Green Ansatz for Deformed Parastatistics*

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Abstract

We make use of the noncocommutative Hopf structure on the deformed parastatistics algebras $\mathfrak{pB}_q(n)$ and $\mathfrak{pF}_q(n)$ in order to construct a deformed version of the Green ansatz.

Green introduced the parafermionic and parabosonic commutation relations [1]

$$\begin{array}{rclrcl} [[a_i^+, a_j^-]_\mp, a_k^+] &=& 2\delta_{jk} a_i^+ \,, & [[a_i^+, a_j^-]_\mp, a_k^-] &=& -2\delta_{ik} a_j^- \,, \\ [[a_i^+, a_j^+]_\mp, a_k^+] &=& 0 \,, & [[a_i^-, a_j^-]_\mp, a_k^-] &=& 0 \,, \end{array} \qquad (1)$$

as a quantization scheme more general than the canonical commutations relations for fermions and bosons keeping the harmony with the quantum mechanical equations of motion. To each degree of freedom one associates a partial hamiltonian $h_i = \frac{1}{2}[a_i^+, a_i^-]_\mp$ and number operator $N_i = a_i^+ a_i^-$. The full hamiltonian h defines equations of motion coinciding with these for the ordinary fermions and bosons as seen from (1)

$$h = \sum\nolimits_i \quad h_i \,, \qquad\qquad [h, a^\pm] = \pm a^\pm \quad (= \frac{d}{dt}\, a^\pm) \,.$$

The relations (1) taken with the upper (lower) sign define a (super)algebra which will be refered to as parafermionic $\mathfrak{pF}$ (parabosonic $\mathfrak{pB}$). We denote $\mathfrak{pF}(n)$ and $\mathfrak{pB}(n)$ the parastatistics algebras with finite number n degrees of freedom.

The unitary representations π_p of the parastatistics algebras $\mathfrak{pB}(n)$ and $\mathfrak{pF}(n)$ with unique vacuum $|0\rangle^{(p)}$ are indexed by a non-negative integer p

$$\pi_p(a_i^-)|0\rangle^{(p)} = 0 \qquad\qquad \pi_p(a_i^-)\pi_p(a_j^+)|0\rangle = p\delta_{ij}|0\rangle^{(p)} \qquad (2)$$

* The talk is based on the joint work [10] with Boyka Aneva.

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called the *order of parastatistics.* In the representation π_p of $\mathfrak{pF}(n)$ ($\mathfrak{pB}(n)$) the hamiltonian h_i* and the number operator N_i are related by $2h_i = 2N_i \mp p$ and the constant $\mp\frac{p}{2}$ plays the role of the energy of the vacuum associated to each degree i. The bosonic $\mathfrak{B}(n)$ and the fermionic $\mathfrak{F}(n)$ algebras arise as special cases when $p = 1$.

It turns out that the parafermi algebra $\mathfrak{pF}(n)$ is isomorphic to the universal enveloping algebra $U(\mathfrak{so}_{2n+1})$[2] orthogonal algebra $so(2n+1)$ while the parabose algebra $\mathfrak{pB}(n)$ is isomorphic to the universal enveloping algebra (UEA) $U(\mathfrak{osp}_{1,2n})$algebra $osp(1|2n)$ [3]

$$\mathfrak{pF}(n) \simeq U(\mathfrak{so}_{2n+1}) \qquad\qquad \mathfrak{pB}(n) \simeq U(\mathfrak{osp}_{1,2n}). \tag{3}$$

The idea of the deformation of the parastatistics algebras [8, 9] is to "quantize" the isomorphism (3), i.e., to deform the relations (1) in such a way that the deformed algebra is isomorphic to the Quantized UEA [4, 5, 6, 7]

$$\mathfrak{pF}_q(n) \simeq U_q(\mathfrak{so}_{2n+1}) \qquad\qquad \mathfrak{pB}_q(n) \simeq U_q(\mathfrak{osp}_{1,2n}). \tag{4}$$

The answer is given by the following theorem extending the results of [8], [9].

