Superbranes and Generic Curved Spacetime

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Abstract

Embedding of a bosonic and/or fermionic p-brane into a generic curved D-dimensional spacetime is considered. In contradistinction to the bosonic p-brane case, when there are no constraints on a generic curving whatsoever, the usual superbrane can be embedded into a curved spacetime of a restricted curving only. A generic curving is achieved by extending the odd sector of a superbrane as to transform w.r.t. $\overline{SL}(D,R)$, i.e. $\overline{Diff}(D,R)$ infinite-component spinorial representations. Relevant constructions in the $D=3$ case are considered.

1. Generic curved target spacetime for superbrane

In the conventional lagrangian formulation for superbranes, the $(p+1)$-dimensional curved (locally reparametrizable) brane world sheet/volume R^{p+1} is embedded in a flat (Poincaré invariance) Minkowski space-time $M^{1,D-1}$.

On the other hand, macroscopic gravity is described classically by Einstein's theory, corresponding to a generic curved Riemannian R^4 manifold (general covariance).

Thus one is faced with an apparent difference in the manifest symmetries of these two theories. This difference is not only of the principal nature, but is crucial for numerous practical questions such as nonperturbative gravitational solutions (Schwarzshild) etc.

One can certainly hope to reconstruct the full general covariance starting from the field theory of superbrane embedded in a flat space. However, preliminary difficulties encountered along this line support a more pragmatic (and in our opinion in fact the only) approach to construct an a priori fully generally-covariant target-space superbrane theory.

1.1. Bosonic brane: Flat to curved space

The (bosonic) p-brane action [1],

$$
\begin{aligned}
S \;&= \int d^{p+1}\xi \,\left(\frac{1}{2}\sqrt{-\gamma}\gamma^{ij}(\xi)\partial_i X^m \partial_j X^n \eta_{mn} - \frac{1}{2}(p-1)\sqrt{-\gamma}\right.\\
&\left. \quad + \frac{1}{(p+1)!}\epsilon^{i_1 i_2 \cdots i_{p+1}}\partial_{i_1}X^{m_1}\partial_{i_2}X^{m_2}\cdots\partial_{i_{p+1}}X^{m_{p+1}}A_{m_1 m_2 \cdots m_{p+1}}(X)\right),
\end{aligned}
$$

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where $i = 0, 1, \dots p$ labels the coordinates $\xi^i = (\tau, \sigma, \rho, \dots)$ of the brane world-volume with metric γ_{ij}, and $\gamma = \det(\gamma_{ij})$; $m = 0, 1, \dots, D-1$ labels the target-space coordinates X^m with metric η_{mn}, and $A_{m_1 m_2 \dots m_{p+1}}$ is a $(p+1)$-form characterizing a Wess-Zumino-like term *can be generalized* in a straight forward way *for a generic curved target space* to read

$$
\begin{aligned}
S &= \int d^{p+1}\xi \left(\frac{1}{2}\sqrt{-\gamma}\gamma^{ij}(\xi)\partial_i X^{\tilde{m}} \partial_j X^{\tilde{n}} g_{\tilde{m}\tilde{n}} - \frac{1}{2}(p-1)\sqrt{-\gamma} \right. \\
&\left. + \frac{1}{(p+1)!}\epsilon^{i_1 i_2 \cdots i_p+1} \partial_{i_1} X^{\tilde{m}_1} \partial_{i_2} X^{\tilde{m}_2} \cdots \partial_{i_{p+1}} X^{\tilde{m}_{p+1}} A_{\tilde{m}_1 \tilde{m}_2 \cdots \tilde{m}_{p+1}}(X) \right),
\end{aligned}
$$

where $\tilde{m} = 0, 1, \dots, D-1$ labels the curved target-space coordinates $X^{\tilde{m}}$ with riemannian metric $g_{\tilde{m}\tilde{n}}$

$$
\begin{array}{rcc}
SO(1, D-1): & X^m & \eta_{mn} \\
\downarrow & \downarrow & \downarrow \\
Diff(D, R): & X^{\tilde{m}} & g_{\tilde{m}\tilde{n}}
\end{array}
$$

1.2. Super brane: Flat to curved space

The super p-brane action reads [2]:

$$
\begin{aligned}
S &= \int d^{p+1}\xi \left(\frac{1}{2}\sqrt{\gamma}\gamma^{ij}(\xi)\Pi_i^m \Pi_j^n \eta_{mn} - \frac{1}{2}(p-1)\sqrt{-\gamma} \right. \\
&\left. + \frac{1}{(p+1)!}\epsilon^{i_1 i_2 \cdots i_{p+1}} \partial_{i_1} Z^{a_1} \partial_{i_2} Z^{a_2} \cdots \partial_{i_{p+1}} Z^{a_{p+1}} B_{a_{p+1} \cdots a_2 a_1} \right) .
\end{aligned}
$$

