R = \pm\frac{1}{2}, \qquad \Delta_R = \frac{1}{2}e_R^2 = \frac{1}{8}. \qquad (5..36)$$

We conclude this section with a discussion of the possibility to reconstruct the Gibbs mean energy from the thermal 2–point functions. Let $\psi_d(z)$ and $\overset{*}{\psi}_d(z)$ be a pair of hermitian conjugate complex chiral fields of (half)integer dimension d in the analytic picture. Let us assume that the stress–energy tensor $T(z)$ contributes to the OPE of $\overset{*}{\psi}_d\psi_d$:

$$\frac{1}{2}\left(\overset{*}{\psi}_d(z_1)\,\psi_d(z_2)+\psi_d(z_1)\,\overset{*}{\psi}_d(z_2)\right) = \frac{N_0}{z_{12}^{2d}}+\frac{N_1}{z_{12}^{2d-2}}\,T(\sqrt{z_1z_2})\,+\,\dots\equiv$$
$$\equiv \frac{N_0}{z_{12}^{2d}}\left(1+\frac{2d}{c}\,z_{12}^2\,T(\sqrt{z_1z_2})+O(z_{12}^4)\right), \qquad (5..37)$$

where the Ward identities (property 7) of Sect. 4.3 imply that $N_1=\frac{2d}{c}\,N_0$, $c:=2z_{12}^4\,\langle 0|\,T(z_1)T(z_2)\,|0\rangle$ being the Virasoro central charge (see [23], Sect. 3.5). It is important for the validity of (5..37) that the product $z_{12}^{2d}\,\big(\,\overset{*}{\psi}_d(z_1)\,\psi_d(z_2)\,+\,\psi_d(z_1)\,\overset{*}{\psi}_d(z_2)\,\big)$ is a *symmetric* dimensionless bilocal field of (z_1,z_2) for both even and odd $2d$. Passing to the compact picture fields, $\psi_d(\zeta_k)=e^{2d\pi i\zeta_k}\psi_d(z_k)$, $T(\zeta_k)=e^{4\pi i\zeta_k}T(z_k)$, $z_k=e^{2\pi i\zeta_k}$ $(k=1,2)$ Eq. (5..38) takes the form

$$\frac{1}{2}\left(\overset{*}{\psi}_d(\zeta_1)\,\psi_d(\zeta_2)+\psi_d(\zeta_1)\,\overset{*}{\psi}_d(\zeta_2)\right)= \qquad (5..38)$$
$$=\frac{N_0}{(2i\sin\pi\zeta_{12})^d}\left(1+\frac{2d}{c}\,(2i\sin\pi\zeta_{12})^2\,T\left(\frac{\zeta_1+\zeta_2}{2}0\right)+O(\sin^4\pi\zeta_{12})\right).$$

This implies the following Laurent expansions in $\sin\pi\zeta_{12}$ and ζ_{12} of the (symmetric under charge conjugation) thermal 2–point function:

$$\frac{1}{2}\left\langle\overset{*}{\psi}_d(\zeta_1)\,\psi_d(\zeta_2)+\psi_d(\zeta_1)\,\overset{*}{\psi}_d(\zeta_2)\right\rangle_q= \qquad (5..39)$$
$$=\frac{N_0}{(2i\sin\pi\zeta_{12})^{2d}}\left(1+\frac{2d}{c}\,\left\langle L_0\right\rangle_q(2i\sin\pi\zeta_{12})^2+O(\sin^4\pi\zeta_{12})\right)$$
$$=\frac{N_0}{(2\pi i\zeta_{12})^{2d}}\left(1+\frac{2d}{c}\,\left\langle\widetilde{L}_0\right\rangle_q\,(2\pi i\zeta_{12})^2+O(\zeta_{12}^4)\right)\,,\quad \widetilde{L}_0:=L_0-\frac{c}{24}\,.$$

(We have already encountered a Laurent expansion of this type in Exercise 3.1 for the case of our basic elliptic functions.) In the case of the complex Weyl field we have $d=1/2$, $c=1$ and $N_0=1$ $(N_1=1)$ reproduces the result for $\left\langle\widetilde{L}_0\right\rangle_q$ (cp. with (5..15) and (5..18)). This gives an interpretation of the passage $L_0\mapsto\widetilde{L}_0$ as an exchange, $\sin\pi\zeta_{12}\mapsto\zeta_{12}$, of the expansion variable which thus makes all coefficients *modular invariant* if the left hand side is. We can also write (5..39) as an expansion in homogeneous elliptic functions: for $2d\geqslant 3$ the stress–energy tensor $T(z)$ occurs in the singular part of the OPE of ψ and then (5..39) implies

$$\frac{1}{2}\left\langle \overset{*}{\psi}_d(\zeta_1)\,\psi_d(\zeta_2)+\psi_d(\zeta_1)\,\overset{*}{\psi}_d(\zeta_2)\right\rangle_q = \sum_{k=0}^{[\![d-\frac{1}{2}]\!]} C_k\, p_{2d-2k}^{\kappa\kappa}(\zeta_{12},\tau) \tag{5..40}$$
$$= (2\pi i)^{-2d} N_0\, p_{2d}^{\kappa\kappa}(\zeta_{12},\tau) + (2\pi i)^{-2d+2} N_1 \left\langle \widetilde{L}_0\right\rangle_q p_{2d-2}^{\kappa\kappa}(\zeta_{12},\tau) + \dots$$

for some constants C_k ($C_0 = (2\pi i)^{-2d} N_0$, $C_1 = (2\pi i)^{-2d+2} N_1 \left\langle \widetilde{L}_0\right\rangle_q$, etc.) and $\kappa := 2d\, mod 2 = 0, 1$ (this follows from the fact that the singular part of each $p_k^{\kappa\lambda}(\zeta_{12},\tau)$ contains only the term ζ_{12}^{-k}).

5.2. Lattice vertex algebras

An important class of vertex algebras involved as building blocks in most known examples of 2D CFT is based on the theory of affine Kac–Moody algebras associated with connected compact Lie groups (see [32] and Sect. 1 of [35]). Each such group is *reductive*: it can be written as the direct product of a (rank r) Abelian group $G = U(1)^r$ and a semi-simple factor which can be (and often are) treated separately. We shall briefly review here the simpler case of a lattice vertex algebra corresponding to G (or rather, to the affine extension of its Lie algebra $\widehat{\mathfrak{g}} = u(1)^{\otimes r}$). The case of nonabelian current algebra has been surveyed by both mathematicians – including Kac's books [31], [32], [34] and physicist [26], [23], [16]).

We will construct first the associative algebra $\mathcal{A}$ generated by the fields' modes together with its vacuum representation $\mathcal{V}$ which will be the linear space of the lattice vertex algebra. Let $\mathfrak{h}$ be an r-dimensional real vector space: denote by $\mathfrak{h}_{\mathbb{C}} := \mathfrak{h} + i\,\mathfrak{h}$ its complexification. We assume that $\mathfrak{h}$ is endowed with an Euclidean scalar product $\left(h \,\middle|\, h'\right) \in \mathbb{R}$ for $h, h' \in \mathfrak{h}$ and with a rank $r = dim\,\mathfrak{h}$ lattice $Q \subset \mathfrak{h}$ (an additive subgroup) such that $\left(\alpha \,\middle|\, \beta\right) \in \mathbb{Z}$ for all $\alpha, \beta \in Q$. Introduce the Heisenberg current Lie algebra $\widehat{\mathfrak{h}}_{\mathbb{C}}$ with generators h_n for $n \in \mathbb{Z}$ and $h \in \mathfrak{h}_{\mathbb{C}}$ such that

$$\left[h_n, h'_m\right] = n\,\delta_{n,-m}\left(h \,\middle|\, h'\right) \ \left((\lambda h + \mu h')_n = \lambda h_n + \mu h'_n,\ \lambda, \mu \in \mathbb{C}\right). \tag{5..41}$$

Its universal enveloping algebra $\mathcal{A}_{\mathfrak{h}}$ contains three subalgebras $\mathcal{S}(\widehat{\mathfrak{h}}_{\mathbb{C}}^{(0,\pm)})$, generated by h_0 and by h_n for $\mp n = 1, 2, \dots$ ($h \in \mathfrak{h}_{\mathbb{C}}$), respectively. Note that each of $\mathcal{S}(\widehat{\mathfrak{h}}_{\mathbb{C}}^{(0,\pm)})$ is a commutative algebra isomorphic to the algebra of symmetric polynomials over the underlying vector spaces and $\mathcal{S}(\widehat{\mathfrak{h}}_{\mathbb{C}}^{(0)})$ is central. Moreover, every element A in $\mathcal{A}_{\mathfrak{h}}$ can be represented as a (finite) sum of *normal order products*: $h^1_{n_1} \dots h^m_{n_m}$ with $n_1 \leqslant \dots \leqslant n_m$, so that we have the isomorphism of vector spaces

$$\mathcal{A}_{\mathfrak{h}} \cong \mathcal{S}(\widehat{\mathfrak{h}}_{\mathbb{C}}^{(+)}) \otimes \mathcal{S}(\widehat{\mathfrak{h}}_{\mathbb{C}}^{(0)}) \otimes \mathcal{S}(\widehat{\mathfrak{h}}_{\mathbb{C}}^{(-)})\,. \tag{5..42}$$

For every α belonging to the lattice Q we will now define the Fock space representation $\mathcal{V}_\alpha$ generated by a vector $|\alpha\rangle$ and the relations

$$h_n|\alpha\rangle = 0, \quad h_0|\alpha\rangle = \big(h \,\big|\, \alpha\big)|\alpha\rangle \quad (n = 1, 2, \dots, \ h \in \mathfrak{h}_{\mathbb{C}}). \tag{5..43}$$

In other words, $\mathcal{V}_\alpha$ is isomorphic to the quotient of $\mathcal{A}_{\mathfrak{h}}$ by the left ideal generated by $(\{h_0 - \big(h \,\big|\, \alpha\big) : h \in \mathfrak{h}_{\mathbb{C}}\} \oplus \widehat{\mathfrak{h}}_{\mathbb{C}}^{(-)})$; so that it is isomorphic, as a vector space, to the symmetric subalgebra $\mathcal{S}(\widehat{\mathfrak{h}}_{\mathbb{C}}^{(+)})$. The vector space of the lattice vertex algebra is defined as the sum of all Fock spaces $\mathcal{V}_\alpha$:

$$\mathcal{V} = \mathop{\oplus}_{\alpha \in Q} \mathcal{V}_\alpha \,. \tag{5..44}$$

Clearly, the algebra $\mathcal{A}_{\mathfrak{h}}$, which is a part of the full algebra $\mathcal{A}$, acts on $\mathcal{V}$ as the direct sum of its actions on $\mathcal{V}_\alpha$. To complete the definition of the algebra $\mathcal{A}$ we further introduce *intertwining operators* E^α on $\mathcal{V}$ for all $\alpha, \beta \in Q$, defined by

$$\begin{aligned} &E^\alpha(\mathcal{V}_\beta) \subseteqq \mathcal{V}_{\alpha+\beta} \,, \qquad\qquad [h_n, E^\alpha] = \delta_{n,0} \, \big(h \,\big|\, \alpha\big) \; E^\alpha \,, \\ &E^\alpha|\beta\rangle = \epsilon\,(\alpha, \beta)\,|\alpha + \beta\rangle \,, \end{aligned} \tag{5..45}$$

where $h \in \mathfrak{h}_{\mathbb{C}}$, $n \in \mathbb{Z}$ and $\epsilon(\alpha, \beta)$ are $U(1)$–factors [42]. Eqs. (5..45) completely determine E^α since the $\mathcal{A}_{\mathfrak{h}}$–representations $\mathcal{V}_\alpha$ are *irreducible*. We will require that the products $E^\alpha E^\beta$ are proportional to $E^{\alpha+\beta}$, $E^0 = \mathbb{I}_{\mathcal{V}}$ and $\epsilon(\alpha, 0) = 1$ (i.e., $E^\alpha|0\rangle = |\alpha\rangle$). Computing then $E^\alpha E^\beta|0\rangle$ and $E^{\alpha+\beta}|0\rangle$ by (5..45) one finds

$$E^\alpha\, E^\beta \,=\, \epsilon(\alpha, \beta)\, E^{\alpha+\beta}. \tag{5..46}$$

The algebra $\mathcal{A}$ is defined as the algebra generated by all E^α and $\mathcal{A}_{\mathfrak{h}}$. The associativity implies the 2*–cocycle relation*:

$$\epsilon(\alpha, \beta)\, \epsilon(\alpha + \beta, \gamma) \,=\, \epsilon(\alpha, \beta + \gamma)\, \epsilon(\beta, \gamma) \quad (\alpha, \beta, \gamma \in Q). \tag{5..47}$$

The *gauge transformation* $E^\alpha \mapsto \eta(\alpha) E^\alpha$ ($|\alpha\rangle \mapsto \eta(\alpha)|\alpha\rangle$, $\eta(0) = 1$), will give rise to a change of the 2-cocycle ϵ by a *coboundary*,

$$\epsilon(\alpha, \beta) \,\mapsto\, \frac{\eta(\alpha)\, \eta(\beta)}{\eta(\alpha + \beta)}\, \epsilon(\alpha, \beta). \tag{5..48}$$

There are further restrictions on $\epsilon(\alpha, \beta)$ coming from the physical requirements of locality and unitarity. The first one means that we should define a vertex algebra structure on the space $\mathcal{V}$ such that the field modes span the algebra $\mathcal{A}$. By the Kac's existence theorem ([32], Chapt. 4) it is enough to introduce a system of mutually local fields whose modes generate $\mathcal{A}$. Note that $\mathcal{A}$ is isomorphic, as a vector space, to the tensor product $\mathcal{A}_{\mathfrak{h}}$

[42] $\epsilon(\alpha, \beta)$ are, in general, nonzero complex numbers but in view of the hermitian structure which we will further define they can be restricted to $U(1)$–factors.