THEOREM [10] *The deformed parafermionic $\mathfrak{pF}_q(n)$ (parabosonic $\mathfrak{pB}_q(n)$) algebra is the associative (super)algebra generated by the generators $a_i^\pm$ and $q^{\pm h_i}$ for $i = 1, \ldots, n$ subject to the relations*

$$q^{h_i} a_j^\pm q^{-h_i} = q^{\pm\delta_{ij}} a_j^\pm\,, \qquad q^{\pm h_i} q^{h_j} = q^{h_j} q^{\pm h_i}\,, \qquad q^{\pm h_i} q^{\mp h_i} = 1\,, \tag{5}$$

$$[\![a_i^+, a_i^-]\!] = \frac{q^{h_i} - q^{-h_i}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} = [2h_i] \qquad \left([x] := \frac{q^{\frac{x}{2}} - q^{-\frac{x}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}\right)\,, \tag{6}$$

$$[\![[\![a_i^+, a_j^-]\!], a_k^+]\!]_{q^{-\delta_{ik}\sigma_{j,k}}} = [2]\ \delta_{jk} a_i^+ q^{\sigma_{i,j} h_j} + (q - q^{-1})\theta_{i,j;k} a_i^+ [\![a_k^+, a_j^-]\!], \tag{7}$$

$$[\![[\![a_i^+, a_j^-]\!], a_k^-]\!]_{q^{-\delta_{jk}\sigma_{i,k}}} = -[2]\delta_{ik} a_j^- q^{-\sigma_{i,j} h_i} - (q - q^{-1})\theta_{j,i;k} [\![a_i^+, a_k^-]\!] a_j^-\,, \tag{8}$$

together with the analogues of the Serre relations

$$[\![a_{i_2}^\pm, [\![a_{i_1}^\pm, a_{i_3}^\pm]\!]]\!]_{q^{-2}} - q^{-1}[\![[\![a_{i_1}^\pm, a_{i_2}^\pm]\!], a_{i_3}^\pm]\!] = 0\,, \qquad i_1 < i_2 \le i_3\,, \tag{9}$$

$$[\![[\![a_{i_1}^\pm, a_{i_3}^\pm]\!], a_{i_2}^\pm]\!]_{q^{-2}} - q^{-1}[\![a_{i_1}^\pm, [\![a_{i_3}^\pm, a_{i_2}^\pm]\!]]\!] = 0\,, \qquad i_1 \le i_2 < i_3\,. \tag{10}$$

The generators $a_i^\pm$ are taken to be even $deg(a_i^\pm) = \bar{0}$ (odd $deg(a_i^\pm) = \bar{1}$) and $[\![a, b]\!]_x = ab - (-1)^{deg(a)deg(b)} x\, ba$. The symbols $\theta_{i,j;k}$, $\sigma_{i,j}$ stay for $\theta_{i,j;k} = \frac{1}{2}\epsilon_{ij}\,\epsilon_{ijk}(\epsilon_{jk} - \epsilon_{ik})$, $\sigma_{i,j} = \epsilon_{ij} + \delta_{ij}$ or $\sigma_{i,j} = \epsilon_{ij} - \delta_{ij}$. (1

1) ϵ stays for the Levi-Civita symbol with $\epsilon_{ij} = 1$ for $i < j$. The tensor $\theta_{i,j;k} = -\theta_{j,i;k}$ is vanishing except for k in between i and j, when it takes values ± 1 for $i < k < j$ and $i > k > j$, respectively.

The Lie (super)algebras $\mathfrak{so}_{2n+1}$ and $\mathfrak{osp}_{1,2n}$ are determined by their Dynkin diagrams

$$\circ_1 \longrightarrow \circ_2 \longrightarrow \quad \longrightarrow \circ_{n-1} \Longrightarrow \circ_n \qquad B(n) \equiv \mathfrak{so}_{2n+1} \tag{11}$$

$$\circ_1 \longrightarrow \circ_2 \longrightarrow \quad \longrightarrow \circ_{n-1} \Longrightarrow \bullet_n \qquad B(0,n) \equiv \mathfrak{osp}_{1,2n} \tag{12}$$

the corresponding Cartan matrices coincide and the relations of QUE algebras are the same modulo the $\mathbb{Z}_2$-grading.

The Chevalley generators $E_i = E_{\alpha_i}$ and $H_i = H_{\alpha_i}$ correspond to the simple roots α_i expressible in the orthogonal basis ε_i of the root space as

$$\alpha_i = \varepsilon_i - \varepsilon_{i+1} \qquad 1 \le i \le n-1, \qquad\qquad \alpha_n = \varepsilon_n \tag{13}$$

This change is invertible which means that ε_i is another basis in the root space

$$\varepsilon_i = \alpha_i + \alpha_{i+1} + \ldots + \alpha_n \tag{14}$$

The generators $a_i^{\pm} = E_{\pm\varepsilon_i}$ and $h_i = H_{\varepsilon_i}$ attached to the roots ε_i provide a "physical" basis for the Lie (super)algebra which is an alternative of the Chevalley basis.