Here, the target space is a supermanifold with super-space coordinates $Z^a = (X^m, \Theta^\alpha)$, $\Pi_i^m = \partial_i X^m - \bar{\Theta}\Gamma^m \partial_i \Theta$, where $m = 0, 1, \cdots, D-1$, $\alpha = 1, 2, \cdots, 2^{[\frac{D-1}{2}]}$, and Γ^m are the corresponding D-dimensional spacetime gamma matrices.

Note that Θ^α transforms w.r.t. fundamental spinorial representation of the $Spin(1, D-1) \simeq \overline{SO}(1, D-1)$ group.

In contradistinction to the bosonic brane case where, while spacetime curving, the $SO(1, D-1)$ group was replaced by the $Diff(D, R)$ one, here in the super brane case, the $Spin(1, D-1)$ group is to be replaced by the covering group of the General Coordinate Transformations group GCT, i.e. $\overline{Diff}(D, R)$.

There are no finite-dimensional representations of the $\overline{Diff}(D, R)$ group for $D \geq 3$ (cf. [3]), and thus *one cannot proceed as in the bosonic case by systematically replacing all local (flat-space) tensorial quantities by the appropriate world (curved-space) ones.*

2. Topology and dimensionality of the $\overline{Diff}(D,R)$ groups

Topology of the $Diff(D,R)$ group, as well as of its $GL(D,R)$ and $SL(D,R)$ linear subgroups, is determined by the topology of its maximal compact subgroup $SO(D)$, which is for $D \geq 3$ double connected ($G = KAN$; the Abelian A and nilpotent N subgroups are contractible to a point and therefore irrelevant for the topology questions). For pin/spin discussion cf. [4].

Thus, in the quantum case, all these groups, for $D \geq 3$, have double valued spinorial representations besides the usual tensorial ones.

2.1. $Diff(D,R)$, $SL(D,R)$ covering groups

The group-subgroup relations of the relevant groups for our considerations is as follows:

$$\begin{array}{ccccccc}
1 & & 1 & & 1 & & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
Z_2 & & Z_2 & & Z_2 & & Z_2 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\overline{Diff}(D,R) & \supset & \overline{GL}(D,R) & \supset & \overline{SL}(D,R) & \supset & Spin(1,D-1) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
Diff(D,R) & \supset & GL(D,R) & \supset & SL(D,R) & \supset & SO(1,D-1) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 & & 1 & & 1 & & 1
\end{array}$$

2.2. $\overline{Diff}(D,R)$, $\overline{SL}(D,R)$ groups of matrices

It turns out that *there are no finite-dimensional complex matrix groups* that contain the $SL(D,R) \supset SO(D)$, $D \geq 3$ group-chain as subgroups [3,5]. Moreover, $\overline{SL}(D,R)$, $D \geq 3$, the double covering of $SL(D,R)$, is a group of infinite matrices. Thus, *all spinorial representations of the* $\overline{Diff}(D,R)$, $\overline{GL}(D,R)$, $\overline{SL}(D,R)$ *groups, for* $D \geq 3$ *are infinite-dimensional*, and when restricted to the spacial $Spin(D-1)$ subgroup they contain all spins.

For example (cf. [6]), the simplest spinorial $\overline{SL}(3,R)$ representation from the (Ladder) Degenerate Series $D^{ladd}_{\overline{SL}(3,R)}(\frac{1}{2})$ contains the following $Spin(3) \simeq SU(2)$ representations:

$$D^{\frac{1}{2}}, \quad D^{\frac{5}{2}}, \quad D^{\frac{9}{2}}, \quad etc.,$$

while the representation $D^{pr}_{\overline{SL}(3,R)}(\frac{1}{2},\sigma_2,\delta_2)$ from the Principal Series contains:

$$D^{\frac{1}{2}}, \quad 2 \times D^{\frac{3}{2}}, \quad 3 \times D^{\frac{5}{2}}, \quad etc.$$

3. Generic curved target-spacetime embedding

In the standard approach to GR, spinors are defined w.r.t. a local tangent spacetime and transform w.r.t. the local Lorentz symmetry group

$Spin(1, D-1)$, i.e. $SL(2,C) \simeq Spin(1,3)$ for $D=4$. The curved spacetime (coordinates x^μ) and the local Minkowskian one (coordinates x^m)are mutually connected by the frame fields $e^a_\mu(x)$ (tetrads for $D=4$). Analogous situation persists in the metric-affine [7] and/or gauge-affine [8] case as well.

