Introduction to noncommutative gauge theories.*

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Abstract

Elementary introduction to noncommutative gauge field theories is presented. The problem of Ultraviolet-Infrared mixing and its implications for physics are discussed.

1. Introduction

Noncommutative field theories have a long history. It was argued by many authors, including W.Heisenberg, E.Schredinger, M.A.Markov, that noncommutativity of space-time coordinates provides a natural possibility to introduce an elementary length, which may improve the ultraviolet behavior of the theory. Extensive studies of this possibility were carried out by H.S.Snyder [1] and V.G.Kadyshevsky with coathors [2]. It appeared however that effective nonlocality, which arises in noncommutative theories, leads to violation of some basic principles, making these models unacceptable. For these reasons noncommutative theories for a long time were abandoned. Last decade these models attracted a new interest partially because of new mathematical development due to A.Connes [3], but mainly because of understanding of their close connections to matrix and string models. It was understood that noncommutative models may serve as effective theories, describing certain regimes of matrix and string models. In particular it was shown [4] that the compactified matrix model, described by the action

$$A = \int dt \, \mathrm{Tr}\,[\sum_{i=1}^{9}(D_t X_i)^2 - \sum_{i<j}[X_i, X_j]^2] \tag{1}$$

where D_t is the covariant derivative, may be described by supersymmetric gauge invariant action on the noncommutative torus. The models describing the strings in the magnetic field in a certain limit may be also described

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by a noncommutative gauge theory [5]. Noncommutative models have important applications to more conventional models like quantum Hall effect [6], [7], [8] and large N QCD [9]. Finally noncommutative models may present an interest from the point of view of modification of the Standard model. They may produce experimentally measurable effects, in particular modifying the dispersion relations. (For discussion see [10]). To avoid contradiction with experiment possible effects of Lorentz invariance breaking due to noncommutativity must be small. Noncommutative theories open new interesting possibilities for investigation of nonperturbative field dynamics. In particular they admit new soliton solutions which may be relevant for interaction models (see e.g. [11]). However in these lectures I mainly concentrate on perturbative aspects of noncommutative theories. The list of references is by no means complete and mainly includes the papers more closely related to the topics discussed in the lectures. For more complete list of references see e.g. [11], [12]. In the second section I discuss the basic classical aspects of the theories on the noncommutative plane. In the third section quantum noncommutative models are considered and some difficulties which arise in these models, in particular ultraviolet-infrared (UV-IR) mixing are discussed. In the fourth section I propose a way to overcome these difficulties and describe a model, which is free of UV-IR mixing and has a smooth commutative limit.

2. Elements of classical noncommutative theories.

Different noncommutative spaces were discussed in the literature, but in these lectures I concentrate on the flat space R_d with some coordinates satisfying the commutation relations

$$[\hat{x}_i, \hat{x}_j] = i\xi\theta_{ij} \tag{2}$$

Here θ_{ij} is a nondegenerate skew-symmetric matrix, $\theta_{ij}\theta_{jk} = -\delta_{ik}$. In the following we shall mainly be interested in the case when only two coordinates x_1 and x_2 are noncommutative. Then the algebra (2) may be reduced to the Heisenberg algebra by rescaling the variables x_1, x_2

$$\hat{x}_1 = \sqrt{\xi}\hat{q}; \quad \hat{x}_2 = \sqrt{\xi}\hat{p}; \quad [\hat{q}, \hat{p}] = i \tag{3}$$

It follows from the Heisenberg algebra that the coordinates x_1 and x_2 cannot be measured simultaneously with arbitrary accuracy and therefore the commutation relations (2) imply the existence of an elementary length and one expects that the corresponding field theory is nonlocal. To see that let us introduce a scalar field on the noncommutative plane as a function $\Phi(\hat{x}_1, \hat{x}_2, x_i)$, where x_i denote all commutative coordinates. We shall deal with the functions Φ as with polynomials in $\hat{x}_1, \hat{x}_2$. To get more general functions one has to complete the algebra of polynomials in some way. Obviously multiplication of $\Phi_1(\hat{x})$ and $\Phi_2(\hat{x})$ defined as the operator product is noncommutative. To work with the fields $\Phi(\hat{x})$ one has to define the operations of differentiation and integration. We define the derivative as the Hermitean operator satisfying the Leibnitz rule. One can take

$$[\hat{\partial}_\mu, \hat{x}_\nu] = \delta_{\mu\nu}; \quad [\hat{\partial}_\mu, \hat{\partial}_\nu] = 0 \tag{4}$$

which implies

$$\hat{\partial}_1\Phi = -\frac{1}{i\xi}[\hat{x}_2, \Phi]; \quad \hat{\partial}_2\Phi = \frac{1}{i\xi}[\hat{x}_1, \Phi] \tag{5}$$