$\otimes\, Span_{\mathbb{C}}\{E^\alpha : \alpha \in Q\}$ due to the relations (5..45) and (5..46). A system of mutually local fields whose modes generate $\mathcal{A}_{\mathfrak{h}}$ is given by the Abelian currents

$$h(z) \;\equiv\; Y(h,z) \;=\; \sum_{n\,\in\,\mathbb{Z}} h_n\, z^{-n-1} \quad \text{for} \quad h \,\in \mathfrak{h}_{\mathbb{C}}\,, \tag{5..49}$$

which obey the canonical commutation relations

$$\big[h(z), h'(w)\big] \;=\; \big(h \,\big|\, h'\big)\; \partial_w \delta(z-w)\,, \quad \delta(z-w) \;:=\; \sum_{n\in\mathbb{Z}} z^n w^{-n-1} \tag{5..50}$$

in accord to (5..41). The fields whose modes will contain the operators E^α are defined as follows:

$$Y_\alpha(z) \;\equiv\; Y(E^\alpha, z) \;:=\; E^\alpha\, z^{\alpha_0}\, e^{-\sum\limits_{n>0} \frac{\alpha_n}{n}\, z^n}\; e^{\sum\limits_{n>0} \frac{\alpha_{-n}}{n}\, z^n} \tag{5..51}$$

where α_n are the modes in the $\widehat{\mathfrak{h}}$ corresponding to an element $\alpha \in Q \subset \mathfrak{h}$. Note that if we introduce the "integral" field $\int \alpha(z)$ by

$$\int \alpha(z) \;:=\; -\sum_{n\,\neq\,0} \frac{\alpha_n}{n} z^{-n} \;=\; \sum_{n\,\neq\,0} \frac{\alpha_{-n}}{n} z^n \tag{5..52}$$

then (5..51) is, by definition the Wick *normal ordered exponent*

$$Y_\alpha(z) \;=\; E^\alpha\, z^{\alpha_0}\; {:}e^{\int \alpha(z)}{:}\,. \tag{5..53}$$

One derives the commutation relations

$$[h(z), Y_\alpha(w)] \;=\; \big(h \,\big|\, \alpha\big)\; \delta(z-w) \tag{5..54}$$

and the operator product expansion formula

$$Y_\alpha(z) Y_\beta(w) = \epsilon(\alpha,\beta)\,(z-w)^{(\alpha|\beta)} E^{\alpha+\beta} z^{\alpha_0} w^{\beta_0} {:}e^{\int \alpha(z) \,+\, \int \beta(w)}{:}\;. \tag{5..55}$$

(Eq. (5..55) follows from the *Weyl property* of the normal ordered exponents

$${:}e^{A(z)}{:}{:}e^{B(w)}{:} \;=\; e^{\langle A(z) B(w) \rangle_0}\; {:}e^{A(z)+B(w)}{:} \tag{5..56}$$

for every two fields $A(z)$ and $B(w)$ whose modes belong to a Heisenberg Lie algebra, $\big\langle\, A(z) B(w) \,\big\rangle_0$ standing for their Wick contraction. Now the locality assumes that there exists a $\mathbb{Z}_2$–factor $(-1)^{p_\alpha}$, $p_\alpha = 0, 1$, such that

$$(z-w)^{N_{\alpha\beta}}\, (Y_\alpha(z)\, Y_\beta(w) - (-1)^{p_\alpha p_\beta}\, Y_\beta(w)\, Y_\alpha(z)) \;=\; 0 \tag{5..57}$$

for $N_{\alpha\beta} \gg 0$ (in fact, for $N_{\alpha\beta} = (|\alpha|^2 + |\beta|^2)/2$) and this combined with (5..56) implies that

$$p_\alpha = |\alpha|^2 \, mod \, 2 \quad (|\alpha| := (\alpha \,|\, \alpha))\,,$$
$$s(\alpha,\beta) := \frac{\epsilon(\alpha,\beta)}{\epsilon(\beta,\alpha)} \; = \; (-1)^{(\alpha|\beta)+|\alpha|^2|\beta|^2}\;. \tag{5..58}$$

Note that the gauge transformations (5..48) leave invariant the statistical factor $s(\alpha,\beta)$. The *stress–energy tensor* of the lattice vertex algebra is defined by the sum of normal products

$$T(z) = \frac{1}{2}\sum_{j=1}^{r} {:}\lambda_j(z)\alpha^j(z){:} \tag{5..59}$$
$$({:}\lambda(z)\alpha(z){:} = \sum_{n>0}\big(\lambda_{-n}z^{n-1}\alpha(z) + \alpha(z)\,\lambda_{n-1}z^{-n}\big))$$

where $\left\{\alpha^j\right\}_{j=1}^{r}$ is basis of the lattice Q and $\{\lambda_j\}_{j=1}^{r}$ is the dual basis

$$\big(\lambda_j \,\big|\, \alpha^k\big) \; = \; \delta_{jk} \tag{5..60}$$

spanning the *dual lattice* Q^*, which consists of all $\lambda \in \mathbb{R}^r$ such that $\big(\lambda \,\big|\, \alpha\big) \in \mathbb{Z}$ for all $\alpha \in Q$. Eq. (5..59) represents the so called "*Sugawara formula*" (for a review and references see [23]). The modes L_n in the Laurent expansion of T, which are expressed in terms of the current modes,

$$T(z) \; = \sum_{n\,\in\,\mathbb{Z}} L_n\, z^{-n-2}\,, \qquad L_n \; = \; \frac{1}{2}\sum_{j\,=\,1}^{r}\sum_{m} {:}\lambda_{jmn-m}\alpha^j_m{:} \tag{5..61}$$

generate the *Virasoro algebra*, characterized by the commutation relations

$$[L_n, L_m] = (n-m)\,L_{n+m} + \frac{c}{12}\,(n^3-n)\,\delta_{n,-m}\,, \qquad [c, L_m] \; = \; 0\,. \tag{5..62}$$

We infer that

$$[L_0, Y_\alpha(z)] \; = \; \left(z\frac{d}{dz} + \frac{1}{2}\,|\alpha|^2\right)\,, \tag{5..63}$$

i.e., the conformal dimension of Y_α is $d_\alpha = |\alpha|^2/2$ (in accord with the value of p_α in (5..58) and the spin–dimension connection (4..53)).

We proceed to define a hermitian structure on a lattice vertex algebra. The fields $h(z)$ having an interpretation of currents whose zero modes h_0 correspond to real *charges* (spanning Q) are, hence, assumed hermitian:

$$(h_n)^* \; = \; h_{-n} \quad (n \in \mathbb{Z}). \tag{5..64}$$

In particular, the hermiticity of h_0 together with the commutation relations (5..45) will require that $(E^\alpha)^* = \sigma(\alpha)E^{-\alpha}$ but we can make all $\sigma(\alpha)$ equal to 1 by a suitable gauge transformation $E^\alpha \mapsto \eta(\alpha)E^\alpha$; so that we have

$$(E^\alpha)^* \,=\, E^{-\alpha}\,. \tag{5..65}$$

This completely determines the hermitian structure on $\mathcal{V}$. It imposes the following additional properties on the 2-cocycle:

$$\epsilon(-\beta,-\alpha) = \overline{\epsilon(\alpha,\beta)}\,, \qquad\qquad \epsilon(\alpha,-\alpha) \,=\, 1 \tag{5..66}$$

(the latter formula uses the fact that $E^\alpha\, E^{-\alpha} = E^\alpha\,(E^\alpha)^* (= \epsilon(\alpha,-\alpha)\,\mathbb{I}_V)$ is a positive operator). To summarize, the 2-cocycle e satisfies the following conditions: (1) $\epsilon\,(\alpha,\beta) \in U(1)$; (2) $\epsilon(\alpha,0) \,=\, \epsilon(0,\alpha) = 1$; (3) Eq. (5..47); (4) Eq. (5..58); and (5) Eq. (5..66). A nontrivial example of such cocycle can be given using an ordering in the lattice Q. An equivalent but different choice (with the same symmetry factor (5..58)) is made in [2]: $\epsilon(\alpha,\beta)$ is assumed real, $\epsilon\,:\, Q\times Q \to \{\pm 1\}$, and bimultiplicative (but it does not satisfy the above condition (5), and the conjugation law reads $(E^\alpha)^* = \epsilon(\alpha,-\alpha)^{-1}E^{-\alpha}$).

The irreducible positive energy (local field) representations of the lattice vertex algebra are labeled by elements of the dual lattice Q^* $(:= \{\lambda\in\mathfrak{h} : \left(\lambda\,\middle|\,\alpha\right) \in \mathbb{Z}$ for all $\alpha\in Q\})$ modulo Q; in other words, they are in one–to–one correspondence with the elements of the *finite Abelian group* Q^*/Q. The *character* of a representation of weight $\lambda\in Q^*$ is given by $(\mu\in\mathfrak{h})$

$$\chi_\lambda(\tau,\mu) = [\eta(\tau)]^{-r}\,\Theta^Q_\lambda(\tau,\mu) = \frac{1}{[\eta(\tau)]^r}\sum_{\gamma\,\in\,\lambda+Q} q^{\frac12\left(\gamma|\gamma\right)}\,e^{2\pi i\left(\gamma|\mu\right)}\,, \tag{5..67}$$

(where η is the Dedekind η-function (3..20), $q = e^{2\pi i\tau}$). If the lattice Q is *even* (i.e. if the norm square, $|\alpha|^2$, of any $\alpha\in Q$ is an even integer) then $\{\chi(\tau,\mu)\}$ span a finite dimensional representation of $\Gamma(1)$:

$$\begin{aligned} \chi_\lambda(\tau+1,\mu) \,&=\, e^{2\pi i\left(\frac{|\lambda|^2}{2}-\frac{r}{24}\right)}\,\chi_\lambda(\tau,\mu)\\ e^{-i\pi\frac{|\mu|^2}{\tau}}\,\chi_\lambda\left(-\frac{1}{\tau},\frac{\mu}{\tau}\right) \,&=\, \sum_{\lambda'\,\in\,Q^*/Q} |Q^*/Q|^{-\frac12}\;e^{-2\pi i\left(\lambda|\lambda'\right)}\chi_{\lambda'}(\tau,\mu)\,, \end{aligned} \tag{5..68}$$

where $|Q^*/Q|$ is the number of elements of (the finite group) Q^*/Q. (For odd lattices $\{\chi_\lambda\}$ span a representation of the index three subgroup Γ_θ (2..59) of $\Gamma(1)$.)

The case of the (even) *self–dual* lattice $Q = E_8$ $(= Q^*)$ is particularly interesting (see [49], [33]); we have a single modular invariant character, χ_0, in this case

$$\chi_0^{E_8}(\tau,0) \,=\, \frac{1}{[\eta(\tau)]^8}\,\Theta_0^{E_8}(\tau,0) \,=\, [j(\tau)]^{\frac13} \tag{5..69}$$

where $j(\tau)$ is the absolute invariant (3..27). We have, in particular,

$$\begin{aligned}\Theta_0^{E_8}(\tau,0) &= \sum_{\gamma\,\in\,E_8} q^{\frac{1}{2}(\gamma|\gamma)} = \frac{1}{2}\left[\vartheta_{00}(0,\tau)^8 + \vartheta_{10}(0,\tau)^8 + \vartheta_{01}(0,\tau)^8\right] \\ &= 240\,G_4(\tau)\,. \end{aligned} \tag{5..70}$$

Remark 5..3. A lattice Q is self dual iff it is *unimodular*, i.e. iff the volume of its fundamental cell (defined as the square root of the absolute value of the determinant of the matrix of inner products of basis vectors (the Gram determinant) of any given basis of Q) is one. Even unimodular lattices only exist in inner products spaces of signature divisible by 8 (see [49] Theorem 5.1). Moreover, even unimodular lattices with indefinite inner product (i.e. with a non-degenerate symmetric bilinear form such that there exist vectors of positive and negative square lengths) are determined up to isomorphism by their rank and signature ([49] Theorem 5.3). In particular, there are unique (isomorphism classes of) even self-dual lattices of type $(25,1)$ and $(9,1)$ corresponding to bosonic and super-string theories, respectively. This is not true for lattices equipped with positive definite (integral) bilinear form. For instance, there are two non-isomorphic positive definite even unimodular lattices of rank 16: Γ_{16} (having a basis of vectors of length squares 2 and 4 – see [49] Lemma 6.5) and $E_8 \oplus E_8$; there are 24 such lattices of rank 24. By contrast, E_8 is the unique (up to isomorphism) even unimodular lattice of signature $(8,0)$. A canonical basis in E_8 is given by the *roots* $\alpha_1, \ldots, \alpha_8$ whose scalar products are given by the *Cartan matrix*: $\big(\alpha_i \,\big|\, \alpha_j\big) = c_{ij} = c_{ji}$, $c_{ii} = 2$, $c_{58} = c_{ii+1} = -1\,(= c_{i+1i})$ for $i = 1, \ldots, 6$, $c_{ij} = 0$ otherwise. The reader will find more information about the E_8 lattice, its automorphism group, and the associated Lie algebra e.g. in [9] and in [31], Chapters 4 and 6.

5.3. The $N = 2$ superconformal model

The $N = 2$ (extended) superconformal model [8] considered as a vertex algebra (in the sense of [32]) is generated by a pair of conjugate to each other local Fermi fields of dimension $\frac{3}{2}$,

$$G^{\pm}(\zeta) = \sum_{\rho\in\mathbb{Z}+\frac{1}{2}} G^{\pm}_{\rho}\, e^{-2\pi i\rho\zeta}\,, \quad \big[G^{\epsilon}_{\rho},\, G^{\epsilon}_{\sigma}\big]_{+} = 0 \quad \text{for} \quad \epsilon = \pm\,. \tag{5..71}$$

Regarded as an (infinite dimensional) Lie superalgebra, the $N = 2$ extended super–Virasoro algebra, $SV(2)$, is spanned by $G^{\pm}$, a $U(1)$ current J, the stress–energy tensor T (of modes L_n) and a central element c. The non–

trivial (anti) commutation relations among their modes read:

$$\begin{aligned}
&\left[G^{\pm}_{n-\frac{1}{2}}, G^{\mp}_{\frac{1}{2}-m}\right]_{+} = 2L_{n-m} \pm (n+m-1)\, J_{n-m} + \frac{c}{3} n\,(n-1)\,\delta_{nm}\,,\\
&[J_n, J_m] = \frac{c}{3} n \delta_{n,-m}\,, \quad \left[J_n, G^{\pm}_{\rho}\right] = \pm G^{\pm}_{n+\rho}\,, \quad [J_n, L_m] = nJ_{n+m}\,,\\
&\left[L_n, G^{\pm}_{\rho}\right] = \left(\frac{n}{2} - \rho\right) G^{\pm}_{n+\rho}\,,\\
&[L_n, L_m] = (n-m)\, L_{n+m} + \frac{c}{12}\left(n^3 - n\right)\delta_{n,-m}\,. \qquad (5..72)
\end{aligned}$$

Applying the KMS condition and using the first equation (5..72) and $\left\langle J_0 \right\rangle_q = 0$, we find the following non–zero Gibbs average of products of $G^{\pm}$–modes

$$\begin{aligned}
&\left\langle G^{+}_{n-\frac{1}{2}} G^{-}_{\frac{1}{2}-n} \right\rangle_q = \frac{q^{n-\frac{1}{2}}}{1+q^{n-\frac{1}{2}}} \left\{ \frac{c}{3}\left(n-\frac{1}{2}\right)^2 + \left\langle 2\widetilde{L}_0 \right\rangle_q \right\},\\
&\widetilde{L}_0 = L_0 - \frac{c}{24}\,.
\end{aligned} \qquad (5..73)$$

This gives the following q–expansion of the 2–point thermal correlation function

$$\begin{aligned}
&\left\langle G^{+}(\zeta_1)\, G^{-}(\zeta_2) \right\rangle_q =\\
&= \frac{2c}{3}\left\{ \frac{3+\cos 2\pi\zeta_{12}}{4\,(2i\sin\pi\zeta_{12})^3} + 2i \sum_{n=1}^{\infty} \frac{\left(n-\frac{1}{2}\right)^2 q^{n-\frac{1}{2}}}{1+q^{n-\frac{1}{2}}}\, \sin(2n-1)\,\pi\zeta_{12} \right\}\\
&\quad + \left\langle \widetilde{L}_0 \right\rangle_q \left\{ \frac{1}{i\sin\pi\zeta_{12}} + 4i \sum_{n=1}^{\infty} \frac{q^{n-\frac{1}{2}}}{1+q^{n-\frac{1}{2}}}\, \sin(2n-1)\,\pi\zeta_{12} \right\}. \qquad (5..74)
\end{aligned}$$

It can be expressed (as in the example of the Weyl field) in terms of $p^{11}_k(\zeta_{12}, \tau)$ (2..15):

$$\left\langle G^{+}(\zeta_1)\, G^{-}(\zeta_2) \right\rangle_q = \frac{ic}{12\pi^3}\, p^{11}_3(\zeta_{12},\, \tau) - \frac{i}{\pi} \left\langle \widetilde{L}_0 \right\rangle_q\, p^{11}_1(\zeta_{12},\, \tau)\,. \qquad (5..75)$$

In this case the Laurent expansion of type (5..39) takes the form:

$$\left\langle G^{+}(\zeta_1) G^{-}(\zeta_2) \right\rangle_q = \frac{ic}{12\pi^3}\, \zeta_{12}^{-3} - \frac{i}{\pi} \left\langle \widetilde{L}_0 \right\rangle_q \zeta_{12}^{-1} + \dots\,. \qquad (5..76)$$

As we are no longer dealing with a free field theory the energy mean $\left\langle \widetilde{L}_0 \right\rangle_q$ is not determined from the thermal 2–point function of $G^{\pm}$. It can be computed, however, using our knowledge of the representation theory of $SV\,(2)$ (see [8]).