In the p-brane case, $Z^a = (X^m, \Theta^\alpha)$ defines a flat tangent superspace over a curved p-brane spacetime at ξ^i.

In a parallel to GR, spinors of a curved spacetime of coordinates X^m are to be defined w.r.t. a "new" tangent spacetime erected at every point X^m. In other words, *in order to define curved target-space spinors one has to construct a flat tangent space to the bosonic spacetime sector of a superbrane at every point* ξ, i.e. to a space that is itself a tangent space. Such a construction simply does not exists, therefore *one can not define spinors of a superbrane in a generic curved spacetime in the standard manner* [9,10]. However, superbranes can be defined (in the standard way) for special spacetimes (e.g. De Sitter, anti De Sitter, ...).

3.1. Restricted curving

Restricted curving is achieved by staying with finite tangent space $Spin(1, D-1)$ spinors, but restricting further curving of $\mathrm{M}^{1,D-1/r\cdot N}$ to such as can be described by that "diagonal" subgroup of $\overline{Diff}(M^{1,D-1/r\cdot N})$ that preserves the orbits of $Spin(1, D-1)$ when acting simultaneously on both even and odd sectors of superspace. In other words, *allow no linear transformations other than* $Spin(1, D-1)$ and adjoin a restricted set of non-linear ones leading to manifolds carrying the action of $Spin(1, D-1)$.

This method inherited from supergravity, has been used extensively in the attempts to curve the "target space" in superstrings and in supermembranes. It allows the highly restricted rheonomic curving undergone by superspace in supergravity in which the group parameters are constrained so that the odd coordinates are not gauged over.

The supertranslations act anholonomically as Lie derivatives ("anholonomized" general coordinate transformations), i.e. as part of the curved-space modified structure group acting as an effective fibre in the appropriate principle bundle.

The superbrane action for the *restricted curving* reads:

$$\begin{aligned} S \;&= \int d^{p+1}\xi \Big(\frac{1}{2}\sqrt{\gamma}\gamma^{ij}(\xi) E_i^{\tilde{a}} E_j^{\tilde{b}} g_{\tilde{m}\tilde{n}} \\ &\quad -\frac{1}{2}(p-1)\sqrt{-\gamma} + \frac{1}{(p+1)!}\epsilon^{i_1 i_2 \cdots i_{p+1}} E_{i_1}^{\tilde{a}_1} E_{i_2}^{\tilde{a}_2} \cdots E_{i_{p+1}}^{\tilde{a}_{p+1}} B_{\tilde{a}_{p+1}\cdots\tilde{a}_2\tilde{a}_1} \Big) \, . \end{aligned}$$

Here, the target space is a supermanifold with super-space coordinates $Z^{\tilde{a}} = (X^{\tilde{m}}, \Theta^\alpha)$, where $\tilde{m} = 0, 1, \ldots, D-1$ and $\alpha = 1, 2, \ldots, 2^{\left[\frac{D-1}{2}\right]}$. Furthermore, $E_i^a = (\partial_i Z^{\tilde{a}}) E_{\tilde{a}}^a(Z)$, where $E_{\tilde{a}}^a$ is the supervielbein and $a = (m\ \alpha)$ is the tangent-space index. In the standard superspace formalism one tends to describe Θ^{α} as a "world" fermionic coordinate, but this time in a very *restricted sense* only.

3.2. Non-linear curving

It is possible to use finite $Spin(1, D-1)$ spinors and represent the quotient $\overline{Diff}(D, R)/Spin(1, D-1)$ non-linearly over the $Spin(1, D-1)$ subgroup, following the pioneering work of Ogievetski and Polubarinov [11]. The result is effectively that of the restricted curving.

In the core of the corresponding non-linear representations is the non-linear realizer field (the metric)

$$g_{\tilde{m}\tilde{n}} = \eta_{mn} e^m_{\tilde{m}} e^n_{\tilde{n}}$$

that defines the linear–to–nonlinear transformation:

$$\begin{aligned} &L(g_{\tilde{m}\tilde{n}}) = exp(ig_{\tilde{m}\tilde{n}} T^{\tilde{m}\tilde{n}}), \qquad T^{\tilde{m}\tilde{n}} \in sl(D, R)/spin(1, D-1), \\ &\overline{Diff}(D, R)/Spin(1, D-1) \\ &\qquad = \overline{Diff}(D, R)/\overline{SL}(D, R) \times \overline{SL}(D, R)/Spin(1, D-1). \end{aligned}$$

Mathematical consistency of a curved superspace, i.e. a mutual relation of the bosonic sector given by non-linear curving and the fermionic sector given by $Spin(1, D-1)$ representations, imposes constraints equivalent to those of the restricted curving.