The integral is defined by the equation

$$\int dx_1 dx_2 \Phi = 2\pi|\xi| \, \mathrm{Tr}\,[\,\Phi] \tag{6}$$

Using the cyclic property of the trace we see that the integral of a complete derivative is equal to zero, provided we integrate "sufficiently good" operators. The problem is that the "matrices" x are infinite dimensional, and not always obey the rules for finite matrices. We shall comment on that later. With this definition one can easily write the simplest action

$$A = \int dx[\frac{1}{2}\frac{\partial\Phi^\dagger}{\partial x_\mu}\frac{\partial\Phi}{\partial x_\mu} - V(\Phi^\dagger\Phi)] \tag{7}$$

where the integration and differentiation with respect to noncommutative coordinates x_1, x_2 are defined by the eqs(5, 6). The action (7) defines a d-dimensional nonlocal field theory. To see that it is convenient to use the representation of operators $\Phi(x)$ in terms of Weyl symbols. To save the place we temporarily drop all commutative coordinates and consider the theory on the two-dimensional noncommutative plane. Let us consider the function

$$\varphi(x_1, x_2) = \frac{1}{2\pi}\int dsdt\phi(s,t)\exp\{isx_1 + itx_2\} \tag{8}$$

The function $\varphi(x_1, x_2)$ is called the Weyl symbol of the operator $\Phi(\hat{x}_1, \hat{x}_2)$ defined by the equation

$$\Phi(\hat{x}_1, \hat{x}_2) = \frac{1}{2\pi}\int dsdt\phi(s,t)\exp\{is\hat{x}_1 + it\hat{x}_2\} \tag{9}$$

The inverse Weyl transformation, which expresses
$\varphi(x_1, x_2)$ in terms of $\Phi(\hat{x}_1, \hat{x}_2)$ looks as follows

$$\begin{aligned}\varphi(x_1, x_2) = \frac{1}{2\pi}\int d\alpha d\beta \, \mathrm{Tr}\,[\exp\{-i\alpha(x_1 - \hat{x}_1) - \\ i\beta(x_2 - \hat{x}_2)\}\Phi(\hat{x}_1, \hat{x}_2)\end{aligned} \tag{10}$$

The validity of the eq.(10) may be checked with the help of the useful relation

$$\begin{aligned}I = \; & \mathrm{Tr}\,[\exp\{i\alpha\hat{x}_1 + i\beta\hat{x}_2\}\exp\{-i\sigma\hat{x}_1 - i\tau\hat{x}_2\} \\ & = 2\pi\delta(\alpha - \sigma)\delta(\beta - \tau)\end{aligned} \tag{11}$$

which is the direct analogue of the δ-function definition in the commutative case. Indeed, using the Baker-Hausdorf formula one can write I in the form

$$I = \int < q_1| \exp\{i\alpha\hat{x}_1\} \exp\{-i\beta\hat{x}_2\} \exp\{-i\frac{\alpha\beta}{2}\}|q_2 > \times$$
$$< q_2| \exp\{i\sigma\hat{x}_1\} \exp\{-i\tau\hat{x}_2\} \exp\{\frac{i\sigma\tau}{2}\}|q_1 > dq_1 dq_2 \quad (12)$$

where $|q_i >$ are eigenvectors of the operator $\hat{x}_2$. The operator $\hat{x}_2$ acting on the vector $|q_i >$ multiplies it by q_i, and the operator $\exp\{i\alpha\hat{x}_1\}$ shifts the vector $|q_i >$ to $|q_i - \alpha >$. Hence

$$I = \int dq_1 dq_2 < q_1|q_2 - \alpha > \exp\{i\beta q_2 - i\frac{\alpha\beta}{2}\} \times$$
$$< q_2|q_1 + \sigma > \exp\{-i\tau q_1 - i\frac{\sigma\tau}{2}\} = \int \delta(q_1 - q_2 + \alpha) \times$$
$$\delta(q_2 - q_1 - \sigma) \exp\{i\beta q_2 - i\tau q_1 - i\frac{\alpha\beta}{2} - i\frac{\sigma\tau}{2}\} dq_1 dq_2$$
$$= 2\pi\delta(\beta - \tau)\delta(\alpha - \sigma) \quad (13)$$

To verify the eq.(10) it is sufficient to substitute into this equation the expression (9) for $\Phi(\hat{x}_1, \hat{x}_2)$ and use the relation (11). The formulae (9, 10) allow to express the quadratic part of the action (7) in terms of Weyl symbols. To have the analogous representation for the potential $V(\Phi^\dagger, \Phi)$ we need also an expression for the Weyl symbol of a product of operators. If the operator Φ_3 is equal to the product $\Phi_1\Phi_2$, its Weyl symbol is expressed in terms of the Weyl symbols φ_1, φ_2 as follows

$$\varphi_3 = (\varphi_1 \star \varphi_2) = \exp\{\frac{i}{2}\xi\theta_{ij}\partial_i^x\partial_j^y\}\varphi_1(x)\varphi_2(y)|_{x=y} \quad (14)$$