The connection of the new basis with the Chevalley basis worked out in [8], [9] reads $q^{h_i} = q^{H_i + \ldots + H_n}$,

$$\begin{aligned} a_i^+ &= E_{\alpha_i + \alpha_{i+1} \ldots + \alpha_n} &&= [E_i, [E_{i+1}, \ldots [E_{n-1}, E_n]_{q^{-1}} \ldots]_{q^{-1}}]_{q^{-1}}\,, \\ a_i^- &= E_{-\alpha_n - \alpha_{n-1} \ldots - \alpha_i} &&= [[\ldots [E_{-n}, E_{-n+1}]_q \ldots, E_{-(i+1)}]_q, E_{-i}]_q\,. \end{aligned} \tag{15}$$

The QUE algebras $U_q(\mathfrak{so}_{2n+1})$ and $U_q(\mathfrak{osp}_{1,2n})$ are endowed with the Drinfeld-Jimbo coalgebraic structure [4], [5] termed in the Chevalley basis

$$\begin{aligned} &\Delta H_i = H_i \otimes 1 + 1 \otimes H_i\,, && S(H_i) = -H_i\,, && \epsilon(H_i) = 0\,, \\ &\Delta E_i = E_i \otimes 1 + q^{H_i} \otimes E_i\,, && S(E_i) = -q^{-H_i} E_i\,, && \epsilon(E_i) = 0\,, \\ &\Delta E_{-i} = E_{-i} \otimes q^{-H_i} + 1 \otimes E_{-i}\,, && S(E_{-i}) = -E_{-i} q^{H_i}\,, && \epsilon(E_{-i}) = 0\,. \end{aligned} \tag{16}$$

The connection between the Chavalley and parastatistics basis (15) of $U_q(\mathfrak{so}_{2n+1})$ and $U_q(\mathfrak{osp}_{1,2n})$ induce a Hopf structure on the operators $a_i^{\pm}$ and q^{h_i} which we are going to explicite.

The QUE algebras $U_q(\mathfrak{g})$ of a Lie (super)algebra $\mathfrak{g}$ is generated by the elements of an upper-triangular and a lower triangular matrices $L^{(+)}$ and $L^{(-)}$

$$R^{(+)} L_1^{(\pm)} L_2^{(\pm)} = L_2^{(\pm)} L_1^{(\pm)} R^{(+)}\,, \qquad R^{(+)} L_1^{(+)} L_2^{(-)} = L_2^{(-)} L_1^{(+)} R^{(+)}\,, \tag{17}$$

where $L_1^{(\pm)} = 1 \otimes L^{(\pm)}$, $L_2^{(\pm)} = L^{(\pm)} \otimes 1$ and $R^{(+)} = PRP$ is the corresponding R-matrix for $U_q(\mathfrak{g})$ [6]. The $L^{(+)}$ and $L^{(-)}$ matrices for $U_q(\mathfrak{so}_{2n+1})$ and

$U_q(\mathfrak{osp}_{1,2n})$ are surprisingly simpe when written down in terms of the new basis. The $n \times n$ minor of the $(n+1) \times (n+1)$ matrix $L^{(+)}$ reads

$$\left(L_{ij}^{(+)}\right)_{1\le i,j\le n+1} = \begin{pmatrix} q^{h_1} & \omega[\![a_1^+, a_2^-]\!] & \omega[\![a^{+1}, a_3^-]\!] & \dots & \omega[\![a_1^+, a_n^-]\!] & ca_1^+ \\ 0 & q^{h_2} & \omega[\![a_2^+, a_3^-]\!] & \dots & \omega[\![a_2^+, a_n^-]\!] & ca_2^+ \\ 0 & 0 & q^{h_3} & \dots & \omega[\![a_3^+, a_n^-]\!] & ca_3^+ \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & q^{h_n} & ca_n^+ \\ 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix} \tag{18}$$