The Neveu–Schwarz sector of positive energy unitary irreducible representations (UIR) of $SV\,(2)$ are described as follows. For each of the discrete set of values of the central charge,

$$c \,=\, c_k \,=\, 3 - \frac{6}{k+2}\,, \quad k \,=\, 1,\,2,\,\ldots, \tag{5..77}$$

there are $\binom{k+2}{k}$ UIR $(k;l,m)$ with representation spaces

$$\begin{aligned} &\mathcal{H}_{lm}(=\mathcal{H}_{lm}^{(k)}), \quad l=0,1,\ldots,k, \quad \tfrac{1}{2}\,(l-m)=0,1,\ldots,l \\ &(m=-l,-l+2,\ldots,l)\,. \end{aligned} \tag{5..78}$$

They are characterized by a *charge* e_m and a *lowest weight* Δ_{lm} given by

$$\begin{aligned} &e_m \,=\, \frac{m}{k+2}\,, \quad \Delta_{lm} \,=\, \frac{l\,(l+2)-m^2}{4\,(k+2)}\,, \quad \text{so that} \\ &\left[e^{2\pi i(J_0-e_m)}-1\right]\mathcal{H}_{lm} \,=\, 0 \,=\, \left[e^{2\pi i(L_0-\Delta_{lm})}-1\right]\mathcal{H}_{lm}\,. \end{aligned} \tag{5..79}$$

Let $\chi_{lm}\,(\tau,\,k)$ be the (restricted) *character* of the UIR $(k;l,m)$:

$$\begin{aligned} &\chi_{lm}\,(\tau,\,\mu;\,k) \,=\, tr_{\mathcal{H}_{lm}}\left(q^{\widetilde{L}_0}q_\mu^{J_0}\right) \quad (q_\mu = e^{2\pi i\mu})\,, \\ &\chi_{lm}\,(\tau,\,k) \,=\, \chi_{lm}\,(\tau,\,0;\,k)\;. \end{aligned} \tag{5..80}$$

Proposition 5..1. *The character (5..80) span a $\binom{k+2}{k}$ dimensional representation of the modular group $\Gamma_\theta(\subset \Gamma\,(1))$ (2..59). They are, in particular, eigenvectors of T^2,*

$$T^2\,\chi_{lm}\,(\tau,\,k) \,:=\, \chi_{lm}\,(\tau+2,\,k) \,=\, e^{2\pi i\,(\Delta_{lm}-c_k)}\chi_{lm}\,(\tau\,,k)\;. \tag{5..81}$$

Each "finite ray" $\left\{\eta\chi_{00}\,(\tau,\,k) : \eta \in \mathbb{C},\, \eta^{4(k+2)} = 1\right\}$ is left invariant by the finite index subgroup $\Gamma_\theta^{(k)}$ of Γ_θ such that

$$(\Gamma_\theta\,\supset)\,\Gamma_\theta^{(k)} \,:=\, \Gamma_0\,(2k+4) \cap \Gamma_\theta \,\supset\, \Gamma\,(2k+4) \tag{5..82}$$

where $\Gamma\,(N)$ and $\Gamma_0\,(N)$ are defined by (2..60) and (2..61), respectively. The group $\Gamma_\theta^{(k)}$ is generated by T^2, $ST^{2k+4}S$ and the central element S^2 of $\Gamma\,(1)$.

Sketch of the proof. We shall first prove that T^{2k+4} acts as a multiple of the unit operator in the (finite dimensional) space spanned by χ_{lm}:

$$T^{2(k+2)}\chi_{lm}(\tau,\,k) \;=\; e^{i\pi\left\{l(l+2)-m^2-\frac{k}{2}\right\}}\,\chi_{lm}(\tau,\,k)\;. \tag{5..83}$$

This follows from the explicit form of c_k (5..77) and Δ_{lm} (5..79), and from the observation that $l\,(l+2)-m^2$ is even in the range (5..78). It implies that each χ_{lm} is an eigenvector of $\Gamma_\theta^{(k)}$. One can prove using [25] that the characters (5..80) transform among themselves under the modular inversion according to the law $\chi_{lm}(-1/\tau,\,k) \;=\; \sum\limits_{l'm'} S_{lml'ml}\chi_{l'm'}(\tau,\,k)$ with

$$S_{lml'm'} \;=\; \frac{2}{k+2}\,\sin\left(\frac{\pi\,(l+1)\,(l'+1)}{k+2}\right)e^{i\pi\frac{mm'}{k+2}}\,. \tag{5..84}$$

□

In order to get a glimpse of the rich variety of physical models captured by the above series of representations of $SV(2)$ we shall briefly discuss the first two of them, corresponding to $k=1,\,2$.

The $k=1$ model is an example of an one dimensional lattice current algebra (with $Q=\mathbb{Z}\sqrt{3}$) considered in Sect. 5.2. It can be viewed as a local extension of the $U(1)$ current algebra – see [12]. There are just three Neveu–Schwarz representations in this case (corresponding to $l=m=0$ and to $l=1$, $m=\pm 1$) which can be labeled by a single quantum number m (giving the charge). Their conformal dimensions are proportional to the squares of the corresponding charges:

$$e_m \;=\; \frac{m}{3}\,,\quad m \;=\; 0,\,\pm 1\,,\quad \Delta_l \;=\; \frac{3}{2}e^2\,(=\;\frac{m^2}{6})\,. \tag{5..85}$$

The latter formula also applies to the basic local fermionic fields $G^\pm$ (with $e=\pm$, $\Delta=3/2$). (The factor 3 in the numerator of Δ_l is the reciprocal of the central term, $c/3=1/3$, in the current commutation relations (5..72).) The characters (5..80) can be computed explicitly in this case in terms of θ–like functions that is a special case of (5..67):

$$\begin{aligned}&\chi_{lm}(\tau,\,\mu;\,3) \;=\; K_m(\tau,\,\mu;\,3)\;,\\ &\eta(\tau)\,K_m(\tau,\,\mu;\,l) \;=\; \textstyle\sum_{n\in\mathbb{Z}}\,q^{\frac{l}{2}\left(n+\frac{m}{l}\right)^2}\,q_\mu^{\,n+\frac{m}{l}}\;.\end{aligned} \tag{5..86}$$

The modular S–matrix (5..84) (for $k=1$) is then recovered from the known transformation law (5..68) for K_m:

$$K_m\left(-\frac{1}{\tau},\,\frac{\mu}{\tau};\,l\right) \;=\; \frac{e^{\frac{2\pi i\mu^2}{l\tau}}}{\sqrt{l}}\,\sum_{m'=1}^{l}\,e^{-2\pi i\frac{mm'}{l}}\,K_{m'}(\tau,\,\mu;\,l)\;. \tag{5..87}$$

The maximal bosonic subalgebra $\mathcal{B}_2$ of the $N = 2$ vertex operator algebra is generated by a pair of opposite charged fields of charges ± 2 (and dimension $3/2 \cdot 2^2 = 6$). The representations of $SV_1\,(2)$ (where we denote by $SV_k\,(2)$ the Lie superalgebra with (anti)commutation relations (5..72), in which the central charge c is replaced by its numerical value c_k (5..77)) splits into two pieces with respect to $\mathcal{B}_2$ and so do the characters K_m (5..86):

$$K_m\,(\tau,\,\mu;\,3) \;=\; K_{2m}\,(\tau,\,2\mu;\,12) + K_{2m+6}\,(\tau,\,2\mu;\,12) \tag{5..88}$$

corresponding to minimal charge and conformal weight

$$\begin{aligned} e\,(2m) &= \frac{m}{3}\,, \qquad & e\,(2m+6) &= \delta^0_m - \frac{2m}{3}\,, \\ \Delta\,(2m) &= \frac{m^2}{6}\,, \qquad & \Delta\,(2m+6) &= \frac{1}{2} + \frac{m^2}{6}\,. \end{aligned} \tag{5..89}$$

For $k = 2$ the fields $G^{\pm}\,(\zeta)$ can be factorized into two commuting factors

$$G^{\pm}\,(\zeta) \;=\; J^{\pm}\,(\zeta)\,\psi\,(\zeta) \tag{5..90}$$

where $J^{\pm}\,(\zeta)$ are $su\,(2)$ currents of charge ± 1 and dimension 1 and $\psi\,(\zeta)$ is the Majorana–Weyl fermion of Remark 5.1. The Neveu–Schwarz representations of $SV_2\,(2)$ involve products of $\mathbb{Z}_2$ twisted representation of the $\widehat{su}_1\,(2)$ current algebra. Their characters can be written in the form:

$$\begin{aligned} \chi_{00}\,(\tau,\,\mu;\,2) \;&=\; \frac{q^{-\frac{1}{48}}}{2} \Big\{ K_0\,(\tau,\,\mu;\,2) \prod_{n=1}^{\infty} \Big(1 + q^{n-\frac{1}{2}}\Big) + \\ &\qquad + K_0\Big(\tau,\,\mu + \frac{1}{2};\,2\Big) \prod_{n=1}^{\infty} \Big(1 - q^{n-\frac{1}{2}}\Big)\Big\} \\ &=\; \frac{q^{-\frac{1}{48}}}{\eta\,(\tau)} \Big\{1 + \big(q_\mu + q_\mu^{-1}\big)\,q^{\frac{3}{2}} + q^2 + \dots\Big\} \end{aligned} \tag{5..91}$$

$$\chi_{1m}\,(\tau,\,\mu;\,2) \;=\; K_{\frac{m}{2}}\,(\tau,\,\mu;\,2)\,q^{-\frac{1}{24}} \prod_{n=1}^{\infty} (1 + q^n)\,, \quad m \;=\; \pm 1 \tag{5..92}$$

$$\Big(e_m \;=\; \frac{m}{4}\,, \quad \Delta_m \;=\; \Big(\frac{m}{4}\Big)^2 + \frac{1}{16} \;=\; \frac{1}{8}\Big)\,,$$

$$\begin{aligned} \chi_{20}\,(\tau,\,\mu;\,2) \;&=\; \frac{q^{-\frac{1}{48}}}{2} \Big\{ K_0\,(\tau,\,\mu;\,2) \prod_{n=1}^{\infty} \Big(1 + q^{n-\frac{1}{2}}\Big) \\ &\qquad - K_0\Big(\tau,\,\mu + \frac{1}{2};\,2\Big) \prod_{n=1}^{\infty} \Big(1 - q^{n-\frac{1}{2}}\Big)\Big\} \\ &=\; \frac{q^{-\frac{1}{48}}}{\eta\,(\tau)} \Big\{q^{\frac{1}{2}} + \big(q_\mu + q_\mu^{-1}\big) + \dots\Big\} \end{aligned} \tag{5..93}$$

$$\chi_{2,2m}(\tau,\,\mu;\,2) \;=\; K_m(\tau,\,\mu;\,2)\,q^{-\frac{1}{48}}\prod_{n=1}^{\infty}\left(1+q^{n-\frac{1}{2}}\right)\,,\quad m \;=\; \pm 1 \quad (5..94)$$

$$\left(e_{2m} \;=\; \frac{m}{2}\,,\quad \Delta_{2m} \;=\; \frac{m^2}{4} \;=\; \frac{1}{4}\right). \qquad (5..95)$$

We leave it to the reader to read off the operator content of the corresponding representations and derive (using (5..87) and (5..35)) Eq. (5..84) for $k = 2$.

6. Free massless scalar field for even D. Weyl and Maxwell fields for $D = 4$

6.1. Free scalar field in $D = 2d_0 + 2$ dimensional space–time

Generalized free fields [63] in a QFT with a unique vacuum can be characterized by having correlation functions expressed as sums of products of 2-point ones. It is important for our purposes that this property remains true, as a corollary, for finite temperature expectation values. We shall accordingly only deal with 2-point functions and the related energy mean values in this section.

A canonical z-picture scalar field $\varphi(z)$ (of conformal dimension $d_0 = \frac{D-2}{2}$) satisfies the Laplace equation $\Delta_z\varphi(z) = 0$ ($\Delta_z = \partial_z^2$), which assumes, in the real compact picture, the form

$$\left(\Delta_u - \left(\frac{\partial}{2\pi\partial\zeta}\right)^2 - d_0^2\right)\varphi(\zeta,\,u;\,d_0) \;=\; 0\,,$$

$$\Delta_u \;=\; \partial_u^2 \;-\; (u\cdot\partial_u)\left((u\cdot\partial_u)+2d_0\right)\,,\quad \partial_u \;=\; \frac{\partial}{\partial u}\,. \qquad (6..1)$$

(Note that the compact picture parametrization $z = e^{2\pi i\zeta}u$ can be interpreted as spherical coordinates with a “logarithmic radius” ζ.)

Remark 6..1. The operator Δ_u in (6.1) is an interior differentiation on the $(D-1)$-sphere $u^2 = 1$ since $\Delta_u\left\{\left(u^2-1\right)f(u)\right\} = \left(u^2-1\right)(\Delta_u f$ $-4\,(u\cdot\partial_u)\,f) = 0$ for $u^2 = 1$.

The Fourier modes of $\varphi(\zeta, u)$ are eigenfunctions of Δ_u:

$$\varphi(\zeta,\,u) \;=\; \sum_{n\in\mathbb{Z}}\varphi_n(u)\,e^{-2\pi i\,n\,\zeta}\,,\quad \left(\Delta_u + n^2 - d_0^2\right)\varphi_n(u) \;=\; 0\,. \qquad (6..2)$$

Note that $\varphi_n(z)$ is a homogeneous function of z of degree $-d_0 - n$; for $n \geqslant d_0$ $\varphi_{-n}(z)$ is a homogeneous harmonic polynomial of degree $n - d_0$. Moreover, we have $\varphi_n = 0$ for $|n| < d_0$. The compact picture 2-point vacuum expectation value,

$$\langle 0|\,\varphi(\zeta_1,\,u_1)\,\varphi(\zeta_2,\,u_2)\,|0\rangle \;=\; e^{-2\pi i\,d_0\,\zeta_{12}} \qquad (6..3)$$
$$\times\;\left(1 - 2\cos 2\pi\alpha\,e^{-2\pi i\,\zeta_{12}} + e^{-4\pi i\,\zeta_{12}}\right)^{-d_0}$$

($\cos 2\pi\alpha := u_1 \cdot u_2$) is proportional to the generating function for the Gegenbauer polynomials in $x = \cos 2\pi\alpha$,

$$\begin{gathered}\left(1 - 2\,x\,t + t^2\right)^{-\lambda} \;=\; \textstyle\sum_{n=0}^{\infty}\, t^n\, C_n^{\lambda}\,(x) \\ \Big(C_n^{\lambda}\,(x) = \sum_{k=0}^{\left[\frac{n}{2}\right]} (-1)^k\, \frac{(\lambda)_{n-k}}{k!}\,\frac{(2x)^{n-2k}}{(n-2k)!}\,\Big)\end{gathered} \tag{6..4}$$

that satisfy the differential equation

$$\left[\left(1 - x^2\right)\frac{d^2}{dx^2} - (2\lambda + 1)\,x\frac{d}{dx} + n\,(n + 2\lambda)\right] C_n^{\lambda}\,(x) \;=\; 0 \tag{6..5}$$

and the orthogonality and normalization conditions

$$\begin{gathered}\frac{2^{2\lambda-1}}{\pi}\int_{-1}^{1} C_n^{\lambda}\,(x)\,C_m^{\lambda}\,(x)\left(1 - x^2\right)^{\lambda-\frac{1}{2}}\,dx \;=\; \frac{\Gamma\,(2\lambda + n)\,\delta_{mn}}{\Gamma^2\,(\lambda)\,n!\,(n + \lambda)}\,, \\ C_n^{\lambda}\,(1) \;=\; \tbinom{2\lambda+n-1}{n}\,.\end{gathered} \tag{6..6}$$