This equation may be obtained similarly to the eq.(7). The symbol $\star$ in the eq(14) denotes so called Mojal product. This product is obviously noncommutative and nonlocal, as the r.h.s. of the eq.(14) contains the derivatives over x and y of arbitrary order. The Moyal product is assosiative

$$(\varphi_1 \star \varphi_2) \star \varphi_3 = \varphi_1 \star (\varphi_2 \star \varphi_3) \quad (15)$$

and cyclic

$$\int (\varphi_1 \star \varphi_2 \star \varphi_3) = \int (\varphi_3 \star \varphi_1 \star \varphi_2) \quad (16)$$

Another important property of the Moyal product concerns the integral of the bilinear form

$$\int dx\varphi_1(x) \star \varphi_2(x) = \int dx\varphi_1(x)\varphi_2(x) \quad (17)$$

To prove this equality it is sufficient to decompose the r.h.s. of the eq.(14) in terms of ξ and note that all terms except for the first one vanish, as integrating by parts one can present them as the product of antisymmetric tensor θ_{ij} and symmetric tensor $\partial_i\partial_j\varphi$. It is true however only if one can neglect the surface terms arising in the process of integration by parts. In other words the functions φ_1, φ_2 must vanish at infinity sufficiently fast. This requirement explains the notion of "good operators" which was introduced after the eqs(5, 6). Now we can write the action (7) in terms of Weyl symbols using for definiteness a particular form of potential V:

$$A = \int dx[\frac{1}{2}\partial_i\varphi^*\partial_i\varphi - \frac{m^2}{2}\varphi^*\varphi - \frac{\lambda}{4}\varphi^* \star \varphi \star \varphi^* \star \varphi] \tag{18}$$

In accordance with the discussion above we replaced the Moyal product in the quadratic terms by the usual commutative one. However in the interaction term the star product is present and the action (18) describes the scalar field theory with a special nonlocal interaction. The action (18) is invariant with respect to the global transformations

$$\varphi(x) \to \exp\{i\alpha\}\varphi(x); \quad \varphi^*(x) \to \exp\{-i\alpha\}\varphi^*(x) \tag{19}$$

One can consider also local transformations

$$\varphi(x) \to U(x) \star \varphi(x) \tag{20}$$

where

$$U(x) = \exp\{i\alpha(x)\}\star = 1 + i\alpha(x) - \frac{1}{2}\alpha(x) \star \alpha(x) + \dots \tag{21}$$

The matrix $U(x)$ is unitary with respect to the Moyal product $U^*(x) \star U(x) = 1$. To preserve the invariance of the theory with respect to the local gauge transformations (20), one has to replace the derivative by the covariant ones

$$\partial_i\varphi \to D_i\varphi = \partial_i\varphi - iA_i \star \varphi \tag{22}$$

where A_i is the gauge field, which transforms as

$$A_i \to U \star A_i \star U^* + iU \star \partial_i U^* \tag{23}$$

The gauge invariant action for the field A_i has a familiar form

$$A = -\frac{1}{4}\int dx F_{ij}F_{ij}$$
$$F_{ij} = \partial_i A_j - \partial_j A_i - i(A_i \star A_j - A_j \star A_i) \tag{24}$$

This construction is easily generalized to $U(N)$ groups. One simply takes the scalar field as the N component column realizing the fundamental representation of the $U(N)$ group. The $N \times N$ matrices U satisfy the unitarity condition $U^*_{\alpha\beta} \star U_{\beta\gamma} = \delta_{\alpha\gamma}$. The Yang-Mills field $A_{\alpha\beta}$ belongs to the Lie algebra of the $U(N)$ group. All the formulae written above are easily generalized to this case with obvious modifications

$$[A_i, A_j]_\star \to [A_i, A_j]^{\alpha\gamma}_\star = A_i^{\alpha\beta} \star A_j^{\beta\gamma} - A_j^{\alpha\beta} \star A_i^{\beta\gamma} \tag{25}$$

Similar formulae may be written for the matter field in adjoint representation

$$\varphi_{\alpha\beta} \to U^*_{\alpha\gamma} \star \varphi_{\gamma\delta} \star U_{\delta\beta} \tag{26}$$

However if one tries to make the same construction for the $SU(N)$ group, one meets the problem. The $SU(N)$ algebra is not closed under Moyal multiplication. If we introduce the fields belonging to $SU(N)$ algebra, $A(x) = A_i(x)T_i$, we see that the star commutator inevitably includes the unit matrix

$$[A(x), A(y)]_\star = A_i(x) \star A_j(y) t^{ijk} T_k + IF(x,y) + \ldots \tag{27}$$