where $\omega = q^{\frac{1}{2}} - q^{-\frac{1}{2}}$. The coefficient $c = q^{-\frac{1}{2}}(q - q^{-1})$. One has $L_{ji}^{(-)} = (L_{ij}^{(+)})^*$.
The coproduct Δ on $U_q(\mathfrak{g})$ is expressed through the succinct formulae [6]

$$\Delta L_{ik}^{(\pm)} = \sum_j L_{ij}^{(\pm)} \otimes L_{jk}^{(\pm)}\,, \tag{19}$$

which allows to obtain the coproduct on the parastatistics basis as multiplication of two upper(lower)-triangular matrices (18). This yields
$\Delta q^{\pm h_i} = q^{\pm h_i} \otimes q^{\pm h_i}$

$$\Delta a_i^+ = a_i^+ \otimes 1 + q^{h_i} \otimes a_i^+ + \omega \sum_{i<j\le n} [\![a_i^+, a_j^-]\!] \otimes a_j^+\,, \tag{20}$$

$$\Delta a_i^- = a_i^- \otimes q^{-h_i} + 1 \otimes a_i^- - \omega \sum_{i<j\le n} a_j^- \otimes [\![a_j^+, a_i^-]\!]\,. \tag{21}$$

With similar calculations one find the antipode and the counit(see [10]).

The representations π_p of the deformed parastatistics algebras $\mathfrak{p}\mathfrak{B}_q(n)$ and $\mathfrak{p}\mathfrak{F}_q(n)$ are again indexed with the order of the parastatistics p. Due to eq.(6) the spectrum of the energy gets deformed and "quantum" version of (2) holds

$$\pi_p(a_i^-)\pi_p(a_j^+)|0\rangle^{(p)} = [p]\delta_{ij}|0\rangle^{(p)}\,. \tag{22}$$

Let us give the simplest non-trivial example, the fundamental representaion π_1 for $\mathfrak{p}\mathfrak{B}_q(n)$. The algebra of the q-deformed bosonic oscillators $\mathfrak{B}_q(n)$ arises as a particular representation π_1 of parabosonic order $p=1$ of the $\mathfrak{p}\mathfrak{B}_q(n)$ (for details see [8])

$$\left.\begin{array}{rclrcl} \underline{a}_i^- \underline{a}_i^+ - q\underline{a}_i^+ \underline{a}_i^- & = & q^{-\underline{N}_i} & \underline{a}_i^- \underline{a}_i^+ - q^{-1}\underline{a}_i^+ \underline{a}_i^- & = & q^{\underline{N}_i} \\ \underline{a}_i^+ \underline{a}_j^+ - q^{\epsilon_{ij}}\underline{a}_j^+ \underline{a}_i^+ & = & 0 & \underline{a}_i^+ \underline{a}_j^- - q^{\epsilon_{ji}}\underline{a}_j^- \underline{a}_i^+ & = & 0 \\ \underline{a}_i^- \underline{a}_j^- - q^{\epsilon_{ij}}\underline{a}_j^- \underline{a}_i^- & = & 0 & \underline{a}_i^- \underline{a}_j^+ - q^{\epsilon_{ji}}\underline{a}_j^+ \underline{a}_i^- & = & 0 \end{array}\right\} \tag{23}$$

We have adopted the notaion $\pi_1(x) = \underline{x}$ and use $\underline{N}_i = \underline{h}_i - \frac{1}{2}$.

Remark. As distinct from the relations of $\mathfrak{pB}_q(n)$ and $\mathfrak{pF}_q(n)$ the relations of the q-deformed bosonic and fermionic oscillators do not define Hopf ideals and consequently there is no Hopf structure on $\mathfrak{B}_q(n)$ and $\mathfrak{F}_q(n)$.

The Green ansatz states that the parafermi (parabose) oscillators a_i^+ and a_i^- can be represented as sums of p fermi (bose) oscillators

$$\pi_p(a_i^+) = \sum_{r=1}^{p} a_{i\,(r)}^+\,, \qquad \pi_p(a_i^-) = \sum_{r=1}^{p} a_{i(r)}^-\,, \tag{24}$$

satisfying quadratic commutation relations of the same type (i.e., fermi for parafermi and bose for parabose) for equal indices (r)

$$[a_{i(r)}^-,\, a_{k\,(r)}^+]_\pm = \delta_{ik}, \qquad [a_{i(r)}^-, a_{k(r)}^-]_\pm = [a_{i\,(r)}^+, a_{k\,(r)}^+]_\pm = 0, \tag{25}$$

and of the opposite type for the different indices

$$[a_{i(r)}^-, a_{k(s)}^-]_\mp = [a_{i\,(r)}^+, a_{k\,(s)}^+]_\mp = [a_{i(r)}^-,\, a_{k\,(s)}^+]_\mp = 0 \qquad r \neq s. \tag{26}$$

The upper (lower) signs stay for the parafermi (parabose) case.