3.3. Generic curving

In the generic curving case we make use of (infinite) world spinors transforming w.r.t. the covering group of the General Coordinate Transformations, $\overline{GCT} = \overline{Diff}(D, R)$. This approach for the superstring was initiated together with Yuval Ne'eman [9]. There are two possible scenarios:

1. *"Minimal" solution – change in the fermionic sector only*:
Here, we replace

$$\Theta^{\alpha}, \quad \alpha = 1, \ldots, 2^{\left[\frac{D-1}{2}\right]}; \qquad \Theta \sim Rep(Spin(D))$$

by a corresponding world spinor

$$\Theta^{\tilde{A}}, \quad \tilde{A} = \frac{1}{2}, \ldots, \infty; \qquad \Theta \sim Rep(\overline{Diff}(D, R)).$$

2. *"Maximal" solution – "world" superspace formulation (generic curved superspace supersymmetry)*:
Here we replace

$$Z^a = (X^m, \Theta^{\alpha}); \qquad X,\ \Theta \sim Rep(Spin(D))$$

by a corresponding curved superspace coordinates

$$Z^{\tilde{I}} = (X^{\tilde{M}}, \Theta^{\tilde{A}}); \qquad X,\ \Theta \sim Rep(\overline{Diff}(D,R)),$$

that are of infinite range for both bosonic and fermionic coordinates. The appropriate replacements are as follows:

$$\begin{array}{ccccccccc}
Spin(1,D-1): & X^m & \eta_{mn} & \Theta^\alpha & \gamma^m & X^m & \eta_{mn} \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
\overline{SL}(D,R): & X^m & \eta_{mn} & \Theta^A & \Gamma^m_{(SL)} & X^M & \eta_{MN} \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
\overline{Diff}(D,R): & X^{\tilde{m}} & g_{\tilde{m}\tilde{n}} & \Theta^{\tilde{A}} & \Gamma^{\tilde{m}}_{(Diff)} & X^{\tilde{M}} & G_{\tilde{M}\tilde{N}}
\end{array}$$

4. Group-theoretical constructions for a generic curved spacetime superbrane embedding

(i) Spinorial and infinite-dimensional tensorial representations of the $\overline{SL}(D,R)$ group.
(ii) Spinorial and infinite-dimensional tensorial representations of the $\overline{Diff}(D,R)$ group
(iii) Dirac-like equation for $\overline{SL}(D,R)$ and $\overline{Diff}(D,R)$ spinors, i.e. the corresponding (infinite) $\Gamma^m_{(SL)}$, $\Gamma^{\tilde{m}}_{(Diff)}$ generalizations of the γ matrices, which a required for the expressions such as:

$$E_i^m \ \rightarrow \ E_i^{\tilde{m}} = \partial_i X^{\tilde{m}} - i\overline{\Theta}^{\tilde{A}} (\Gamma^{\tilde{m}}_{(Diff)})_{\tilde{A}\tilde{B}} \partial_i \Theta^{\tilde{B}},$$

(iv) Infinite super algebras that generalize the Virasoro and Neveu-Schwarz-Ramond ones and contain respectively the $SL(D,R)$ and $\overline{SL}(D,R)$ tensorial and spinorial adjoint representations as subalgebras, thus providing for a complete superspace supersymmetry formulation.

5. $\overline{SL}(D,R)$ Spinorial representations - $Spin(D)$ multiplicity free case

The $\overline{SL}(D,R)$ group can be contracted (a la Wigner-Inönü) w.r.t. its $\overline{SO}(D)$ subgroup to yield the semidirect-product group $T' \wedge \overline{SO}(D)$. T' is an Abelian group generated by operators U_{mn}, which form an $\overline{SO}(D)$ second rank symmetric operator with commutation relations

$$[J,\ J] \subset J, \qquad [J,\ U] \subset U, \qquad [U,\ U] = 0.$$

An efficient way of constructing explicitly the $\overline{SL}(D,R)$ infinite-dimensional representations is based on the decontraction formula, which is an inverse of the Wigner-Inönü contraction. According to the decontraction formula, the following operators [12]

$$T_{mn} = pU_{mn} + \frac{i}{2\sqrt{U\cdot U}}\left[C_2(\overline{SO}(D)),\ U_{mn}\right],$$

together with J_{mn} form the $\overline{SL}(D,R)$ algebra. The parameter p is an arbitrary complex number, and $C_2(\overline{SO}(D))$ is the $\overline{SO}(D)$ second-rank Casimir operator.