It is impossible to construct a $SU(N)$invariant noncommutative gauge theory. Although generalization of gauge invariance to the noncommutative case is straight forward, there are some important distinctions between commutative and noncommutative theories. 1. As one sees from eqs(22-24), even noncommutative $U(1)$ theory is nonabelian and there is a direct interaction of noncommutative "photons". 2. Only $U(N)$ models allow gauge invariant noncommutative formulation. $SU(N)$ noncommutative gauge theories do not exist. 3. No simple local observables can be constructed in noncommutative gauge theories. In the commutative case trace of the squared curvature tensor is gauge invariant. However in the noncommutative theory

$$\mathrm{Tr}\,[\,F_{\mu\nu} \star F_{\mu\nu}] \to \;\mathrm{Tr}\,[\,\omega \star F_{\mu\nu} \star F_{\mu\nu} \star \omega^*] \tag{28}$$

and only $\int dx\, \mathrm{Tr}\,[\,F_{\mu\nu}F_{\mu\nu}]$ is gauge invariant. This fact is related to the existence of certain relations between internal and space-time symmetries in noncommutative theories. For example using the definition of derivative in the noncommutative theory

$$\hat{\partial}_i f(\hat{x}) = [-\frac{i\theta^{-1}_{ij}}{\xi}\hat{x}^j, f] \tag{29}$$

one can write the shift transformation in the form

$$f(\hat{x}^i + \varepsilon^i) = \exp\{-i\frac{\theta^{-1}_{ij}}{\xi}\varepsilon^i\hat{x}^j\} f(\hat{x}) \exp\{i\frac{\theta^{-1}_{ij}}{\xi}\varepsilon^i\hat{x}^j\} \tag{30}$$

which clearly looks similar to internal symmetry transformation.

3. Quantum noncommutative theories

Quantization of noncommutative field theories presents a serious and still unsolved problem. Due to nonlocality of the action canonical formalism is not applicable and a construction of a unitary and causal scattering matrix for such models is an open problem. However, if noncommutativity is restricted to the space-space sector, the corresponding action includes only first order time derivatives and the canonical quantization is straightforward. Of course the interaction is still nonlocal in space, but it does not prevent using the standard canonical formalism. In fact in the standard Yang-Mills theory the same situation occurs when it is quantized in Coulomb gauge. In the following we assume that noncommutative coordinates are x_1 and x_2 and the time coordinate is commutative. As a simplest example we consider the Hermitean scalar field on the noncommutative plane, described by the action

$$A = \int dx[\frac{1}{2}\partial_\mu\varphi\partial_\mu\varphi - \frac{1}{2}m^2\varphi^2 - \frac{\lambda}{4}\varphi \star \varphi \star \varphi \star \varphi] \tag{31}$$

The free propagator has a standard form

$$D(p) = (p^2 - m^2)^{-1} \tag{32}$$

and the interaction is conveniently written in terms of Fourier components

$$\varphi(x) = \frac{1}{(2\pi)^4}\int \exp\{ipx\}\phi(p)d^4p \tag{33}$$

$$\begin{aligned}\int d^4x(\varphi \star \varphi \star \varphi \star \varphi) = \int \frac{d^4p}{(2\pi)^4}\delta^4(p_1 + \dots p_4) \\ \times \exp\{\frac{i}{2}\sum_{i<j} p_i \wedge p_j\}\phi(p_1)\dots\phi(p_4)\end{aligned} \tag{34}$$

$$p \wedge q = \xi p_\alpha \theta^{\alpha\beta} q_\beta \tag{35}$$

In distinction of the commutative case the interaction includes the oscillating factors $\exp\{\frac{i}{2}\sum_{i<j} p_i \wedge p_j\}$, and does not possess a symmetry with respect to permutations of p_i. The invariance holds only under cyclic permutations of p_i. The formula (35) allows to establish the parallel between the diagram technique for the model (31) and the double line N^{-1} expansion technique [13]. To make this parallel more transparent let us make the change of variables $p_j = k_j - k_{j-1}$. Then the interaction (35) acquires the form

$$\begin{aligned}4^4 \int d^4k\delta^4(k_1 + \dots k_4)\exp\{\frac{i}{2}k_4 \wedge k_1\}\phi(k_1 - k_4) \\ \dots \exp\{\frac{i}{2}k_3 \wedge k_4\}\phi(k_1 - k_3)\end{aligned} \tag{36}$$