Every representation of $\mathfrak{pF}(n)$ and $\mathfrak{pB}(n)$ arises as tensor power of the fundamental, oscillator representation. But the tensor power of representations can be endowed with structure of representation with the help of the coproduct. In this way the Green ansatz arises through iteration of the coproduct [11](see also [12]). The same logic allows to define the Green ansatz as we are going to show.

Let us denote the iteration of the coproduct by $\Delta^{(p)}$

$$\Delta^{(1)} = id, \quad \Delta^{(2)} = \Delta, \quad \ldots, \quad \Delta^{(p)} = (\underbrace{\Delta \otimes 1 \otimes \ldots \otimes 1}_{p-1}) \circ \Delta^{(p-1)} \tag{27}$$

Let us consider the parabose algebra $\mathfrak{pB}(n)$. The element $\Delta^{(p)}(a_i^\pm) \in (\mathfrak{pB}(n))^{\otimes p}$. is evaluated with π_1 on each slot of the tensor power on the bosonic Fock space

$$\pi_1^{\otimes p} \circ \Delta^{(p)}(a_i^+) = \sum_{r=1}^{p} \underbrace{1 \otimes \ldots \otimes 1}_{r-1} \otimes \pi_1(a_i^+) \otimes \underbrace{1 \otimes \ldots \otimes 1}_{p-r} := \sum_{r=1}^{p} a_{i\,(r)}^\pm \tag{28}$$

The summands are to be identified with the Green components $a_{i\,(r)}^\pm$. Indeed, it is evident that the elements of the kind

$$1^{\otimes(r-1)} \otimes \pi_1(a_i^\pm) \otimes 1^{\otimes(p-r)}$$

satisfy the bosonic commutation relations for equal indices (r) and it is easy to see that they do anticommute for different (r), because of the $\mathbb{Z}_2$-grading of the underlying tensor product for $\mathfrak{pB}(n)$. Analogously one checks for $\mathfrak{pF}(n)$. Thus we saw that the relations (25) and (26) are satisfied from the elements $1^{\otimes(r-1)} \otimes \pi_1(a_i^{\pm}) \otimes 1^{\otimes(p-r)}$ hence these provide realization of the Green components. We have seen that the Green ansatz (24) is summarized in

$$\pi_p(a_i^+) = \pi_1^{\otimes p} \circ \Delta^{(p)}(a_i^+) \qquad \pi_p(a_i^-) = \pi_1^{\otimes p} \circ \Delta^{(p)}(a_i^-) \tag{29}$$

This sophisticated way to write the Green ansatz shows up that every representation is obtain as a product of the fundamental one and gives us a hint for generalization after the deformation.

DEFINITION *The system of quadratic exchange relations stemming from the commutativity of the diagrams*

$$\begin{array}{ccc} \mathfrak{pF}_q(n) & \xrightarrow{\Delta^{(p)}} & \mathfrak{pF}_q(n)^{\otimes p} \\ & \pi_p \searrow & \downarrow \pi_1^{\otimes p} \\ & & \mathfrak{F}_q(n)^{\otimes p} \end{array} \qquad \begin{array}{ccc} \mathfrak{pB}_q(n) & \xrightarrow{\Delta^{(p)}} & \mathfrak{pB}_q(n)^{\otimes p} \\ & \pi_p \searrow & \downarrow \pi_1^{\otimes p} \\ & & \mathfrak{B}_q(n)^{\otimes p} \end{array}$$

will be referred to as deformed Green ansatz of parastatistics of order p.