For the representation Hilbert space we take the homogeneous space of L^2 functions of the maximal compact subgroup $\overline{SO}(D)$ parameters. The $\overline{SO}(D)$ representation labels are given either by the Dynkin labels $(\lambda_1,\lambda_2,\ldots,\lambda_r)$ or by the highest weight vector which we denote by $\{j\} = \{j_1, j_2, \ldots, j_r\}$, $r = \left[\frac{D}{2}\right]$.

The $\overline{SL}(D,R)$ commutation relations are invariant w.r.t. an automorphism defined by:

$$s(J) = +J, \qquad s(T) = -T.$$

This enables us to define an 's-parity' to each $\overline{SO}(D)$ representation of an $\overline{SL}(D,R)$ representation. In terms of Dynkin labels we find

$$\begin{aligned}
s(D_2) &= (-)^{\frac{1}{2}(\lambda_1+\lambda_2-\epsilon)},\\
s(D_{n\geq 3}) &= (-)^{\lambda_1+\lambda_2+\ldots+\lambda_{n-2}+\frac{1}{2}(\lambda_n-\lambda_{n-1}-\epsilon)}\\
s(B_1) &= (-)^{\frac{1}{2}(\lambda_1-\epsilon)}\\
s(B_{n\geq 2}) &= (-)^{\lambda_1+\lambda_2+\ldots+\lambda_{n-1}+\frac{1}{2}(\lambda_n-\epsilon)}
\end{aligned}$$

where $\epsilon = 0$ $(+1)$ if λ is even (odd).

For the $\frac{1}{2}(D+2)(D-1)$-dimension representation of $\overline{SO}(D)$, i.e. for $(20\ldots0) = \square\square$, one has $s(20\ldots0) = +1$. A basis of an $\overline{SO}(D)$ representation is provided by the Gel'fand - Zetlin pattern characterized by the maximal weight vectors of the subgroup chain $\overline{SO}(D) \supset \overline{SO}(D-1) \supset \overline{SO}(2)$. We write the basic vectors as $\left|\begin{smallmatrix}\{j\}\\ \{m\}\end{smallmatrix}\right\rangle$, where $\{m\}$ corresponds to $\overline{SO}(D-1) \supset \overline{SO}(D-2) \supset \overline{SO}(2)$ subgroup chain weight vectors.

The Abelian group generators $\{U\} = U^{[\square\square\}}_{\{\mu\}}$ can be, in the case of multiplicity free representations, written in terms of the $\overline{SO}(D)$-Wigner functions as follows $U^{\{\square\square\}}_{\{\mu\}} = D^{\{\square\square\}}_{\{0\}\{\mu\}}(\phi)$. It is now rather straightforward to determine the noncompact operators matrix elements, which read [5,12]

$$\left\langle \begin{matrix}\{j'\}\\ \{m'\}\end{matrix} \left| T^{\{\square\square\}}_{\{\mu\}} \right| \begin{matrix}\{j\}\\ \{m\}\end{matrix} \right\rangle = \begin{pmatrix}\{j'\} & \{\square\square\} & \{j\}\\ \{m'\} & \{\mu\} & \{m\}\end{pmatrix} < \{j'\}||T^{\{\square\square\}}||\{j\} >,$$

$$< \{j'\}||T^{\{\square\square\}}||\{j\} >= \sqrt{dim\{j'\}dim\{j\}}\left\{p+\frac{1}{2}(C_2(\{j'\})-C_2(\{j\}))\right\}$$
$$\times\begin{pmatrix} \{j'\} & \{\square\square\} & \{j\} \\ \{0\} & \{0\} & \{0\} \end{pmatrix}.$$

$\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$ is the appropriate "$3j$" symbol for the $\overline{SO}(D)$ group. For the multiplicity free $\overline{SL}(D,R)$ representations each $\overline{SO}(D)$ sub-representation appears at most once and has the same s-parity.

6. $\overline{Diff}(D,R)$ representations for world spinor fields

The world spinor fields transform w.r.t. $\overline{Diff}(D,R)$ as follows

$$(D(a,\bar{f})\Psi_M)(x) = (U_{\overline{Diff}_0(D,R)}(\bar{f}))^N_M \Psi_N(f^{-1}(x-a)),$$
$$(a,\bar{f}) \in T_D \wedge \overline{Diff}_0(D,R),$$

where $\overline{Diff}_0(D,R)$ is the homogeneous part of $\overline{Diff}(D,R)$, while f is the element corresponding to $\bar{f}$ in $Diff(D,R)$. The $D_{\overline{Diff}_0(D,R)}$ representations can be reduced to direct sum of infinite-dimensional $\overline{SL}(D,R)$ representations. We consider here those representations of $\overline{Diff}_0(D,R)$ that are nonlinearly realized over the maximal linear subgroup $\overline{SL}(D,R)$.