If one introduces the notation

$$\Phi(k_i,k_j)=\exp\{\frac{i}{2}k_i\wedge k_j\}\phi(k_j-k_i) \qquad (37)$$

where $\Phi(k_i,k_j)$ is a Hermitean infinite dimensional matrix, the interaction (35) looks as the interaction in the model describing the matrix fields φ_{ij}. In analogy with the N^{-1} expansion for matrix models one can separate all the diagrams into planar ones, which may be drawn on the plane without self intersections, and nonplanar diagrams. In planar diagrams the phase factors depending on the internal momenta cancel and the remaining phase factor for the diagram with external momenta $p_1\dots p_n$ has a form $V(p_1\dots p_n)=\exp\{\frac{i}{2}\sum_{i<j}p_i\wedge p_j\}$. Therefore the divergent structures of these diagrams up to constant factors are the same as in the commutative case and the ultraviolet infinities may be removed by renormalization of parameters in the classical Lagrangian. Any nonplanar diagram for each line crossing acquires an extra factor, depending on internal momenta, $\exp\{\frac{i}{2}k_i\wedge k_j\}$. This oscillating factor improves the convergence of nonplanar diagrams. However as we saw, the hope for ultraviolet finiteness of noncommutative theories is false. Planar diagrams generate the usual ultraviolet divergencies. Moreover as we shall see, noncommutativity introduces new, infrared divergencies, which make the existence of quantum noncommutative models questionable. To analyze these new divergencies we continue to consider the simple model (31). The lowest order contribution to the vacuum polarization in this model is given by the tadpole diagram, and after Wick rotation may be written in the form

$$\Pi(p)=\frac{\lambda}{4(2\pi)^4}\int d^4k\cos^2(k\wedge p)[\frac{1}{k^2+m^2}-\sum_{j=1,2}\frac{C_j}{k^2+M_j^2}] \qquad (38)$$

where to make the integral ultraviolet convergent we introduced the Pauli-Villars regularization. The coefficients C_1,C_2 satisfy the Pauli-Villars conditions

$$C_1+C_2=1;\quad C_1M_1^2+C_2M_2^2=m^2 \qquad (39)$$

The trigonometric factor $\cos^2(k\wedge p)$ may be presented as

$$\cos^2(k\wedge p)=\frac{1}{2}(1+\cos(2k\wedge p)) \qquad (40)$$

where the first term corresponds to the planar diagram and in the limit $M_i\to\infty$ produces the usual ultraviolet infinities, whereas the second term corresponds to nonplanar diagram and is finite in the limit when the regularization is removed. The nonplanar part may be calculated exactly with the result

$$\Pi_{np}=\frac{\lambda}{16\pi^2}[\frac{m}{\xi|\tilde{p}|}K_1(2m\xi|\tilde{p}|)-\sum_{i=1,2}C_i\frac{M_iK_1(2M_i\xi|\tilde{p}|)}{\xi|\tilde{p}|}] \qquad (41)$$

where K_1 is the Kelvin function and $\tilde{p}_\mu = \theta_{\mu\nu} p_\nu$. For large z the function $K_1(z) \sim \sqrt{\pi(2z)^{-1}} \exp\{-z\}$, whereas for small z, $K_1(z) \sim (z)^{-1} + \ldots$. In the limit $M_i \to \infty$

$$\Pi_{np}(p) \sim \frac{\lambda}{16\pi^2} \frac{m}{\xi|\tilde{p}|} K_1(2m\xi|\tilde{p}|) + \ldots \tag{42}$$

Contrary to naive expectations there is no smooth transition to the commutative theory: in the limit $\xi \to 0$ the radiative corrections blow up. For a finite ξ the polarization operator has a pole singularity at $p = 0$, which change drastically dispersion relations, and when several polarization operators are inserted as subgraphs to more complicated diagrams it leads to nonintegrable infrared singularity making the theory inconsistent. On the other hand, if one keeps the ultraviolet regularization fixed, the limit $\xi \to 0$ does exist and coincides with the result of the regularized commutative theory. This phenomenon, which is called ultraviolet-infrared (UV-IR)-mixing, is due to the absence of absolute convergence of the integrals corresponding to nonplanar diagrams. The limits $\xi \to 0$ and $M \to 0$ are noncommutative. The oscillating factor $\cos(2k \wedge p)$ improves the UV convergence of the integral for all $p \neq 0$. However for $p = 0$ the divergency arises again and manifest itself as the infrared pole of the polarization operator (39). There is a correspondence between the superficial degree of the ultraviolet divergency and infrared behavior of the integral. The diagrams which in the absence of oscillating factor were quadratically or linear UV divergent, acquire the infrared poles of the second and the first order respectively. Logarithmically ultraviolet divergent integrals produce logarithmic infrared singularities. Logarithmic infrared singularities do not lead to inconsistency of the theory as the corresponding integrals still make sense. However the existence of a smooth limit at $\xi \to 0$ is not guaranteed even in this case and corrections due to noncommutativity may be large for small ξ, making difficult a physical interpretation of such theories. The vacuum polarization diagram in the scalar model considered above is superficially quadratically UV divergent, which results in the second order pole at $p = 0$. Insertion of several such graphs into higher loop diagram produces the infrared singularity $p^{2n}, \quad n \geq 2$, which is not integrable. The theory does not exist beyond one loop approximation. One may hope that the situation is better in noncommutative gauge models as gauge invariance prevents appearance of quadratically and linearly divergent counterterms. However the superficial degree of UV divergency of the polarization operator in gauge theory still is equal to two and in the noncommutative version the infrared poles are present. Explicit calculation of the vacuum polarization in the noncommutative $U(1)$ gauge model gives the result