It follows that $\varphi_n(u)$ obey the commutation relations

$$\left[\varphi_n\,(u_1)\,,\,\varphi_m\,(u_2)\right] \;=\; \frac{n}{|n|}\delta_{n,-m}\,C_{|n|-d_0}^{d_0}\,(\cos 2\pi\alpha) \quad \text{for} \quad |n| \geqslant d_0\,. \tag{6..7}$$

For $D = 4, d_0 = 1$ (the case studied in [64]) these *canonical commutation relations* assume an elementary explicit form:

$$[\varphi_n(u_1)\,,\varphi_m(u_2)] = \frac{\sin 2\pi n\alpha}{\sin 2\pi\alpha}\,\delta_{n,-m} \quad (D = 4,\;\, d_0 = 1,\;\, \cos 2\pi\alpha = u_1 \cdot u_2). \tag{6..8}$$

The calculations (4..82)–(4..84) in this case give

$$\begin{aligned} q^n \left\langle\, \varphi_n\,(u_1)\,\varphi_{-n}\,(u_2)\,\right\rangle_q &= \left\langle\, \varphi_{-n}\,(u_2)\,\varphi_n\,(u_1)\,\right\rangle_q = \\ &= \left\langle\, \varphi_n(u_1)\,\varphi_{-n}(u_2)\,\right\rangle_q - C_{n-d_0}^{d_0}(u_1 \cdot u_2)\end{aligned}$$

and hence

$$\left\langle \varphi_n(u_1)\,\varphi_{-n}(u_2)\right\rangle_q = \frac{C_{n-d_0}^{d_0}(u_1 \cdot u_2)}{1 - q^n} = q^{-n}\left\langle \varphi_{-n}(u_2)\,\varphi_n(u_1)\right\rangle_q \tag{6..9}$$

for $n \geqslant d_0$. Inserting this in the Fourier expansion of the 2-point function,

$$\left\langle\, \varphi\,(\zeta_1,\,u_1)\,\varphi\,(\zeta_2,\,u_2)\,\right\rangle_q \;=\; \sum_{|n|\,\geqslant\,d_0} \left\langle\, \varphi_n\,(u_1)\,\varphi_{-n}\,(u_2)\,\right\rangle_q\, e^{-2\pi i\,n\,\zeta_{12}}\,. \tag{6..10}$$

we find

$$\begin{aligned}\left\langle \varphi(\zeta_1,\, u_1)\, \varphi(\zeta_2,\, u_2) \right\rangle_q \;&=\; (-4 \sin \pi\zeta_+ \sin \pi\zeta_-)^{-d_0} \\ &\quad +\; 2 \sum_{n\,=\,d_0}^{\infty} \frac{q^n}{1-q^n} \cos 2\pi n \zeta_{12}\; C^{d_0}_{n-d_0}(\cos 2\pi\alpha) \\ &=\; \frac{1}{(4\pi)^{2d_0}} P_{d_0}(\zeta_{12}; u_1, u_2; \tau) \qquad (6..11)\end{aligned}$$

where according to Eq. (4..81) P_{d_0} is the basic elliptic function (4..85). We thus have found in particular, the q–expansion of the functions P_k since the above arguments are valid for any field dimension d.

In the case $D = 4$ ($d_0 = 1$), using the canonical normalization (4..39) we deduce

$$\begin{aligned}\langle 0|\, \varphi(\zeta_1, u_1)\varphi(\zeta_2, u_2)\, |0\rangle \;&=\; -\frac{1}{8\pi \sin \pi\zeta_+ \sin \pi\zeta_-} \\ &=\; \frac{1}{4\pi \sin 2\pi\alpha} \left(\mathrm{cotg}\pi\zeta_+ - \mathrm{cotg}\pi\zeta_-\right) \\ &=\; \frac{1}{4\pi \sin 2\pi\alpha} \sum_{n\,\in\,\mathbb{Z}} \left(\frac{1}{\zeta_+ + n} - \frac{1}{\zeta_- + n} \right) \qquad (6..12)\end{aligned}$$

and the passage to the thermal 2–point function consists of replacing the sum with the doubly periodic Eisenstein-Weierstrass series (for $D = 4$)

$$\left\langle \varphi(\zeta_1,\, u_1)\, \varphi(\zeta_2,\, u_2) \right\rangle_q \;=\; \frac{1}{4\pi \sin 2\pi\alpha} \left(p_1(\zeta_+,\, \tau) - p_1(\zeta_-,\, \tau)\right) \qquad (6..13)$$

(which corresponds to Eq. (2..52)). Similarly, for $D = 6$ ($d_0 = 2$) we have

$$\begin{aligned}\langle 0|\, \varphi(\zeta_1,\, u_1)\, \varphi(\zeta_2,\, u_2)\, |0\rangle \;&=\; (2 \sin \pi\zeta_+ \sin \pi\zeta_-)^{-2} \\ &=\; \frac{1}{16 \sin^2 2\pi\alpha} \left\{ (\sin \pi\zeta_-)^{-2} + (\sin \pi\zeta_+)^{-2} \right. \\ &\qquad \left. +\; 2\mathrm{cotg}\, 2\pi\alpha \left(\mathrm{cotg}\, \pi\zeta_- - \mathrm{cotg}\, \pi\zeta_+ \right) \right\}\,, \\ \left\langle \varphi(\zeta_1,\, u_1)\, \varphi(\zeta_2,\, u_2) \right\rangle_q \;&=\; (4\pi \sin 2\pi\alpha)^{-2} \left\{ p_2(\zeta_-,\, \tau) + p_2(\zeta_+,\, \tau) \right. \\ &\qquad \left. +\, 2\pi\mathrm{cotg}\, 2\pi\alpha \left(p_1(\zeta_-,\, \tau) - p_1(\zeta_+,\, \tau) \right) \right\}\,. \qquad (6..14)\end{aligned}$$

To compute the mean thermal energy by Eq. (4..71) we have to find the dimensions $d_b(n)$ of the space $Span_{\mathbb{C}} \left\{ \varphi_{-n}(u)|0\rangle \right\}$ of the energy n one particle states ($d_f(n) = 0$ in this case). According to the properties of the mode expansion of $\varphi(\zeta, u)$ every such space is isomorphic to the space of homogeneous harmonic polynomials in u of degree $n - d_0$ for $n = d_0, d_0 + 1, \ldots$. Recalling that the generating function for the dimensions of the spaces is

$\frac{1-t^2}{(1-t)^D}$ $(= \sum_{n=d_0}^{\infty} d_b(n)\ t^{n-d_0}$, $D = 2d_0 + 2)$ we find $d_b(n) = n^2$ for $D = 4$ and for even $D > 4$:

$$d_b(n) \;=\; \frac{2n^2}{(2d_0)!} \prod_{k=1}^{d_0-1} \left(n^2 - k^2\right) \;=\; \sum_{k=0}^{d_0} c_k^{(D)} n^{2k} \,. \tag{6..15}$$

Thus, using the q–expansions (3..13) we find

$$\begin{aligned} \left\langle H \right\rangle_q \left(\equiv \frac{tr_{\mathcal{V}}\left(H\, q^H\right)}{tr_{\mathcal{V}}\left(q^H\right)} \right) &= \sum_{n=1}^{\infty} \frac{n\, d\,(n)\; q^n}{1-q^n} \\ &= \sum_{k=1}^{d_0+1} c_{k-1}^{(D)} \frac{B_{2k}}{4k} + \sum_{k=1}^{d_0+1} c_{k-1}^{(D)} G_{2k}\,(\tau) \end{aligned} \tag{6..16}$$

for even $D > 4$ (B_{2k} being the Bernoulli numbers). In particular, for the physical 4-dimensional case we have:

$$\left\langle H + E_0 \right\rangle_q \;=\; G_4(\tau) \quad \text{for} \quad E_0 \;=\; \frac{1}{240}\,. \tag{6..17}$$

Remark 6..2. All thermal correlation functions have well defined restrictions to coinciding u_a on the $D-1$ sphere. This is seen directly from our formulae and is a consequence of a general observation by Borchers [7]. The result is an expression for the corresponding correlation functions in a chiral CFT. For instance the chiral (2D) restriction of the 2-point function (6..13),

$$\begin{aligned} \mathcal{W}(\zeta_{12},\tau) \;&=\; -\,(2\pi)^{-2}\, p_2(\zeta_{12},\tau) \\ &=\; -\,(2\sin\pi\zeta_{12})^{-2} + 2\sum_{n=1}^{\infty} \frac{n\, q^n}{1-q^n} \cos 2\pi n\,\zeta_{12} \end{aligned} \tag{6..18}$$

coincides with the thermal 2-point function of a chiral $U(1)$ current. The importance of this remark stems from the fact that it is easier to verify, say, Wightman positivity in the 1-dimensional (chiral) case, thus obtaining a necessary condition for the existence of a consistent higher dimensional theory.

6.2. Weyl fields

We begin by introducing the 2×2 matrix representation of the quaternionic algebra (see also Appendix C) which will prove useful for studying both spinor and antisymmetric tensor z-picture fields:

$$\begin{gathered} Q_k \;=\; -i\,\sigma_k \;=\; -Q_k^+ \quad (k \;=\; 1,\,2,\,3)\,, \quad Q_4 \;=\; \mathbb{I} \\ \sigma_1 \;=\; \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 \;=\; \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 \;=\; \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \end{gathered} \tag{6..19}$$

characterized by the anticommutation relations

$$Q_\alpha^+ Q_\beta + Q_\beta^+ Q_\alpha = 2\,\delta_{\alpha\beta} = Q_\alpha Q_\beta^+ + Q_\beta Q_\alpha^+ \quad \text{for} \quad \alpha,\, \beta = 1,\, \dots,\, 4\,. \tag{6..20}$$

Here and below we denote (as in Sect. 4.4) the hermitian matrix conjugation by a superscript "+". The matrices

$$i\,\sigma_{\alpha\beta} \;=\; \frac{1}{2}\left(Q_\alpha^+\, Q_\beta - Q_\beta^+\, Q_\alpha\right)\,, \quad i\,\widetilde{\sigma}_{\alpha\beta} \;=\; \frac{1}{2}\left(Q_\alpha\, Q_\beta^+ - Q_\beta\, Q_\alpha^+\right) \tag{6..21}$$

are the selfdual and antiselfdual antihermitian *spin*(4) Lie algebra generators. We shall also use the notation

$$\begin{aligned} \not{z} &= \sum_{\alpha=1}^{4} z^\alpha\, Q_\alpha, & \not{z}^+ &= \sum_{\alpha=1}^{4} z^\alpha\, Q_\alpha^+, \\ \not{\partial}_z &= \sum_{\alpha=1}^{4} Q_\alpha\, \partial_{z^\alpha}, & \not{\partial}_z^+ &= \sum_{\alpha=1}^{4} Q_\alpha^+\, \partial_{z^\alpha}\,, \end{aligned} \tag{6..22}$$

Note that in the definition of $\not{z}^+$ we do not conjugate the coordinates z^α. Then Eqs. (6..20) are equivalent to

$$\not{z}_1^+\, \not{z}_2 + \not{z}_2^+\, \not{z}_1 \;=\; \not{z}_1\, \not{z}_2^+ + \not{z}_2\, \not{z}_1^+ \;=\; 2\, z_1 \cdot z_2 \quad \left(\not{z}^+\, \not{z} \;=\; \not{z}\, \not{z}^+ \;=\; z^{\,2}\right). \tag{6..23}$$

The Weyl generalized free fields of dimension $d = 1/2,\, 3/2,\, \dots$ are two mutually conjugate complex 2–component fields,

$$\chi\,(z)^+ \;=\; \left(\chi_1^*\,(z)\,,\, \chi_2^*\,(z)\right) \quad \text{and} \quad \chi\,(z) \;=\; \begin{pmatrix} \chi_1\,(z) \\ \chi_2\,(z) \end{pmatrix}\,, \tag{6..24}$$

transforming under the elementary induced representations of *spin* (4) corresponding to the selfdual and antiselfdual representations σ and $\widetilde{\sigma}$ (6..21), respectively. In particular, the action of the Weyl reflection j_W is,

$$\begin{aligned} \chi\,(z) \;&\longmapsto\; \frac{\not{z}}{\left(z^{\,2}\right)^{d+\frac{1}{2}}}\; \chi\,(z)\ \left(\equiv\; \pi(z,R)\,\chi\,(z)\,\right), \\ \chi^+\,(z) \;&\longmapsto\; \chi^+\,(z)\; \frac{\not{z}}{\left(z^{\,2}\right)^{d+\frac{1}{2}}}\ \left(\equiv\; \pi^+(z,R)\,\chi^+\,(z)\,\right). \end{aligned} \tag{6..25}$$

The conformal invariant 2–point functions, characterizing the fields, have the following matrix representation

$$\langle 0|\, \chi\,(z_1)\, \chi^+\,(z_2)\, |0\rangle \;=\; \frac{\not{z}_{12}^+}{\left(z_{12}^2\right)^{d+\frac{1}{2}}}\,, \tag{6..26}$$

$$\langle 0|\,\chi_\alpha\,(z_1)\,\chi_\beta\,(z_2)\,|0\rangle \;=\; \langle 0|\,\chi^+_\alpha\,(z_1)\,\chi^+_\beta\,(z_2)\,|0\rangle \;=\; 0\,.$$

In particular, the invariance under the reflection $I(z)$ (4..56) is ensured by the equality

$$\frac{\not{z}_1}{z_1^2}\,\not{z}^+_{12}\,\frac{\not{z}_2}{z_2^2} \;=\; \frac{\not{z}^+_1}{z_1^2} - \frac{\not{z}^+_2}{z_2^2}\,. \tag{6..27}$$

The conjugation law reads

$$\chi^+\,(\overline{z})^+ \;=\; \frac{\not{z}}{\left(z^{\,2}\right)^{d+\frac{1}{2}}}\;\chi\left(\frac{z}{z^{\,2}}\right)\,. \tag{6..28}$$

The canonical, $d = 3/2$, z-picture Weyl field ψ and its subcanonical counterpart χ satisfy a first and a third order partial differential equation, respectively:

$$\not{\partial}_z\,\psi\,(z) \;=\; 0\,, \qquad\qquad \Delta_z\,\not{\partial}_z\,\chi\,(z) \;=\; 0\,. \tag{6..29}$$

Their vacuum correlation functions are diagonal in "the moving frame" representation defined as follows. For given non-collinear unit real vectors $u_1,\,u_2 \in \mathbb{S}^{D-1}(\subset \mathbb{R}^D)$ such that $u_1\cdot u_2 = \cos 2\pi\alpha$ let v and $\overline{v}$ be the unique complex vectors (in $\mathbb{C}^D$) for which

$$u_1 \;=\; e^{\pi\,i\,\alpha}\,v + e^{-\pi\,i\,\alpha}\,\overline{v}\,, \qquad u_2 \;=\; e^{-\pi\,i\,\alpha}\,v + e^{\pi\,i\,\alpha}\,\overline{v}\,. \tag{6..30}$$