Provided the relevant $\overline{SL}(D,R)$ representations are known, one can first define the corresponding general/special affine spinor fields, $\Psi_A(x)$, and than make use of the infinite-component pseudo-frame fields $E^A_{\tilde{A}}(x)$ (linear-to-nonlinear mapping) [13,14],

$$\Psi_{\tilde{A}}(x) = E^A_{\tilde{A}}(x)\Psi_A(x), \qquad E^A_{\tilde{A}}(x) \sim \overline{Diff}_0(D,R)/\overline{SL}(D,R)$$

where $\Psi_{\tilde{A}}(x)$ and $\Psi_A(x)$ are the world (curved-space) and local Affine (flat-space) spinor fields respectively.

Their infinitesimal transformations are

$$\delta E^A_{\tilde{A}}(x) = i\epsilon^a_b(x)\{Q^b_a\}^A_B E^B_{\tilde{A}}(x) + \partial_\mu \xi^\nu e^a_\nu e^\mu_b \{Q^a_b\}^A_B E^B_{\tilde{A}}(x),$$

where ϵ^a_b and ξ^μ are group parameters of $\overline{SL}(D,R)$ and $\overline{Diff}(D,R)/\overline{Diff}_0(D,R)$ respectively, while e^a_ν are the standard n-bine frame fields.

The transformation properties of the world spinor fields themselves are given as follows:

$$\delta\Psi^{\tilde{A}}(x) = i\{\epsilon^a_b(x)E^{\tilde{A}}_A(x)(Q^b_a)^A_B E^B_{\tilde{B}}(x) + \xi^\mu[\delta^{\tilde{A}}_{\tilde{B}}\partial_\mu + E^{\tilde{A}}_B(x)\partial_\mu E^B_{\tilde{B}}(x)]\}\Psi^{\tilde{B}}(x).$$

The $(Q^b_a)^{\tilde{A}}_{\tilde{B}} = E^{\tilde{A}}_{A}(x)(Q^b_a)^A_B E^B_{\tilde{B}}(x)$ is the holonomic form of the $\overline{SL}(D,R)$ generators given in terms of the corresponding anholonomic ones. The $(Q^b_a)^{\tilde{A}}_{\tilde{B}}$ and $(Q^b_a)^A_B$ act in the spaces of spinor fields $\Psi_{\tilde{A}}(x)$ and $\Psi_A(x)$ respectively.

The above outlined construction allows one to define a fully $\overline{Diff}(D,R)$ covariant Dirac-like wave equation for the corresponding world spinor fields provided a Dirac-like wave equation for the $\overline{SL}(D,R)$ group is known. In other words, one can lift up an $\overline{SL}(D,R)$ covariant equation of the form

$$(ie^{\tilde{m}}_m(\Gamma^m_{(SL)})^B_A \partial_{\tilde{m}} - \mu)\Psi_B(x) = 0,$$

to a $\overline{Diff}(n,R)$ covariant equation

$$(ie^{\tilde{m}}_m E^A_{\tilde{A}}(\Gamma^m_{(SL)})^B_A E^{\tilde{B}}_B \partial_m - \mu)\Psi_{\tilde{B}}(x) = 0,$$

where the former equation exists provided a spinorial $\overline{SL}(D,R)$ representation for Ψ is given, such that the corresponding representation Hilbert space is invariant w.r.t. $\Gamma^m_{(SL)}$ action. Thus, the crucial step towards a Dirac-like world spinor equation is a construction of the vector operator $\Gamma^m_{(SL)}$ in the space of $\overline{SL}(D,R)$ spinorial representations [5,15].

7. $\Gamma^m_{(SL)}$ for a Dirac-like world spinor equation

It is well known that one can satisfy the commutation relations

$$[M_{mn}, \Gamma_p] = i(\eta_{mp}\Gamma_n - \eta_{np}\Gamma_m), \qquad M_{mn} \in spin(1, D-1),$$

in the Hilbert space of $Spin(1,D-1)$ irreducible representations. However, in order for an $Spin(1,D-1)$ vector to be an $\overline{SL}(D,R)$ vector as well, it has to satisfy additionally the following commutation relations

$$[T_{mn}, \Gamma_p] = i(\eta_{mp}\Gamma_n + \eta_{np}\Gamma_m), \qquad T_{mn} \in sl(D,R)/spin(1,D-1).$$

This is a much harder task to achieve [16], and in principle, one can find nontrivial solutions only for particular representation spaces.