$$\Pi_{\mu\nu}(p) = \frac{g^2}{2\pi^2} \frac{\tilde{p}_\mu p_\nu}{\xi^2 |\tilde{p}|^2} + \ldots \tag{43}$$

In a similar way the most singular part of the three point function may be written in the form

$$\Gamma_{\mu\nu\rho}(p, q) \sim \cos(\xi \tilde{p} q) \{ \frac{\tilde{p}_\mu \tilde{p}_\nu \tilde{p}_\rho}{\xi |\tilde{p}|^2} + sym \} \tag{44}$$

where "sym" denotes symmetrization with respect to arguments $p,\mu; q,\nu; -(p+q),\rho$. Note that the presence of the infrared singularities does not contradict gauge invariance. The Generalized Ward Identity (GWI) for $\Pi_{\mu\nu}(p)$ implies transversality of this function. This property holds for the function (43) as $(\tilde{p}_\mu p_\mu) = 0$. Presence of the infrared singular terms at the one loop level leads to the following modification of dispersion relations

$$\omega(p) = \overrightarrow{p}^2 - \frac{2g^2}{\pi^2\xi^2(p_1^2+p_2^2)} \tag{45}$$

This dispersion relation does not admit a simple particle interpretation and obviously breaks Lorentz invariance. In analogy with the scalar model considered above, insertion of the polarization operator (43) to higher loop diagrams produces nonintegrable infrared singularities. These examples show that using of noncommutative models in quantum field theory meets serious difficulties and is rather problematic at least in the framework of perturbation theory ([14], [15], [16], [17]). The situation is somewhat better in supersymmetric noncommutative theories. Supersymmetric models have only logarithmic ultraviolet divergencies and one expects that in the noncommutative case only logarithmic infrared singularities appear. Explicit calculations of one loop diagrams confirm this hypothesis ([18], [19], [20]). Logarithmic infrared singularities are integrable, hence supersymmetric noncommutative models do exist for a finite ξ. However the existence of a smooth limit at $\xi \to 0$ in these models is problematic. The absence of such a limit would mean that even for small ξ breaking of Lorentz invariance is strong and these theories are not suitable for description of the real world. Moreover the study of one-loop dispersion relations in the $N = 1$ Yang -Mills theory with softly broken supersymmetry shows that radiative corrections in this model produce a tachyonic pole [10]

$$\omega(\overrightarrow{p}^2) = \overrightarrow{p}^2 - \frac{g^2M^2}{2\pi^2} \tag{46}$$

Softly broken supersymmetric $N = 4$ Yang-Mills theory has no tachyonic pole, but the one loop result is also unacceptable from experimental point of view. Failure of straight forward noncommutative generalizations of gauge invariant theories stimulated me to study more sophisticated models, in particular to revise the common approach when the Yang-Mills theory is used for description of spin 1 particles. In the next section I describe a gauge invariant noncommutative model, which is free of nonintegrable infrared singularities and has a smooth commutative limit at $\xi \to 0$. Contrary to naive expectations this model describes spin zero particle.

4. Consistent noncommutative $U(1)$-model.

Experience obtained in commutative theories suggests that the appearance of divergencies which cannot be removed by renormalizing parameters of the classical Lagrangian is a signal that the underlying classical theory is not complete and should be modified in such a way that possible divergency have the same structure as the terms presented in the classical action. Motivated by this observation we propose a modified noncommutative $U(1)$ invariant action which is free of the difficulties discussed in the previous section. We require the modified action to be gauge invariant, power counting renormalizable and introduce nonlocality only via star product. This action should describe a model which has no nonintegrable infrared singularities and in the commutative limit $\xi \to 0$ is Lorentz invariant. Such an action looks as follows [21]:

$$A = \int d^4x[-\frac{1}{4}F_{\mu\nu}F_{\mu\nu} + \lambda(x)\theta_{\mu\nu}F_{\mu\nu}(x)] \tag{47}$$