Their compact picture 2-point functions have the form:

$$\langle 0|\,\chi\,(\zeta_1,\,u_1)\,\chi^+\,(\zeta_2,\,u_2)\,|0\rangle \;=\; \frac{1}{2i}\left(\frac{\not{v}^+}{\sin\pi\zeta_-} + \frac{\overline{\not{v}}^+}{\sin\pi\zeta_+}\right)\,, \tag{6..31}$$

$$\begin{aligned}\langle 0|\,\psi\,(\zeta_1,\,u_1)\,\psi^+\,(\zeta_2,\,u_2)\,|0\rangle &= \frac{1}{2i\sin\pi\zeta_-\,\sin\pi\zeta_+}\left(\frac{\not{v}^+}{\sin\pi\zeta_-} + \frac{\overline{\not{v}}^+}{\sin\pi\zeta_+}\right)\\ = \frac{i}{8\,\sin 2\pi\alpha}&\left(\not{v}^+\left(\frac{\cos\pi\zeta_-}{\sin^2\pi\zeta_-} - \frac{\operatorname{cotg} 2\pi\alpha}{\sin\pi\zeta_-} + \frac{1}{\sin 2\pi\alpha\,\sin\pi\zeta_+}\right)\right.\\ &\left.-\,\overline{\not{v}}^+\left(\frac{\cos\pi\zeta_+}{\sin^2\pi\zeta_+} + \frac{\operatorname{cotg} 2\pi\alpha}{\sin\pi\zeta_+} - \frac{1}{\sin 2\pi\alpha\,\sin\pi\zeta_-}\right)\right)\,,\end{aligned} \tag{6..32}$$

where $\zeta_\pm = \zeta_{12}\pm\alpha$ (as in previous sections). Let N be the "charge operator" defined by $[N,\chi^+\,(z)] = \chi^+\,(z)$, $[N,\,\chi\,(z)] = -\chi\,(z)$ (and similarly for ψ). We introduce the *grand canonical mean value*

$$\left\langle\, A\,\right\rangle_{q,\,\mu} \;:=\; \frac{tr_{\mathcal{H}}\left(A\,q^H\,e^{2\pi i\,\mu\,N}\right)}{tr_{\mathcal{H}}\left(q^H\,e^{2\pi\,i\,\mu\,N}\right)}\,. \tag{6..33}$$

Then the grand canonical 2-point correlation functions assume the form:

$$\left\langle\chi(\zeta_1,u_1)\,\chi^+(\zeta_2,u_2)\right\rangle_{q,\mu} = \frac{1}{2\pi\,i}\left(p_1^{11}(\zeta_-,\tau,\mu)\,\not{v}^+ + p_1^{11}(\zeta_+,\tau,\mu)\,\overline{\not{v}}^+\right) \tag{6..34}$$

$$\left\langle \psi(\zeta_1,\, u_1)\, \psi^+(\zeta_2,\, u_2) \right\rangle_{q,\mu} = \frac{i}{8\pi \sin 2\pi\alpha} \Big(\not{\psi}^+ \Big(p_2^{11}(\zeta_-, \tau, \mu) - \operatorname{cotg} 2\pi\alpha\; p_1^{11}(\zeta_-, \tau, \mu) + \frac{p_1^{11}(\zeta_+, \tau, \mu)}{\sin 2\pi\alpha} \Big) - \overline{\not{\psi}}^+ \Big(p_2^{11}(\zeta_+, \tau, \mu) + \operatorname{cotg} 2\pi\alpha\; p_1^{11}(\zeta_+, \tau, \mu) - \frac{p_1^{11}(\zeta_-, \tau, \mu)}{\sin 2\pi\alpha} \Big) \Big)\,, \tag{6..35}$$

where

$$p_k^{\kappa\lambda}(\zeta, \tau, \mu) = \lim_{M\to\infty} \lim_{N\to\infty} \sum_{m=-M}^{M} \sum_{n=-N}^{N} \frac{e^{\pi i m (2\mu+\kappa)}\, e^{\pi i n \lambda}}{(\zeta + m\tau + n)^k}\,, \quad k = 1, 2, \dots \tag{6..36}$$

(for the properties of these extended p-functions, see Appendix A).

We observe that the $\alpha \to 0$ ($u_1 = u_2$) limit of (6..34) reproduces the Gibbs 2-point function (5..29) of the chiral Weyl field for $D = 2$ (as $\not{\psi}^+ + \overline{\not{\psi}}^+ = \not{u}^+$ is equal to the unit matrix in the frame $u = (\boldsymbol{0}, 1)$. It is noteworthy that it is modular invariant for $\mu = 0, 1/2$; for instance, $p_1^{11}(\zeta, \tau, 0) = p_1^{11}(\zeta, \tau)$ obeys (2..49) with $k = 1$ and $\gamma \in \Gamma_\theta$ (see (2..50) and Exercise 2.14). By contrast, the $d = 3/2$ restricted 2-point function

$$\left\langle \psi(\zeta_1, u)\psi^+(\zeta_2, u) \right\rangle_q = \frac{i}{(2\pi)^3} \left(p_3^{11}(\zeta_{12}, \tau) + \frac{\pi^2}{2}\, p_1^{11}(\zeta_{12}, \tau) \right) \tag{6..37}$$

is a linear superposition of modular functions of weight 3 and 1, and hence, is not modular invariant.

Remark 6..3. Comparing with the $N = 2$ superconformal model of Sect. 5.3 we observe that one would have had the same problem had one assumed that the pair of $d = 3/2$ fields $G^\pm$ satisfy c-number anticommutation relations (instead of (5..72)). In $D = 4$, however, one is bound to consider the $d = 3/2$ field ψ as a free field unless one is prepared to give up Wightman positivity. Indeed, the unique conformally invariant 2-point function (6..26) for $d = 3/2$ implies (in a positive metric Hilbert space framework) the free field equation (6..29) for ψ. It appears intriguing to try to construct an indefinite metric model for a $d = 3/2$ Weyl field with a modular invariant 2-point function.

We conclude this subsection with a consideration of the Gibbs energy distributions for Weyl fields.

The temperature 2-point function (6..37) can be used to derive the mean-value of the conformal Hamiltonian in the equilibrium (Gibbs) state at hand. Denote by $\mathfrak{w}(\zeta_{12}, \tau)$ the function $\mathcal{W}_{3/2}$ (6..37), regularized by subtracting its (third and first order) poles at $\zeta_{12} = 0$. Then the canonical expression for the conformal Hamiltonian of a free Weyl field tells us that its Gibbs energy distribution is given by

$$\left\langle H \right\rangle_q = \frac{1}{2\pi i} \lim_{\zeta\to 0} \frac{\partial}{\partial \zeta}\, \mathfrak{w}(\zeta, \tau)\,. \tag{6..38}$$

We shall verify this formula by a direct computation that will also apply to deriving the energy distribution of the subcanonical field χ.

The positive charge 1–particle state–space is a direct sum of energy eigenspaces corresponding to eigenvalues $E_0+n+3/2$, $n=0,1,\ldots$, where E_0 is the vacuum energy. Each such eigenspace is spanned by vectors of the form $\psi^{+}_{-n-3/2}(u)|0\rangle$ and carry the irreducible representation $((n{+}1)/2,\, n/2)$ of $Spin(4)$ of dimension $(n+2)(n+1)$. The dimension of the full 1–particle space, including charge -1 states, is twice as big. It follows that

$$\langle H\rangle_q \,=\, E_0 + \sum_{n=0}^{\infty} \frac{2\left(n+\frac{3}{2}\right)(n+1)\,(n+2)\,q^{n+\frac{3}{2}}}{1+q^{n+\frac{3}{2}}} \tag{6..39}$$

which is verified to coincide with (6..38) (for a suitable choice of E_0). To express this Gibbs average in terms of modular forms we use the identity $n(n+1)\left(n+\frac{1}{2}\right)=\frac{1}{8}\left((2n+1)^3-(2n+1)\right)$ with the result

$$\langle H\rangle_q \,=\, \frac{1}{4}\left(G_4\Big(\frac{\tau+1}{2}\Big)-8\,G_4(\tau)\right) \,-\, \frac{1}{4}\left(G_2\Big(\frac{\tau+1}{2}\Big)-2\,G_2(\tau)\right)\,, \tag{6..40}$$

where we have set

$$E_0\,(=\,\langle 0|\,H\,|0\rangle\,) \,=\, \frac{7}{4}\,\frac{B_4}{8}-\frac{1}{4}\,\frac{B_2}{4} \,=\, -\frac{17}{960}. \tag{6..41}$$

We now proceed to derive the Gibbs distribution of the conformal Hamiltonian $H_{1/2}$ of the subcanonical $d=1/2$ Weyl field. Using the implication of the third order equation (6..29) on the modes of χ we deduce that the energy $n+1/2$ eigenspace of positive charge is isomorphic in this case to the (pseudoorthogonal) direct sum of three irreducible $Spin(4)$ representations,

$$\left(\frac{n{+}1}{2},\,\frac{n}{2}\right)\oplus\left(\frac{n{-}1}{2},\,\frac{n}{2}\right)\oplus\left(\frac{n{-}1}{2},\,\frac{n{-}2}{2}\right) \tag{6..42}$$

of total dimension

$$\begin{aligned} d_{\frac{1}{2}}(n) \,&=\, (n+2)(n+1)+n(n+1)+n(n-1)=3n(n+1)+2 \\ &=\, \frac{3(2n+1)^2+5}{4}\,. \end{aligned} \tag{6..43}$$

It follows that the Gibbs energy average is given in this case by

$$\begin{aligned} \left\langle\, H_{\frac{1}{2}}\,\right\rangle_q \,&=\, \langle 0|\,H_{\frac{1}{2}}\,|0\rangle + \sum_{n>0} \frac{2\left(n+\frac{1}{2}\right)d_{\frac{1}{2}}(n)\,q^{n+\frac{1}{2}}}{1+q^{n+\frac{1}{2}}} \,= \\ &=\, \frac{3}{4}\left(G_4\Big(\frac{\tau+1}{2}\Big)-8\,G_4(\tau)\right) \,+\, \frac{5}{4}\left(G_2\Big(\frac{\tau+1}{2}\Big)-2\,G_2(\tau)\right) \end{aligned} \tag{6..44}$$

for

$$\langle 0|\,H\,|0\rangle \,=\, -\frac{3}{4}\,\frac{B_4}{8}\left(1-2^3\right)-\frac{5}{4}\,\frac{B_2}{4}\,(1-2) \,=\, \frac{29}{960}\,. \tag{6..45}$$

6.3. The free Maxwell field

It is convenient to write the Maxwell field as a 2-form,

$$F\,(z) \;=\; \frac{1}{2}\,F_{\alpha\beta}(z)\;dz^{\alpha}\wedge dz^{\beta}\,, \tag{6..46}$$

which makes clear not just its transformation properties but also its conjugation law:

$$\Big(F_{\alpha\beta}(z)\;dz^{\alpha}\wedge dz^{\beta}\Big)^{*} = F_{\alpha\beta}(z^{*})\;d\overline{z}^{\alpha}\wedge d\overline{z}^{\beta}\quad \Big(z^{*}=\frac{\overline{z}}{\overline{z}^{\,2}},\;\;(dz^{\alpha})^{*}=d\,(z^{*})^{\alpha}\Big). \tag{6..47}$$

The free field is characterized by its 2-point function

$$\langle 0|\;F_{\alpha_1\beta_1}(z_1)\,F_{\alpha_2\beta_2}(z_2)\,|0\rangle \;:=\; \frac{r_{\alpha_1\alpha_2}(z_{12})\,r_{\beta_1\beta_2}(z_{12})-r_{\alpha_1\beta_2}(z_{12})\,r_{\beta_1\alpha_2}(z_{12})}{\left(z_{12}^{2}\right)^{2}}\,, \tag{6..48}$$

$r_{\alpha\beta}\,(z) := \delta_{\alpha\beta}-2\,\dfrac{z_{\alpha}z_{\beta}}{z^{\,2}}$. It is verified to satisfy the *Maxwell equations*

$$dF\,(z)\;=\;0\,,\quad d\,{*}(F)\,(z)\;=\;0\,, \tag{6..49}$$

$*$ being *Hodge* conjugation $*(F)_{\alpha\beta}(z)\;:=\;\varepsilon_{\alpha\beta\rho\sigma}\,F^{\rho\sigma}(z)$.

To compute the (compact picture) finite temperature correlation functions $\big\langle\,F_{\alpha_1\beta_1}\,(\zeta_1,u_1)\;F_{\alpha_2\beta_2}\,(\zeta_2,u_2)\,\big\rangle_q$ we use again the diagonal frame in which, $2v=(0,\,0,\,-i,\,1)$, $u_{1,2}=(0,\;\,0,\,\pm\sin\pi\alpha,\;\cos\pi\alpha)$; then there exist linear combinations of the field components

$$\begin{array}{ll}\sqrt{2}\,F_1^{\pm}=F_{23}\pm F_{14}\,, & \sqrt{2}\,F_2^{\pm}=F_{31}\pm F_{24}\,,\\ \sqrt{2}\,F_3^{\pm}=F_{12}\pm F_{34}\,, & \sqrt{2}\,F_{\pm}^{\varepsilon}=F_1^{\varepsilon}\pm iF_2^{\varepsilon}\,,\end{array}\qquad (\varepsilon=\pm)\text{ such that} \tag{6..50}$$

$$\begin{aligned}
&\langle 0|\;F_{+}^{+}(\zeta_1,u_1)\,F_{-}^{-}(\zeta_2,u_2)\,|0\rangle \;=:\;\mathcal{W}_0\,(\zeta_{12},\alpha)=\\
&=\frac{1}{4\sin^{3}2\pi\alpha}\,(\operatorname{cotg}\pi\zeta_{-}\operatorname{cotg}\pi\zeta_{+})-\frac{1}{4\sin 2\pi\alpha}\left(\frac{\cos\pi\zeta_{+}}{\sin^{3}\pi\zeta_{+}}-\frac{\operatorname{cotg}2\pi\alpha}{\sin^{2}\pi\zeta_{+}}\right);\\
&\langle 0|\;F_{-}^{+}(\zeta_1,u_1)\,F_{+}^{-}(\zeta_2,u_2)\,|0\rangle \;=\;\mathcal{W}_0\,(\zeta_{12},-\alpha)\,;\\
&\langle 0|\;F_{3}^{+}(\zeta_1,u_1)\,F_{3}^{-}(\zeta_2,u_2)\,|0\rangle \;=\\
&=\frac{1}{4\sin^{2}2\pi\alpha}\left(\frac{1}{\sin^{2}\pi\zeta_{+}}+\frac{1}{\sin^{2}\pi\zeta_{-}}+2\operatorname{cotg}2\pi\alpha\,(\operatorname{cotg}\pi\zeta_{+}-\operatorname{cotg}\pi\zeta_{-})\right)
\end{aligned} \tag{6..51}$$

($\zeta_\pm = \zeta_{12} \pm \alpha$). The corresponding finite temperature correlation functions are:

$$\left\langle F_+^+(\zeta_1,u_1)\,F_-^-(\zeta_2,u_2)\right\rangle_q =: \mathcal{W}_q(\zeta_{12},\alpha) = \frac{1}{4\sin^3 2\pi\alpha}\left(p_1(\zeta_-,\tau) - p_1(\zeta_+,\tau)\right)$$
$$-\frac{1}{4\sin 2\pi\alpha}\left(\frac{1}{2\pi}p_3(\zeta_+,\tau) - \mathrm{cotg}\, 2\pi\alpha\, p_2(\zeta_+,\tau)\right);$$
$$\left\langle F_-^+(\zeta_1,u_1)\,F_+^-(\zeta_2,u_2)\right\rangle_q = \mathcal{W}_q(\zeta_{12},-\alpha)\,; \quad \left\langle F_3^+(\zeta_1,u_1)\,F_3^-(\zeta_2,u_2)\right\rangle_q =$$
$$= \frac{1}{4\sin^2 2\pi\alpha}\left(p_2(\zeta_+,\tau) + p_2(\zeta_-,\tau) + 2\mathrm{cotg}\, 2\pi\alpha\left(p_1(\zeta_+,\tau) - p_1(\zeta_-,\tau)\right)\right). \tag{6..52}$$