Example: For $SL(3,R)$ finite-dimensional reps., one can satisfy the above algebraic conditions only in the special case of a reducible representation of Young tableaux $[2q+1,q] \oplus [2q+1,q+1]$.

The multiplicity free (ladder) unitary (infinite-dimensional) irreducible representations

$$D^{ladd)}_{SL(3,R)}(0,\sigma_2), \qquad \{j\} = \{0,2,4,\ldots\},$$

and

$$D^{ladd)}_{SL(3,R)}(1,\sigma_2), \qquad \{j\} = \{1,3,5,\ldots\},$$

can be viewed as limiting cases of the series of finite-dimensional representations $[0,0]$, $[2,0]$, $[4,0]$, ..., and $[1,0]$, $[3,0]$, $[5,0]$, ... respectively.

Upon the coupling with the $SL(3,R)$ vector representation $[1,0]$, one has $[1,0]\otimes[2n,0] \supset [2n+1,0]$, and $[1,0]\otimes[2n+1,0] \supset [2n+2,0]$, $(n=0,1,2,\ldots)$. It seems possible to represent the vector operator Γ^m in the Hilbert space of the $D^{ladd)}_{SL(3,R)}(0,\sigma_2)\oplus D^{ladd)}_{SL(3,R)}(1,\sigma_2)$ representation. However, the resulting representations obtained after the Γ^m action have different values of the Casimir operators and thus define new (mutually orthogonal) Hilbert spaces.

7.1. Algebraic solution for Γ^m

A rather efficient way to impose additional algebraic constraints on the vector operator Γ consists in embedding it into a non-Abelian Lie-algebraic structure. The minimal semi-simple Lie algebra that contains both the $sl(D,R)$ algebra and the corresponding vector operator Γ is given by the $sl(D+1,R)$ algebra. There are two $SL(D,R)$ vector operators: A^m and B_m, $m=1,2,\ldots D$, in the $sl(D+1,R)$ algebra that transform w.r.t. $[1,0]$ and $[1,1,\ldots,1]$ representations of $SL(D,R)$ respectively. Components of each of them mutually commute, while their commutator yields the $SL(D,R)$ generators themselves, i.e.

$$[A^m,A^n]=0,\quad [B_m,B_m]=0,\quad [A^m,B_n]=iQ^m_n.$$

Now, due to the $sl(D+1,R)$ algebra constraints, any irreducible representation (or an arbitrary combination of them) of $SL(D+1,R)$ defines a Hilbert space that is invariant under the action of an $SL(D,R)$ vector operator Γ^m proportional to A or B.

7.2. Γ^m construction in the $D=3$ case

$\overline{SL}(3,R)$ is embedded into $\overline{SL}(4,R)$, and a reduction of the spinorial irreducible representations (multiplicity free Discrete Series) of the latter group down to $D=3$ is as follows [15]:

$$\begin{aligned} D^{disc}_{\overline{SL}(4,R)}(j_0,0) \quad &\supset \quad \bigoplus_{j=1}^{\infty} D^{disc}_{\overline{SL}(3,R)}(j_0;\sigma_2(j),\delta_1(j)) \\ D^{disc}_{\overline{SL}(4,R)}(0,j_0) \quad &\supset \quad \bigoplus_{j=1}^{\infty} D^{disc}_{\overline{SL}(3,R)}(j_0;\sigma_2(j),\delta_1(j)) \end{aligned}$$

The vector operator is either $\Gamma\sim A$ or $\Gamma\sim B$. The explicit form of the $A+B$ operator (in the spherical basis of the $Spin(4)=SU(2)\otimes SU(2)$ group) is well known, while the above embedding approach yields a closed expressions for the $A-B$ operator as well. In particular,

$$\left\langle \begin{array}{c} J' \\ M' \end{array} \middle| (A-B)_\alpha \middle| \begin{array}{c} J \\ M \end{array} \right\rangle$$

$$= i\sqrt{6}(-)^{J'-M'}\sqrt{(2J'+1)(2J+1)}\begin{pmatrix} J' & 1 & J \\ -M' & \alpha & M \end{pmatrix}$$
$$\times \begin{Bmatrix} j_1' & 1 & j_1 \\ j_2' & 1 & j_2 \\ J' & 1 & J \end{Bmatrix} < j_1'j_2'||Z||j_1j_2 >,$$

where, $< j_1'j_2'||Z||j_1j_2 >$ are known reduced matrix elements of the $\overline{SL}(4,R)$ noncompact operators $Z_{\alpha\beta}$.