Here $\lambda(x)$ is the Lagrange multiplier, which transforms according to adjoint representation of the gauge group. Variation of the action (47) with respect to $\lambda(x)$ produces the constraint equation

$$\theta_{\mu\nu}F_{\mu\nu} = 0 \tag{48}$$

which decreases the number of physical degrees of freedom. Although the action (47) is written in terms of the vector field A_μ, it describes a spin zero particle. Below we shall show that in this model the nonintegrable infrared singularities do not contribute to physical matrix elements and in the limit $\xi \to 0$ Lorentz invariance is restored. We assume that the noncommutative coordinates are x_1 and x_2, so the standard quantization procedure may be applied. The quantization is performed in the axial gauge $A_1 = 0$ [22]. In this gauge the constraint (48) reduces to

$$\partial_1 A_2 = 0 \tag{49}$$

where we took into account that the only nonzero elements of the matrix $\theta_{\mu\nu}$ are θ_{12} and θ_{21}. Assuming the usual asymptotic condition $A_\mu(x) \to 0$ when $\vec{x} \to \infty$ we see, that the eq.(49) implies $A_2 = 0$. The classical action (47) in the chosen gauge has a form

$$A = \frac{1}{2}(\partial_0 A_3 - \partial_3 A_0 + ig[A_0, A_3]_\star)^2 + \frac{1}{2}\sum_{i=1,2}(\partial_i A_0)^2 - \frac{1}{2}\sum_{i=1,2}(\partial_i A_3)^2 \tag{50}$$

Introducing the canonical momentum

$$p_3 = \frac{\partial L}{\partial \dot{A}_3} = F_{03} \tag{51}$$

we may write the action in the form

$$A = \int \{p_3\dot{A}_3 - \frac{1}{2}p_3^2 + A_0 D_3 p_3 + \frac{1}{2}\sum_{i=1,2}(\partial_i A_0)^2 - \frac{1}{2}\sum_{i=1,2}(\partial_i A_3)^2\}dx \tag{52}$$

Here D_3 is the covariant derivative. The variable A_0 can be excluded by solving the corresponding time independent equation

$$A_0 = (\nabla)^{-2} D_3 p_3, \quad (\nabla)^2 = \partial_1^2 + \partial_2^2 \tag{53}$$

Substituting this solution to the action (47) we get

$$A = \int \{p_3 \dot{A}_3 - \frac{1}{2} p_3^2 - \frac{1}{2} D_3 p_3 \nabla^{-2} D_3 p_3 - \frac{1}{2} \sum_{i=1,2} (\partial_i A_3)^2\} dx \tag{54}$$

This equation describes a scalar particle with the Hamiltonian

$$H = \frac{p_3^2}{2} + \frac{1}{2} D_3 p_3 \nabla^{-2} D_3 p_3 - \frac{1}{2} \sum_{i=1,2} (\partial_i A_3)^2 \tag{55}$$

and the canonical quantization is applied directly. The action (47) is not invariant with respect to the Lorentz transformation, however in the classical commutative limit $\xi \to \infty$ the invariance of the theory is restored. Indeed in the commutative limit the action acquires a form

$$A = \int \{p_3 \dot{A}_3 - \frac{1}{2} p_3^2 - \frac{1}{2} \partial_3 p_3 \nabla^{-2} \partial_3 p_3 - \frac{1}{2} \sum_{i=1,2} (\partial_i A_3)^2\} dx \tag{56}$$

In terms of the rescaled variables

$$\pi(k) = [1 + k_3^2 (k_1^2 + k_2^2)^{-1}]^{\frac{1}{2}} \tilde{p}_3(k) \tag{57}$$

$$\phi(k) = [1 + k_3^2 (k_1^2 + k_2^2)^{-1}]^{-\frac{1}{2}} \tilde{A}_3(k) \tag{58}$$

it is the standard Lorentz invariant action describing the free scalar particle:

$$A = \int \{\pi \dot{\phi} - \frac{\pi^2}{2} - \frac{1}{2} \sum_{i=1}^{3} (\partial_i \phi)^2\} dx \tag{59}$$

However the quantum corrections may break this smooth transition as it happens in the standard noncommutative $U(1)$ model. Below we shall show that in our model the smooth transition to the free scalar particle exists in the quantum case as well. All quantum corrections vanish in the limit $\xi \to 0$. The scattering matrix may be presented as usual in the form of the path integral of the $\exp\{iA\}$, where A is given by the eq.(54). It is more convenient to write it introducing the integration over A_0:

$$S = \int \exp\{i \int [p_3 \dot{A}_3 - \frac{1}{2} p_3^2 + A_0 D_3 P_3 +$$
$$\frac{1}{2} \sum_{i=1,2} (\partial_i A_0)^2 - \frac{1}{2} \sum_{i=1,2} (\partial_i A_3)^2] dx\} dA_3 dA_0 \tag{60}$$

where the usual asymptotic conditions for A_3 are assumed. Performing the Gaussian integration over P_3 we get

$$S = \int \exp\{i \int [\frac{1}{2} F_{03}^2 + \frac{1}{2} \sum_{i=1,2} [(\partial_i A_0)^2 - (\partial_i A_3)^2]] dx\} dA_3 dA_0 \qquad (61)$$