In order to find the temperature energy mean value for the Maxwell field we have to compute the dimension $d_F(n)$ of the 1–particle state space of conformal energy n, spanned by $F_{\alpha\beta;\,-n}(z)|0\rangle$ where the mode $F_{\alpha\beta;\,-n}(z)$ is a homogeneous (harmonic) polynomial of degree $n-2$, satisfying the Maxwell equations. To this end we display the $SO(4)$ representation content of the modes satisfying the Maxwell equations. Decomposing the antisymmetric tensor $F_{\alpha\beta}$ into selfdual and antiselfdual parts, $(1,0)\oplus(0,1)$, we see that the full space of homogeneous skewsymmetric–tensor valued polynomials in z of degree $n-2$ generically splits into a direct sum of three conjugate pairs of $SU(2)\times SU(2)$ representations; for instance,

$$(1,0)\otimes\left(\frac{n-2}{2},\,\frac{n-2}{2}\right) = \left(\frac{n}{2},\,\frac{n-2}{2}\right)\oplus\left(\frac{n-2}{2},\,\frac{n-2}{2}\right)\oplus\left(\frac{n-4}{2},\,\frac{n-2}{2}\right)$$

(for $n > 3$). Maxwell equations imply that only two of the resulting six representations, those with maximal weights, appear in the energy n 1–particle space: $\left(\frac{n}{2},\,\frac{n-2}{2}\right)\oplus\left(\frac{n-2}{2},\,\frac{n}{2}\right)$. Thus,

$$d_F(n) = 2\left(n^2-1\right) \tag{6..53}$$

and then find

$$\left\langle H_F\right\rangle_q = 2G_4(\tau) - 2G_2(\tau)\,, \quad \langle 0|\,H_F|0\rangle = -2\,\frac{B_4}{8} + 2\,\frac{B_2}{4} = \frac{11}{120}\,. \tag{6..54}$$

If A_α is the gauge potential, such that $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$, then we say that that $A_\alpha \mapsto A_\alpha + l_\alpha$ is a *conformal gauge transformation* if $l_\alpha(z)$ is purely *longitudinal* generalized free field,

$$\partial_\alpha\, l_\beta = \partial_\beta l_\alpha, \quad \text{so that} \quad \langle 0|\, l_\alpha(z_1) l_\beta(z_2)\,|0\rangle = C\,\frac{r_{\alpha\beta}}{z_{12}^2} \tag{6..55}$$

which, hence, satisfies the third order equation

$$\Delta\,\partial\cdot l = 0\,. \tag{6..56}$$

One can write $l_\alpha(z) = \partial_\alpha s(z)$ where s has a logarithmic 2–point function but there is no conformal scalar field whose gradient is $l_\alpha(z)$. The energy n pure gauge 1–particle state space is spanned by vectors of the form $l_{-n}^{\alpha_1\alpha_2\ldots\alpha_n} z_{\alpha_2} \ldots z_{\alpha_n} |0\rangle$ where $l_{-n}^{\alpha_1\ldots\alpha_n}$ is a symmetric rank n tensor (due to (6..56)). (We are using interchangeably – for writing convenience – upper and lower Euclidean indices.) Taking (6..56) into account one computes the dimension of this space to be

$$d_l(n) \,=\, \binom{n+3}{3} - \binom{n-1}{3} \,=\, (n+1)^2 + (n-1)^2 \,=\, 2(n^2+1)\,. \quad (6..57)$$

It follows that the total energy mean value is a modular form of weight four:

$$\left\langle\, H_F + H_l \,\right\rangle_q \,=\, 4\, G_4(\tau)\,. \quad (6..58)$$

7. The thermodynamic limit

7.1. Compactified Minkowski space as a "finite box" approximation

We shall now substitute z in Eq. (4..24) by z/R thus treating $\mathbb{S}^{D-1}$ and $\mathbb{S}^1$ in the definition of $\overline{M}$ as a sphere and a circle of radius $R\,(>0)$. Performing further the Minkowski space dilation $(2R)^{X_{-1D}} : x^\mu \mapsto x^\mu/2R$, $\mu = 0, \ldots, D-1$ (see Eq. (4..22)) on the (real) variable $(z =)\, x$ in (4..24) we find $z(x;R) = R\, z(x/2R)$ or

$$\boldsymbol{z}(x;R) = \frac{\boldsymbol{x}}{2\omega\left(\dfrac{x}{2R}\right)}\,, \qquad z_D(x;R) - R = \frac{ix^0 - \dfrac{x^2}{2R}}{2\omega\left(\dfrac{x}{2R}\right)}\,,$$
$$2\,\omega\left(\frac{x}{2R}\right) = 1 + \frac{x^2}{4R^2} - i\,\frac{x^0}{R}\,. \quad (7..1)$$

The stability subgroup of $z\,(x;R) \,=\, 0\,(\in T_+)$ in $\mathcal{C}$ is conjugate to the maximal compact subgroup $\mathcal{K} \subset \mathcal{C}$:

$$\mathcal{K}(2R) \,=\, (2R)^{X_{-1D}}\, \mathcal{K}\, (2R)^{-X_{-1D}}\,,$$
$$\mathcal{K} \,\equiv\, \mathcal{K}\big(1\big) \,\cong\, U\,(1) \times Spin\,(D)/_{\mathbb{Z}_2}\,. \quad (7..2)$$

In particular, the hermitian $U\,(1)$–generator $H\,(2R)$, which acts in the z–coordinates (7..1) as the Euler vector field $z \cdot (\partial/\partial z)$, is conjugate to $H \equiv H(1)$,

$$H\,(2R) = (2R)^{X_{-1D}}\, H\, (2R)^{-X_{-1D}}\,, \qquad H \,\equiv\, H\big(1\big)\,. \quad (7..3)$$

For large R and finite x the variables $(\boldsymbol{z}, z_D - R)$ approach the (Wick rotated) Minkowski space coordinates $\big(\boldsymbol{x}, ix^0\big)$. In particular, for $x^0 = 0 (= \zeta)$, the real $(D-1)$–sphere $z^2 = R^2$ can be viewed as a $SO(D)$–invariant "box" approaching for $R \to \infty$ the flat space $\mathbb{R}^{D-1}$. Thus the conformal compactification of Minkowski space also plays the role of a convenient tool for studying the thermodynamic limit of thermal expectation values. This interpretation is justified in view of the following:

Proposition 7..1. *The asymptotic behaviour of $z\,(x;R) - Re_D$ ($e_D = (\mathbf{0},1)$) and of the associated Hamiltonian for large R is:*

$$\boldsymbol{z}\,(x;R) = \boldsymbol{x} + O\,\Big(\frac{\|x\|^2}{R}\Big)\,, \quad z_D\,(x;R) - R = i\,x^0 + O\,\Big(\frac{\|x\|^2}{R}\Big)\,, \tag{7..4}$$

$$H_R := \frac{H\,(2R)}{R} = P_0 + \frac{1}{4R^2}\,K_0\ \Big(= P_0 + O\,\Big(\frac{1}{R}\Big)\ \in i\,c\,\Big)\,, \tag{7..5}$$

where $\|x\| := \sqrt{(x^0)^2 + |\boldsymbol{x}|^2}$ for $x = \big(x^0, \boldsymbol{x}\big) \in M$ and iP_0 is the real conformal algebra generator of the Minkowski time (x^0) translation (see Sect. 4.2.). The operator H_R is the physical conformal Hamiltonian (of dimension inverse length).

Proof. Eq. (7..4) is obtained by a straightforward computation. To derive Eq. (7..5) one should use (4..30) and the equations

$$\lambda^{X_{-1D}}\,P_0\,\lambda^{-X_{-1D}} = \lambda\,P_0\,, \qquad \lambda^{X_{-1D}}K_0\lambda^{-X_{-1D}} = \lambda^{-1}K_0\,;$$

hence,

$$H\,(2R) = (2R)^{X_{-1D}}\ H\ (2R)^{-X_{-1D}} = R\,P_0 + \frac{1}{4R}\,K_0\,.$$

□

Remark 7..1. The observation that the universal cover of $\overline{M}$, the Einstein universe $\widetilde{M} = \mathbb{R} \times \mathbb{S}^{D-1}$ (for $D = 4$), which admits a *globally causal structure*, is locally undistinguishable from M for large R has been emphasized over 30 year ago by Irving Segal (for a concise exposé and further references – see [61]). For a fixed choice, X_{-1D}, of the dilation generator in (7..2) he identifies the Minkowski energy P_0 with the scale covariant component of H_R. With this choice M is osculating $\overline{M}$ (and hence $\widetilde{M}$) at the north pole $(\boldsymbol{z}, z_D) = (\mathbf{0}, R)$ (respectively, $\zeta = 0$, $u = e_D$), identified with the origin $x = 0$ in M. (The vector fields associated with H_R and P_0 coincide at this point.)

Using the Lie algebra limit $\lim_{R\to\infty} H_R = P_0$ implied by (7..5), one can approximate the Minkowski energy operator P_0 for large R by the physical conformal Hamiltonian H_R. As we shall see below, the fact that in all considered free field models in dimension $D = 4$ the conformal mean energy is a linear combination of modular forms $G_{2k}(\tau)$ with highest weight $2k = 4$, has a remarkable corollary: the *density* $\mathcal{E}$ of the physical mean energy has a limit reproducing the *Stefan–Boltzmann* law

$$\mathcal{E}(\beta) := \lim_{R\to\infty} \frac{\langle H_R \rangle_{q_\beta}}{\mathcal{V}_R} = \frac{C}{\beta^4} \quad \text{for} \quad q_\beta := e^{-\beta} \tag{7..6}$$

where C is some constant, $\beta = 1/(kT)$ is the inverse absolute temperature T (multiplied by the Boltzmann constant k) and $\mathcal{V}_R := 2\pi^2 R^3$ is the volume of the 3–sphere of radius R at a fixed time (say $x^0 = 0 = \zeta$). We will calculate this limit for two cases: the model of a free scalar filed in $D = 4$ (see Sect. 6.1.) which we will further denote by φ and the Maxwell free field model introduced in Sect. 6.3..

Proposition 7..2. *For the free scalar field φ in dimension $D = 4$ we have the following behaviour of the mean energy density for $\dfrac{R}{\beta} \gg 1$*

$$\begin{aligned} \mathcal{E}_R^{(\varphi)}(\beta) &:= \frac{1}{\mathcal{V}_R} \frac{tr_{\mathcal{V}}\, H_R\, e^{-\beta H_R}}{tr_{\mathcal{V}}\, e^{-\beta H_R}} \\ &= \left(\frac{\pi^2}{30} - \frac{1}{480\pi^2} \frac{\beta^4}{R^4} + O\big(e^{-4\pi^2 \frac{R}{\beta}}\big) \right) \frac{1}{\beta^4}\,. \end{aligned} \tag{7..7}$$

The corresponding result for of the Maxwell free field $F_{\mu\nu}$ is

$$\mathcal{E}_R^{(F)}(\beta) = \left(\frac{\pi^2}{15} - \frac{1}{6} \frac{\beta^2}{R^2} + \frac{1}{4\pi^3} \frac{\beta^3}{R^3} - \frac{11}{240\pi^2} \frac{\beta^4}{R^4} + O\big(e^{-4\pi^2 \frac{R}{\beta}}\big) \right) \frac{1}{\beta^4}\,. \tag{7..8}$$

Proof. The hermitian operators H and $H(2R)$ are unitarily equivalent due to Eq. (7..3). This leads to the fact that $tr_{\mathcal{V}}\, q^{H(2R)}$ and $tr_{\mathcal{V}}\, H(2R)\, q^{H(2R)}$ do not depend on R. Then Eqs. (6..17) and (6..54) imply that in the two models under consideration we have

$$\begin{aligned} \mathcal{E}_R^{(\varphi)}(\beta) &= \frac{G_4\big(\frac{i\beta}{2\pi R}\big) - \frac{1}{240}}{R\mathcal{V}_R}\,, \\ \mathcal{E}_R^{(F)}(\beta) &= \frac{2G_4\big(\frac{i\beta}{2\pi R}\big) - 2G_2\big(\frac{i\beta}{2\pi R}\big) - \frac{11}{120}}{R\mathcal{V}_R}\,. \end{aligned} \tag{7..9}$$

Using further the relations

$$G_2(\tau) = \frac{1}{\tau^2} G_2\big(\frac{-1}{\tau}\big) - \frac{i}{4\pi\tau}\,, \qquad G_4(\tau) = \frac{1}{\tau^4} G_4\big(\frac{-1}{\tau}\big) \tag{7..10}$$

(which are special cases of (3..15)) we find

$$\mathcal{E}_R^{(\varphi)}(\beta) = \frac{1}{\beta^4}\left(8\pi^2 G_4\left(\frac{2\pi i R}{\beta}\right) - \frac{\beta^4}{480\pi^2 R^4}\right), \tag{7..11}$$

$$\mathcal{E}_R^{(F)}(\beta) = \frac{1}{\beta^4}\Bigg(16\pi^2 G_4\left(\frac{2\pi i R}{\beta}\right) + \frac{4\beta^2}{R^2} G_2\left(\frac{2\pi i R}{\beta}\right) + \frac{\beta^3}{4\pi^3 R^3} - \frac{11\beta^4}{240\pi^2 R^4}\Bigg). \tag{7..12}$$

Finally, to obtain Eq. (7..7) one should apply the expansion (3..13) implying that

$$G_2\left(\frac{2\pi i R}{\beta}\right) = -\frac{1}{24} + O\left(e^{-4\pi^2 \frac{R}{\beta}}\right), \quad G_4\left(\frac{2\pi i R}{\beta}\right) = \frac{1}{240} + O\left(e^{-4\pi^2 \frac{R}{\beta}}\right).$$

□

Remark 7..2. In order to make comparison with the familiar expression for the black body radiation it is instructive to restore the dimensional constants h and c setting $H_R = \dfrac{hc}{R} H(2R)$ (instead of (7..4)). The counterpart of (7..9) and (3..13) then reads

$$\langle H_R \rangle_q = \frac{hc}{R}\left(G_4\left(\frac{ihc\beta}{R}\right) - E_0\right) = \frac{hc}{R}\sum_{n=1}^{\infty} \frac{n^3 e^{-n\frac{hc\beta}{R}}}{1 - e^{-n\frac{hc\beta}{R}}}. \tag{7..13}$$

Each term in the infinite sum in the right hand side is a constant multiple of Plank's black body radiation formula for frequency

$$\nu = n\frac{c}{R}. \tag{7..14}$$

Thus, for finite R, there is a minimal frequency, c/R. Using the expansion in (7..13) one can also find an alternative integral derivation of the limit mean energy density $\mathcal{E}_R^{(\varphi)}(\beta)$ (7..9):

$$\mathcal{E}_R^{(\varphi)}(\beta) = \frac{1}{2\pi^2 h^3 c^3 \beta^4}\sum_{n=1}^{\infty} \frac{\left(n\frac{hc\beta}{R}\right)^3 e^{-n\frac{hc\beta}{R}}}{1 - e^{-n\frac{hc\beta}{R}}} \frac{hc\beta}{R} \underset{R\to\infty}{\longrightarrow} \frac{\pi^2}{30 h^3 c^3 \beta^4} \tag{7..15}$$

since the sum in the right hand side goes to the integral $\int\limits_0^\infty \frac{t^3 e^{-t}}{1-e^{-t}}\, dt = \frac{\pi^4}{15}$.