Finally, we can write an $\overline{SL}(3,R)$ covariant spinorial wave equation in the form

$$(i\Gamma^m\partial_m - \mu)\Psi(x) = 0,$$
$$\Psi \sim D^{disc}_{\overline{SL}(4,R)}(j_0,0),\ D^{disc}_{\overline{SL}(4,R)}(0,j_0),$$
$$\Gamma^m = \frac{1}{2}(J^{(1)m} - J^{(2)m} + (A-B)^m), \qquad m = 0,1,2$$

The matrix elements of all operators defining the $\overline{SL}(3,R)$ vector operator Γ^m in the infinite-component representation of the field $\Psi(x)$ are explicitly constructed.

References

[1] E. Bergshoeff, E. Sezgin and P.K. Townsend, "Supermembranes and eleven-dimensional supergravity" *Phys. Lett. B* **189** (1987) 75.

[2] M.J. Duff, "Supermembranes", Lectures given at the Theoretical Advanced Study Institute in Elementary Particle Physics (TASI 96), Boulder, 1996, hep-th/9611203.

[3] Y. Ne'eman and Dj. Šijački, "$\overline{GL}(4,R)$ Group-Topology, Covariance and Curved-Space Spinors", *Int. J. Mod. Phys.* **A 2** (1987) 1655.

[4] M. Berg, C. DeWitt-Morette, S. Gwo and E. Kramer, "The Pin Groups in Physics: C, P, and T", *Rev. Math. Phys.* **13** (2001) 953.

[5] Dj, Šijački, "Affine Particles and Fields" *Int. J. Geom. Meth. Mod. Phys.* **2** (2005) 189.

[6] Dj. Šijački, "The Unitary Irreducible Representations of $\overline{SL}(3,R)$", *J. Math. Phys.* **16** (1975) 298.

[7] F.W. Hehl, G.D. Kerlick and P. von der Heyde, "On a New Metric Affine Theory of Gravitation", *Phys. Lett.* **B 63** (1976) 446; F.W. Hehl, J.D. McCrea, E.W. Mielke and Y. Ne'eman, "Metric Affine Gauge Theory of Gravity: Field Equations, Noether Identities, World Spinors, and Breaking of Dilation Invariance", *Phys. Reports* **258** (1995) 1.

[8] Y. Ne'eman and Dj. Šijački, "Unified Affine Gauge Theory of Gravity and Strong Interactions with Finite and Infinite $\overline{GL}(4,R)$ Spinor

Fields", *Ann. Phys. (N.Y.)* **120** (1979) 292; Y. Ne'eman and Dj. Šijački, "Gravity from Symmetry Breakdown of a Gauge Affine Theory", *Phys. Lett.* **B 200** (1988) 489.

[9] Y. Ne'eman and Dj. Šijački, "Spinors for Superstring in a Generic Curved Space", *Phys. Lett.* **B 174** (1986) 165.

[10] Y. Ne'eman and Dj. Šijački, "Curved Space-Time and Supersymmetry Treatments for p-Extendons", *Phys. Lett.* **B 206** (1988) 458.

[11] V. O. Ogievetskii and I. V. Polubarinov, *JETP* **48** (1965) 1625.

[12] Dj. Šijački, "$\overline{SL}(n, R)$ Spinors for Particles, Gravity and Superstrings", in *Spinors in Physics and Geometry*, A. Trautman and G. Furlan eds. (World Scientific Pub., 1988) 191; Dj, Šijački, "Generic Curved Space Superextendon Theories", in *Supermembranes and Physics in 2+1 Dimensions*, eds. M. Duff, C. Pope and E. Sezgin (World Scientific Pub., 1990) 213.

[13] Y. Ne'eman and Dj. Šijački, "$\overline{SL}(4, R)$ World Spinors and Gravity", *Phys. Lett.* **B 157** (1985) 275.

[14] Dj. Šijački, "World Spinors Revisited", *Acta Phys. Polonica* **B 29** (1998) 1089.

[15] Dj. Šijački, "$\overline{SL}(4, R)$ Embedding for a 3D World Spinor Equation", *Class. Quant. Grav.* **21** (2004) 4575.

[16] I.Kirsch and Dj. Šijački, "From Poincaré to Affine Invariance: How does the Dirac Equation Generalize?", *Class. Quant. Grav.* **19** (2002) 3157.