The propagator is determined by the inverse of the quadratic form of the free action (61) and looks as follows

$$D_{\mu\nu}(k) = \frac{1}{k^2 + i\epsilon} \{g^{\mu\nu} - \frac{k_\mu k_\nu}{k_1^2 + k_2^2}\}; \quad \mu, \nu = 0, 3 \qquad (62)$$

As all other elements of $D_{\mu\nu}$ are zero, it obviously satisfies the constraint equation

$$\tilde{k}_\mu D_{\mu\nu}(k) = 0 \qquad (63)$$

Due to gauge invariance of the action (47) the Green functions satisfy GWI:

$$\int \exp\{iA + \int [\frac{1}{2\alpha}(A_1)^2 + J_\mu A_\mu] dx\}$$

$$\{\frac{1}{\alpha} \partial_1 A_1(x) - \partial_\mu J_\mu(x) + ig[A_\mu(x), J_\mu(x)]_\star\} dA_\mu = 0 \qquad (64)$$

For the two-point function one has

$$-\frac{i}{\alpha} < \partial_1 A_1(x) A_\nu(y) >= \partial_\nu \delta(x - y) \qquad (65)$$

Fourier transform of the polarization operator satisfies the transversality condition

$$p_\mu \Pi_{\mu\nu} = 0 \qquad (66)$$

Let us consider the polarization operator in more details. It is easy to see that the pole singular terms in this function depend only on $\tilde{p}$. Indeed, the polarization operator has a form

$$\Pi_{\mu\nu}(p) = \int \prod_i dk_i \sin(k_1 \tilde{p} \xi) \sin(k_n \tilde{p} \xi) [f_{\mu\nu}(k_i, p_0, p_3, \tilde{p})] \qquad (67)$$

Here the integrand is a rational function with the dimension p^{-4n+2} , multiplied by the product of trigonometric functions. Let us present $\Pi_{\mu\nu}(p)$ as follows

$$\Pi_{\mu\nu}(p) = \int [f_{\mu\nu}(k_i, p_0, p_3, \tilde{p}) - f_{\mu\nu}(k_i, 0, 0, \tilde{p})] \times$$

$$\prod_i \sin(k_1 \tilde{p} \xi) \sin(k_n \tilde{p} \xi) dk_i + \int f_{\mu\nu}(k_i, 0, 0, \tilde{p}) \times$$

$$\prod_i \sin(k_1 \tilde{p} \xi) \sin(k_n \tilde{p} \xi) dk_i \qquad (68)$$

The first term cannot produce the second order infrared pole, as the integral of the rational part diverges at most linearly. In fact for symmetry reasons the divergency is logarithmic. Hence the pole infrared singularities may be present only in the second term, which depends only on $\tilde{p}$. The only possible structure compatible with the Ward identities is

$$\Pi^{pole}_{\mu\nu}(p) = \tilde{p}_\mu \tilde{p}_\nu \Pi(\tilde{p}) \tag{69}$$

Now we recall that the free propagator of the field A_μ, given by the eq.(62), satisfies the transversality condition (63). Therefore a possible infrared pole singularity of $\Pi_{\mu\nu}(p)$ does not contribute and is irrelevant for calculation of diagrams describing a physical process. The only allowed infrared singularity is logarithmic $\sim \ln(\tilde{p}^2\xi)$. This singularity is integrable and does not make the theory inconsistent. Similar arguments may be applied to the three point Green function. As before the only infrared singularity which may contribute to observable is logarithmic. All higher Green functions have at most logarithmic ultraviolet superficial divergencies and cannot produce infrared poles. So we proved that the the $U(1)$ noncommutative model described by the action (47) is free from nonintegrable infrared singularities and allows calculation of radiation corrections of arbitrary order. Note that all interaction vertices in the commutative limit $\xi \to 0$ vanish as ξ. Taking into account that possible singularities in ξ are $\sim \ln(\xi)$, we conclude that the limit $\xi \to 0$ exists in the quantum version of our model and describes the free scalar particle. Our conclusion does not change if the interaction with a spinor field in adjoint representation is included. The model is free of infrared pole singularities and in the limit $\xi \to 0$ is Lorentz invariant. If the interaction includes also spinor fields in the fundamental representation, infrared poles are still absent, but logarithmic infrared singularities are present in the limit $\xi \to 0$ Lorentz invariance may be broken.

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