Remark 7..3. We observe that the constant C in (7..6) in both considered models is equal to $c_1/(30\,\pi^2)$, where c_1 is the coefficient to the G_4–modular form in $\langle H \rangle_q$ (see Eq. (6..16)). If we use in the definition (7..5) of H_R the Hamiltonian $H\,(2R) + E_0'$ instead of $H\,(2R)$, $\widetilde{H}_R := (H\,(2R) + E_0')/R$, then this will only reflect on the (non–leading) terms $(c_4\,\beta^4)/R^4$ in (7..7) (7..8) replacing them by $((E_0' - E_0)\,\beta^4)/(2\pi^2\,R^4)$, where E_0 is the "vacuum energy" for the corresponding models (i.e., E_0 is $1/240$ and $11/120$ for the fields φ and $F_{\mu\nu}$, respectively).

7.2. Infinite volume limit of the thermal correlation functions

We shall study the $R \to \infty$ limit on the example of a free scalar field φ in four dimensions.

Denote by $\varphi^M(x)$ (the canonically normalized) $D = 4$ free massless scalar field with 2–point function

$$\langle 0|\, \varphi^M(x_1)\, \varphi^M(x_2)\, |0\rangle \;=\; (2\pi)^{-2} \left(x_{12}^2 + i\, 0\, x_{12}^0 \right)^{-1} \tag{7..16}$$

($x_{12} = x_1 - x_2$, $x_{12}^2 = \boldsymbol{x}_{12}^2 - \left(x_{12}^0\right)^2$, see also (4..39)). We define, in accord with Proposition 7.1., a finite volume approximation of its thermal correlation function by

$$\left\langle \varphi^M(x_1)\, \varphi^M(x_2) \right\rangle_{\beta,R} := \frac{tr_{\mathcal{V}}\ \varphi^M(x_1)\, \varphi^M(x_2)\ e^{-\beta H_R}}{tr_{\mathcal{V}}\ e^{-\beta H_R}} \tag{7..17}$$

and will be interested in the thermodynamic limit,

$$\left\langle \varphi^M(x_1)\, \varphi^M(x_2) \right\rangle_{\beta,\infty} := \lim_{R\to\infty} \left\langle \varphi^M(x_1)\, \varphi^M(x_2) \right\rangle_{\beta,R} . \tag{7..18}$$

Proposition 7..3. *The limit (7..18) (viewed as a meromorphic function) is given by*

$$\left\langle \varphi^M(x_1)\, \varphi^M(x_2) \right\rangle_{\beta,\infty} = \frac{\sinh 2\pi \frac{|\boldsymbol{x}_{12}|}{\beta}}{8\pi\beta\, |\boldsymbol{x}_{12}|} \left(\cosh 2\pi \frac{|\boldsymbol{x}_{12}|}{\beta} - \cosh 2\pi \frac{x_{12}^0}{\beta} \right)^{-1}, \tag{7..19}$$

($|\boldsymbol{x}_{12}| := \sqrt{\boldsymbol{x}_{12}^2} \equiv \sqrt{\left(x_{12}^1\right)^2 + \left(x_{12}^2\right)^2 + \left(x_{12}^3\right)^2}$).

We shall *prove* this statement by relating $\varphi^M(x)$ to the compact picture field $\varphi\,(\zeta, u)$ ($\equiv \phi^{(1)}\,(\zeta, u)$) whose thermal 2–point function was computed in Sect. 6..

First, we use Eq. (4..38) to express $\varphi^M(x)$ in terms of the z–picture field (corresponding to the R–depending chart (7..1))

$$2\pi\,\varphi^M(x) \,=\, \frac{1}{2\omega\big(\dfrac{x}{2R}\big)}\;\varphi_R(z\,(x;R)) \tag{7..20}$$

(since $dz^{\,2} = \omega\big(\frac{x}{2R}\big)^{-2}\,\frac{dx^{\,2}}{4}$, cp. (4..25)). The factor 2π in front of φ^M accounts for the different normalization conventions for the x– and z–picture fields (we set $\langle 0|\,\varphi(z_1)\;\varphi(z_2)\,|0\rangle = \big(z_{12}^{\,2}\big)^{-1}$ instead of (7..16)).

As a second step we express $\varphi_R(z)$ – and thus $\varphi^M(x)$ – in terms of the compact picture field $\varphi_R(\zeta,u)$:

$$\begin{aligned} \varphi_R(\zeta,u) \,&:=\, Re^{2\pi i\zeta}\varphi\big(Re^{2\pi i\zeta}u\big)\,,\\ 2\pi\,\varphi^M\,(x) \,&=\, \frac{1}{2R\,\big|\,\omega\big(\dfrac{x}{2R}\big)\big|}\,\varphi_R\big(\zeta\big(\frac{x}{2R}\big)\,,u\big(\frac{x}{2R}\big)\big)\,. \end{aligned} \tag{7..21}$$

Here ζ and u are determined as functions of $\dfrac{x}{2R}$ from

$$e^{2\pi i\zeta}u = \frac{z\,(x;R)}{R} = z\big(\frac{x}{2R}\big)$$

($z(x)$ is given by (4..24) for $z = x \in M$); in deriving the second equation in (7..21) we have used the relation

$$e^{4\pi i\zeta} = \frac{z\,(x;R)^2}{R^2} = \overline{\omega\big(\frac{x}{2R}\big)}\,\omega\big(\frac{x}{2R}\big)^{-1}\;.$$

Next we observe that $\varphi_R(\zeta,u)$ are mutually conjugate (for different R) just as $H\,(2R)$ in Eq. (7..3). (To see this one can use an intermediate "dimensionless" coordinates $\widetilde{z}(x;R) = z/R = z\big(x/2R\big)$, which differs from (4..24) just by the dilation $(2R)^{X_{-1D}}$.) It follows that its vacuum and thermal 2–point function with respect to the Hamiltonian $H(2R)$ do not depend on R and coincide with (6..13). Thus

$$4\pi^2\,\big\langle\varphi^M(x_1)\,\varphi^M(x_2)\big\rangle_{\beta,R} \,=\, \frac{p_1(\zeta_{12}{+}\alpha,\tau_R) - p_1(\zeta_{12}{-}\alpha,\tau_R)}{16\pi R^2\,|\omega_1\omega_2|\,\sin 2\pi\alpha} \tag{7..22}$$

for $\omega_k = \omega\big(\dfrac{x_k}{2R}\big)$, $\zeta_{12} = \zeta\big(\dfrac{x_1}{2R}\big) - \zeta\big(\dfrac{x_2}{2R}\big)$, $\cos 2\pi\alpha = u\big(\dfrac{x_1}{2R}\big)\cdot u\big(\dfrac{x_2}{2R}\big)$, $\tau_R = \dfrac{i\beta}{2\pi R}$. In order to perform the $R\to\infty$ limit we derive the large

R behaviour of $|\omega_k|$, ζ_{12} and α:

$$\begin{aligned} 2\pi\zeta_{12} &= \frac{x_{12}^0}{R}\left(1+O\left(\frac{\|x_1\|^2+\|x_2\|^2}{R^2}\right)\right), \\ 2\pi\alpha &= \frac{|\boldsymbol{x}_{12}|}{R}\left(1+O\left(\frac{\|x_1\|^2+\|x_2\|^2}{R^2}\right)\right), \\ 4\,|\omega_k|^2 &= 1+O\left(\frac{\|x_k\|^2}{R^2}\right), \end{aligned} \tag{7..23}$$

($\|x\| := \sqrt{(x^0)^2+|\boldsymbol{x}|^2}$) following from

$$\begin{aligned} \cos 2\pi\zeta_k &= \frac{1+\left(\frac{x_k}{2R}\right)^2}{2\,|\omega_k|}\,, & \sin 2\pi\zeta_k &= \frac{x_k^0}{2R\,|\omega_k|}\,, \\ \boldsymbol{u} &= \frac{\boldsymbol{x}_k}{2R\,|\omega_k|}\,, & u_4 &= \frac{1-\left(\frac{x_k}{2R}\right)^2}{2R\,|\omega_k|}\,, \end{aligned} \tag{7..24}$$

$$4\sin^2\pi\alpha = (u_1-u_2)^2 = \frac{|\boldsymbol{x}_{12}|^2}{R^2}\left(1+O\left(\frac{\|x_1\|^2+\|x_2\|^2}{R^2}\right)\right).$$

To evaluate the small τ_R (large R) limit of the difference of p_1–functions in (7..22) we use (2..23), (2..49) and (3..15) to deduce

$$p_1(\zeta,\tau) = \frac{1}{\tau}\left(p_1\left(\frac{\zeta}{\tau},\frac{-1}{\tau}\right) - 2\pi i\zeta\right). \tag{7..25}$$

Eq. (7..25) implies, on the other hand, that

$$p_1\left(\frac{\zeta_{12}\pm\alpha}{\tau_R},\frac{-1}{\tau_R}\right) \underset{R\to\infty}{\approx} p_1\left(\frac{x_{12}^0\pm|x_{12}|}{i\beta},\frac{i2\pi R}{\beta}\right) \underset{R\to\infty}{\longrightarrow} \pi i\,\coth\left(\pi\,\frac{x_{12}^0\pm|x_{12}|}{\beta}\right). \tag{7..26}$$

Inserting (7..23)–(7..26) into (7..22) we complete the proof of (7..19) and hence of Proposition 7.2..

Remark 7..4. The physical thermal correlation functions should be, in fact, defined as distributions which amounts to giving integration rules around the poles. To do this one should view (7..19) as a boundary value of an analytic function in x_{12} for $x_{12}^0 \to x_{12}^0 - i\varepsilon$, $\varepsilon > 0$, $\varepsilon \to 0$ (cf. (7..16)). It is not difficult to demonstrate that the limit $\varepsilon \to +0$ and $R \to \infty$ in (7..18) commute. Using (7..24) we can also compute the $\frac{1}{R\beta}$ correction to (7..19):

$$\left\langle \varphi^M(x_1)\, \varphi^M(x_2) \right\rangle_{\beta,R} \underset{R \gg \beta}{\approx} \left\langle \varphi^M(x_1)\, \varphi^M(x_2) \right\rangle_{\beta,\infty} - \frac{1}{4\pi^2 \beta R}. \qquad (7..27)$$

To obtain the Fourier expansion of the result we combine Eqs. (7..22), (7..23) with the q–series (2..52) and set (as in Remark 7.1.)

$$\frac{n}{R} \to p\,, \quad \frac{1}{R} \to dp\,, \quad \sum_{n=1}^{\infty} \frac{1}{R} f\big(\frac{n}{R}; x, \beta\big) \underset{R \to \infty}{\longrightarrow} \int_0^{\infty} f(p; x, \beta)\, dp. \qquad (7..28)$$

The result is

$$(2\pi)^2 \left\langle \varphi^M(x_1)\, \varphi^M(x_2) \right\rangle_{\beta,\infty} = \frac{1}{x_{12}^2 + i\,0 x_{12}^0} \qquad (7..29)$$

$$+ \frac{2}{|\boldsymbol{x}_{12}|} \int_0^{\infty} \frac{e^{-\beta p}}{1 - e^{-\beta p}} \cos\left(p x_{12}^0\right) \sin\left(p\,|\boldsymbol{x}_{12}|\right)\, dp\,.$$

To conclude: the conformal compactification $\overline{M}$ of Minkowski space M can play a dual role.

On one hand, it can serve as a *symmetric* finite box approximation to M in the study of finite temperature equilibrium states. In fact, any finite inverse temperature β actually fixes a Lorentz frame (cf. [11]) so that the symmetry of a Gibbs state is described by the 7–parameter "Aristotelian group" of (3–dimensional) Euclidean motions and time translations. In the passage from M to $\overline{M}$ the Euclidean group is deformed to the (stable) compact group of 4–dimensional rotations while the group of time translations is compactified to $U(1)$. Working throughout with the maximal (7–parameter) symmetry allows to write down simple explicit formulae for both finite R and the "thermodynamic limit".

On the other hand, taking $\overline{M}$ (and its universal cover, $\widetilde{M} = \mathbb{R} \times \mathbb{S}^3$) not as an auxiliary finite volume approximation but as a model of a static space–time, we can view R as a (large but) finite length and use the above discussion as a basis for studding finite R corrections to the Minkowski space formulae. It is a challenge from this second point of view to study the conformal symmetry breaking by considering massive fields in $\widetilde{M}$.

Guide to references

Among the books on elliptic functions and modular forms we have mostly referred in these notes to the conference proceedings [22] (see, in particular, the beginning of Sect. 2), to the readers-friendly text [47] (which gives a flavor of the work of the founding fathers on the subject) and, to a lesser extent, to Weil's book [68] which provides both a valuable contribution to the history and an elegant exposition of the theory of Eisenstein's series.

The reader who enjoys learning about history of mathematics for its own sake is probably aware without our recommendation of the entertaining essays of Bell [3] but may find also interesting the emotionally told story of 19th century mathematics by one of its participants [37]. We have also referred, in passing, to books on the history of topics that are (to a varying degree) periferal to our subject ([28], [69]).

The books by Serge Lang [39] and [40] provide a systematic background on the mathematical part of these lectures on a more advanced level. The electronically available lecture notes by Milne [48] are recommended for a ready to use treatment of the Riemann-Roch theorem (applied in Sect. 3.1 to the classification of modular forms). The rather engaging exposition in [57] is mostly directed towards the solution of algebraic equations but also contains an elementary introduction to the arithmetic theory of elliptic curves (Chapter 5) including an idea about Wiles' proof of Fermat last theorem. More systematic on number theoretic applications are the earlier texts [59], [62]. Mumford's book [50] on theta functions is a classic.

A rigorous treatment of the applications of elliptic functions and modular forms to 2-dimensional conformal field theory has been given in [72]. The mathematical theory of chiral vertex algebras was anticipated in work by Frenkel and Kac [20] and developed by Borcherds [5], [6]. Nowadays, it is the subject of several books: [21], [32], [19].

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Appendix A. Elliptic functions in terms of Eisenstein series

The family of functions $p_k^{\kappa\lambda}(\zeta,\tau)$, $k=1,2,\ldots$, $\kappa,\lambda=0,1$, encountered throughout these lectures, is uniquely determined by the following set of properties.

(i) $p_k^{\kappa\lambda}(\zeta,\tau)$ are meromorphic functions in $(\zeta,\tau)\in\mathbb{C}\times\mathfrak{H}$ which have exactly one pole at $\zeta=0$ of order k and residue 1 in the domain $\big\{\alpha\tau+\beta:\alpha,\beta\in[0,\,1)\big\}\subset\mathbb{C}$ for all $\tau\in\mathfrak{H}$ and $k=1,2,\ldots$;

(ii) $p_{k+1}^{\kappa\lambda}(\zeta,\tau)=-\frac{1}{k}\frac{\partial}{\partial\zeta}p_k^{\kappa\lambda}(\zeta,\tau)$ for $k=1,2,\ldots$;

(iii) $p_k^{\kappa\lambda}(\zeta+1,\tau)=(-1)^{\lambda}p_k^{\kappa\lambda}(\zeta,\tau)$ for $k=1,2,\ldots$;

(iv) $p_k^{\kappa\lambda}(\zeta+\tau,\tau)=(-1)^{\kappa}p_k^{\kappa\lambda}(\zeta,\tau)$ for $k+\kappa+\lambda>1$;

(v) $p_k^{\kappa\lambda}(-\zeta,\tau)=(-1)^{k}p_k^{\kappa\lambda}(\zeta,\tau)$ for $k=1,2,\ldots$.

One cannot require that the function $p_1(\zeta,\tau)(=p_1^{00})$ is doubly periodic (as it has a simple pole in the fundamental parallelogram) and we have chosen it to have a single period, 1, in ζ.

For $k>2$ the above functions are readily determined: they can be written as absolutely convergent Eisenstein(-Weierstrass) series,

$$p_k^{\kappa\lambda}(\zeta,\tau)\;=\sum_{m,\,n\,\in\,\mathbb{Z}}\frac{(-1)^{\kappa m+\lambda n}}{(\zeta+m\tau+n)^k}\quad(k>2)\,,\tag{A.1}$$

for ζ outside the lattice $\mathbb{Z}\tau+\mathbb{Z}$. Condition ($ii$) then determines each of the four functions $p_1^{\kappa\lambda}$ up to an additive linear in ζ term. Conditions (iv) and (v) guarantee their uniqueness (provided that they exist). The existence is established by the explicit construction (4..80) which can be rewritten in the form

$$p_1^{\kappa\lambda}(\zeta,\tau,\mu)=\lim_{M\to\infty}\sum_{n=-M}^{M}\frac{\pi\cos^{1-\lambda}\big[\pi\big(\zeta+n\tau\big)\big]}{\sin\big[\pi\big(\zeta+n\tau\big)\big]}\,e^{\pi i n(2\mu+\kappa)}.\tag{A.2}$$

The resulting expressions for p_1 and p_2 are related to the corresponding Weierstrass functions by (2..23), (2..24).

The above p-functions admit an extension, needed when dealing with a chemical potential, in which the character $(-1)^{\kappa}$ in condition (iv) is replaced by a more general one:

$$p_k^{\kappa\lambda}\,(\zeta+\tau,\tau,\mu) \;=\; (-1)^k\; e^{-2\pi i\mu}\, p_k^{\kappa\lambda}\,(\zeta,\tau,\mu)\;. \tag{A.3}$$

The resulting p_1-functions have a manifestly meromorphic representation in terms of ratios of Jacobi ϑ functions:

$$\begin{aligned} p_1^{\kappa\lambda}(\zeta,\,\tau,\,\mu) \;=\; & \frac{(\partial_\zeta\,\vartheta_{11})(0,\tau)}{\vartheta_{1-\lambda\,1-\kappa}(\mu,\tau)}\,\frac{\vartheta_{1-\lambda\,1-\kappa}(\zeta{+}\mu,\tau)}{\vartheta_{11}(\zeta,\tau)} \\ & -\;(1{-}\lambda)\,\pi\,\mathrm{cotg}\,\pi\big(\mu{+}\,\frac{\kappa}{2}\big) \end{aligned} \tag{A.4}$$

(see Proposition A.1. of [55]).

Appendix B. The action of the conformal Lie algebra on different realizations of (compactified) Minkowski space

In this section we will sketch of the proof of the relations (4..22), (4..30) and (4..32) between the three (complex) bases of the conformal Lie algebra $\mathfrak{c}$ used in Sect. 4: the basis $\{X_{ab}\}$ in the projective description, the familiar generators in the Minkowski (x-space) chart, and the T_α, C_α and H generators of the z-picture. We begin with an important observation.

Proposition B.1. *The correspondence between the conformal Lie algebra generators $X\in\mathfrak{c}$ and first order linear differential operators $\mathcal{O}_X$, given by*

$$[X,\phi(u)] \;=\; \mathcal{O}_X\phi(u)\,,\quad u=x,\,z \tag{B.1}$$

($\phi(u)=\{\phi_A(u)\}$) is a Lie algebra **anti***homomorphism, i.e. for $X,\,Y\in\mathfrak{c}$:*

$$[-\mathcal{O}_X,-\mathcal{O}_Y] \;=\; -\mathcal{O}_{[X,Y]}\,. \tag{B.2}$$

As a corollary, the correspondence $X\mapsto-\mathcal{O}_X$ is a Lie algebra homomorphism.

Proof. Using the Jacobi identity for the double commutator we find

$$\begin{aligned} [[X,Y]\,,\phi(u)] \;&=\; [X,[Y,\phi(u)]]-[Y,[X,\phi(u)]]=[X,\mathcal{O}_Y(\phi(u))] \\ &\quad-\;[Y,\mathcal{O}_X(\phi(u))]=\mathcal{O}_Y([X,\phi(u)])-\mathcal{O}_X([Y,\phi(u)]) \\ &=\;-\,[\mathcal{O}_X,\mathcal{O}_Y]\,\phi(u)\,. \end{aligned} \tag{B.3}$$

□

Note that the derivatives' parts of the above operators $\mathcal{O}_{X,Y}$ are some vector fields $\mathfrak{D}_{X,Y}$ and the correspondence $X\mapsto\mathfrak{D}_X$ is again antihomomorphism.

In order to derive Eq. (4..22) let us define the (physical) generators of translations iP_μ, special conformal transformations iK_μ and dilations $i\mathcal{D}$ by

$$[iP_\mu,\phi(x)] = \frac{\partial}{\partial x^\mu}\,\phi(x), \qquad [i\mathcal{D},\phi_A] \,=\, x^\nu \frac{\partial}{\partial x^\nu}\,\phi(x) + d_\phi \phi(x)\;,$$
$$[iK_\mu,\phi(x)] = \Big(x^{\,2}\frac{\partial}{\partial x^\mu} - x_\mu x^\nu \frac{\partial}{\partial x^\nu}\Big)\,\phi(x) + M(x)\,\phi(x)\,, \tag{B.4}$$

where $M(x)$ is some x-dependent (matrix) function (whose explicit form is not essential for the present calculations) and d_ϕ is the field dimension. We note that P_0 is the physical (hermitian) energy operator (e^{itP_0} generating the unitary time evolution) and is, hence, positive definite in the state space $\mathcal{V}$. Using the Klein–Dirac compactification formulae (4..15) mapping the Minkowski space M into the quadric Q (4..13) we can find a representation on Q of the vector fields corresponding to iP_μ, iK_μ and $i\mathcal{D}$: $-\frac{\partial}{\partial x^\mu}$, $-x^{\,2}\frac{\partial}{\partial x^\mu}$ $+\, 2x_\mu x^\nu \frac{\partial}{\partial x^\nu}$ and $-x^\nu \frac{\partial}{\partial x^\nu}$, respectively (accordingly to Proposition B.1). In order to express the generators iP_μ, iK_μ and $i\mathcal{D}$ in terms of X_{ab} we should, in addition, factorize with respect to the Euler field $\xi^a \frac{\partial}{\partial \xi^a}$ (noting that the scalar functions on $\overline{M}$ (4..13) are lifted to homogeneous functions of degree zero on Q on which the Euler field vanishes).

Similarly, in order to prove that (4..30) and (4..32) agree with the relations (4..48)–(4..51) we use the imbedding analogous to (4..15)

$$z \mapsto \big\{\lambda \vec{\xi}_z\big\} \in \overline{M}_{\mathbb{C}}, \quad \vec{\xi}_z \,=\, z^\alpha \vec{e}_\alpha + \frac{1+z^{\,2}}{2}\,\vec{e}_{-1} + i\,\frac{1-z^{\,2}}{2}\,\vec{e}_0 \quad \text{or,}$$
$$z^\alpha \,=\, \frac{\xi^\alpha}{\xi^{-1} - i\xi^0}\,. \tag{B.5}$$

(Note that (4..15) and (B.5) reproduce (4..24).)

Appendix C. Clifford algebra realization of $spin(D,2)$ and the centre of $Spin(D,2)$

Let β_μ, $\mu = 0,1,\dots,2r-1$ be $2^r \times 2^r$ complex matrices generating the Clifford algebra *Cliff* $(2r-1,1)$; more precisely, we assume the relations

$$[\beta_\mu,\beta_\nu]_+ \,=\, 2\eta_{\mu\nu} \;\; (\eta_{\mu\nu} \,=\, diag(-1,1,\dots,1)), \;\; \beta^*_\mu \,=\, \beta^\mu \,=\, \eta^{\mu\nu}\,\beta_\nu \tag{C.1}$$

(where we skip – here and in what follows – the corresponding unit matrix). These Clifford units give rise to two collections of $2r+2$ matrices $\{\beta_a\}$ and

$\{\check{\beta}_a\}$, $a = -1, 0, \ldots, 2r$, by setting $\check{\beta}_\mu = \beta_\mu$, for $\mu = 0, 1, \ldots, 2r-1$,

$$\beta_{2r} \,=\, i^{r-1}\,\beta_0\,\beta_1\ldots\beta_{2r-1} \,=\, \check{\beta}_{2r}, \quad \beta_{-1} \,=\, \mathbb{I} \,=\, -\check{\beta}_{-1}. \tag{C.2}$$

The resulting pair of matrix valued vectors are characterized by the bilinear relations

$$\beta_a\,\check{\beta}_b + \beta_b\,\check{\beta}_a \,=\, 2\,\eta_{ab} \,=\, \check{\beta}_a\,\beta_b + \check{\beta}_b\,\beta_a. \tag{C.3}$$

The two 2^r–dimensional spinorial representation $S_\pm$ of $spin(2r,2)$ are then generated by

$$\begin{aligned} S_+(X_{ab}) \,&\equiv\, \gamma_{ab} \,:=\, \frac{1}{4}\,(\beta_b\,\check{\beta}_a - \beta_a\,\check{\beta}_b)\,, \\ S_-(X_{ab}) \,&\equiv\, \check{\gamma}_{ab} \,:=\, \frac{1}{4}\,(\check{\beta}_b\,\beta_a - \check{\beta}_a\,\beta_b)\,. \end{aligned} \tag{C.4}$$

The matrices β_a and $\check{\beta}_a$ can be also used to construct the single spinorial representation of the conformal group $Spin(2r+1,2)$ corresponding to an odd dimensional space—time. To this end we introduce the *Cliff* $(2r+1,2)$ algebra generators

$$\begin{gathered} \Gamma_a \,=\, \begin{pmatrix} 0 & \beta_a \\ \check{\beta}_a & 0 \end{pmatrix}, \quad a = -1, 0, 1, \ldots, 2r, \quad \Gamma_{2r+1} \,=\, \begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix}, \\ [\Gamma_A, \Gamma_B]_+ \,=\, 2\eta_{AB}, \quad A, B = -1, 0, \ldots, 2r+1 \quad (\eta_{2r+1,2r+1} = 1); \end{gathered} \tag{C.5}$$

then the $2^{r+1} \times 2^{r+1}$–dimensional spinorial representation of the $spin(2r+1,2)$ generators assumes the form

$$S(X_{AB}) \,=\, \Gamma_{AB} \,=\, \frac{1}{4}\,[\Gamma_B, \Gamma_A]\,. \tag{C.6}$$

For $a \neq b$ the generators $\gamma_{ab} = \frac{1}{2}\,\beta_b\,\check{\beta}_a$ (C.4) satisfy

$$(2\gamma_{ab})^2 \,=\, (\beta_b\,\check{\beta}_a)^2 \,=\, -\eta_{aa}\,\eta_{bb} \;\; \text{(in general, } (2\gamma_{ab})^2 = \eta_{ab}^2 - \eta_{aa}\,\eta_{bb}). \tag{C.7}$$

It follows that the *valence element* v of the centre Z of $Spin(2r,2)$ is the common central element of $U(1)$ and $Spin(2r)$:

$$v \,:=\, e^{2\pi\gamma_{-10}} \,=\, \cos\pi + 2\gamma_{-10}\sin\pi \,=\, -\mathbb{I} \,=\, e^{2\pi\gamma_{\alpha\beta}}. \tag{C.8}$$

Note that v is mapped on the group unit by the corresponding (two-to-one) map $Spin(2r,2) \to SO(2r,2)$. On the other hand, for even r the centre of $Spin(2r,2)$ is $\mathbb{Z}_4$ and it is generated by the product

$$c_1 \,=\, e^{\pi\gamma_{-10}}\,e^{\pi\gamma_{12}}\ldots e^{\pi\gamma_{2r-1,2r}} \,=\, \beta_0\,\beta_2\,\beta_1\ldots\beta_{2r}\,\beta_{2r-1} \,=\, i^{r-3}. \tag{C.9}$$

(In the last equation we have used the definition (C.2) of β_{2r}.) Clearly, in parallel with Bott periodicity property we have

$$\begin{aligned} &c_1^2 = v \quad \text{for } r \text{ even}, \qquad && c_1^2 = \mathbb{I} \quad \text{for } r = 1 \ mod\ 4, \\ &c_1 = \mathbb{I} \quad \text{for } r = 3 \ mod\ 4. \end{aligned} \tag{C.10}$$

Remark C.1. The term "valence" for the element v of $Z(G)$ originates from the fact that it coincides with the unit operator for single valued representations of the pseudoorthogonal group $SO_0(2r,2)$ and corresponds to multiplication by $-\mathbb{I}$ on its double valued representations (i.e. for exact representations of $Spin(2r,2)$).

We note that the centre of $Spin(2r+1,2)$ is $\mathbb{Z}_2$ with non–trivial element v (C.8). The centre of $Spin(2r,2)$ for odd r is $\mathbb{Z}_2 \times \mathbb{Z}_2$ (see Appendix A to [35]).

We end up with some additional remarks about the case $\mathcal{C} = SU(2,2)$ of chief interest.

Here is a realization of the matrices β_a and γ_{ab} for $D = 4$ in terms of the quaternion units of Sect. 7:

$$\begin{aligned} &\beta_0 = \begin{pmatrix} -i\mathbb{I} & 0 \\ 0 & i\mathbb{I} \end{pmatrix} = -\beta^0, \qquad && \beta_j = \begin{pmatrix} 0 & Q_j \\ Q_j^+ & 0 \end{pmatrix}, \\ &\beta_4 = i\beta_0\beta_1\beta_2\beta_3 = \begin{pmatrix} 0 & -\mathbb{I} \\ -\mathbb{I} & 0 \end{pmatrix} \end{aligned} \tag{C.11}$$

($Q_j Q_k = \epsilon_{jkl} Q_l - \delta_{jk}$, $Q_j^+ = -Q_j$, $j,k,l = 1,2,3$);

$$\gamma_{31} = \frac{1}{2}\begin{pmatrix} Q_2 & 0 \\ 0 & Q_2 \end{pmatrix}, \quad \gamma_{j4} = \frac{1}{2}\begin{pmatrix} Q_j & 0 \\ 0 & -Q_j \end{pmatrix}, \quad \gamma_{-1\alpha} = \frac{1}{2}\beta_\alpha \tag{C.12}$$

($j = 1,2,3$, $\alpha = 1,2,3,4$).

The positive energy unitary irreducible representations $U(g)$ of $SU(2,2)$ [43] are labeled by triples $(d; j_1, j_2)$ of non-negative half integers (j_1 and j_2 being the spins giving the IR of the semisimple part $Spin(4) = SU(2) \times SU(2)$ of the maximal compact subgroup $\mathcal{K}$). The lowest energy subspace $\mathcal{V}_d(j_1, j_2)$ has dimension $(2j_1+1)(2j_2+1)$. The triples are restricted by the conditions:

$$\begin{aligned} &d + j_1 + j_2 \in \mathbb{N}; \qquad && d \geqslant 2 + j_1 + j_2 \ \text{ for } \ j_1 j_2 \neq 0\,; \\ & && d \geqslant 1 + j_1 + j_2 \ \text{ if } \ j_1 j_2 = 0\,. \end{aligned} \tag{C.13}$$

The fields which give rise to representations with $d + j_1 + j_2$ ($j_1 j_2 = 0$) satisfy free field equations. The symmetric tensor fields, corresponding to $j_1 = j_2 = \ell/2$, $d = \ell + 2$, $\ell = 1,2,\ldots$, are conserved. It follows from the first relation (C.13) that the representation $U_{d;j_1,j_2}(v)$ of the valence element (C.8) of $SU(2,2)$ is given by

$$U_{d;j_1,j_2}(v) = (-1)^{2j_1+2j_2} = (-1)^{2d}. \tag{C.14}$$

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