Let us show the consistency of the condition (22) with the deformed Green ansatz. The vacuum state $|0\rangle^{(p)}$ of the representation π_p is to be identified with the tensor power of the oscillator ($p = 1$) vacuum, $|0\rangle^{(p)} = |0\rangle^{\otimes p}$. Evaluating the image of the graded commutator (6)

$$\Delta^{(p)}[\![a_i^+, a_i^-]\!] = [\![\Delta^{(p)}a_i^+, \Delta^{(p)}a_i^-]\!] = \frac{(q^{h_i})^{\otimes p} - (q^{-h_i})^{\otimes p}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} \,, \tag{30}$$

on the vacuum state $|0\rangle^{\otimes p}$ in the oscillator representations $\pi_1^{\otimes p}$ we get the defining condition (22) of the deformed π_p

$$\begin{aligned} &\mp\pi_1^{\otimes p} \circ \Delta^{(p)}[\![a_i^+, a_i^-]\!]|0\rangle^{(p)} = \pi_p(a_i^-)\,\pi_p(a_i^+)|0\rangle^{(p)} = [p]|0\rangle^{(p)} \\ &\Big(= \frac{q^{\frac{p}{2}} - q^{-\frac{p}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}|0\rangle^{(p)} \Big) \end{aligned}$$

since $\pi_1(q^{h_i}) = q^{N_i \mp \frac{1}{2}}$ for $\mathfrak{pF}_q(n)$, $\mathfrak{pB}_q(n)$ respectively, which shows the consistency.
The mapping $\pi_1^{\otimes p} \circ \Delta^{(p)}$ can be split up into the following Green components

$$\begin{aligned} a^+_{i\,(r)} &= \sum_{k_1,\ldots,k_r} \underline{L}^{(+)}_{ik_1} \otimes \underline{L}^{(+)}_{k_1k_2} \otimes \ldots \otimes \underline{L}^{(+)}_{k_{r-1}k_r} \otimes \underline{a}^+_{k_r} \otimes 1 \ldots \otimes 1 \,, \\ a^-_{j(r)} &= \sum_{k_1,\ldots,k_{p-r}} 1 \otimes \ldots \otimes 1 \otimes \underline{a}^-_{k_1} \otimes \underline{L}^{(-)}_{k_1k_2} \otimes \underline{L}^{(-)}_{k_2k_3} \otimes \ldots \otimes \underline{L}^{(-)}_{k_{p-r}j} \,, \end{aligned} \tag{31}$$

where the upper (lower) triangularity of the matrices $L^{(+)}(L^{(-)})$ infers that only the terms subject to the inequalities $i \leq k_1 \leq \ldots \leq k_r \leq n$ are non-zero (respectively $n \geq k_1 \geq \ldots \geq k_{p-r} \geq j$).
At the end of the day, after lengthy calculations with the deformed oscillator algebra we obtain the following system of relations

$$\begin{array}{lll} [a^+_{i(r)}, a^+_{j(s)}]_\mp = \mp(q-q^{-1})a^+_{j(r)}a^+_{i(s)}\,, & [a^-_{i(r)}, a^-_{j(s)}]_\mp = 0, & i<j\,, \\ [a^-_{i(r)}, a^-_{j(s)}]_\mp = \pm(q-q^{-1})a^-_{j(r)}a^-_{i(s)}\,, & [a^{+i}_{(r)}, a^{+j}_{(s)}]_\mp = 0\,, & i>j\,, \end{array} \tag{32}$$

$$[a^+_{i(r)}, a^+_{i(s)}]_{\mp q} = 0\,, \qquad [a^-_{i(r)}, a^-_{i(s)}]_{\mp q^{-1}} = 0\,, \tag{33}$$

$$[a^-_{i(r)}, a^{+j}_{j(s)}]_\mp = 0\,, \tag{34}$$

when $r \neq s$ and when the Green indices coincide one gets

$$\begin{array}{ll} [a^+_{i(r)}, a^+_{j(r)}]_{\pm q^{\mp\epsilon_{ij}}} = 0\,, & [a^-_{i(r)}, a^-_{j(r)}]_{\pm q^{\mp\epsilon_{ij}}} = 0\,, \\ [a^-_{i(r)}, a^+_{j(r)}]_{\pm q^{\mp 1}} = q^{\mp\frac{1}{2}}Q^{(-)}_{ji(r)}\,, & [a^-_{i(r)}, a^+_{j(r)}]_{\pm q^{\pm 1}} = q^{\pm\frac{1}{2}}Q^{(+)}_{ji(r)}\,, \end{array} \tag{35}$$

where the operators $Q^{(\pm)}_{ji(r)}$ are quadratic in the Green components

$$\begin{array}{l} q^{\mp\frac{1}{2}}Q^{j(-)}_{i(r)} = (q-q^{-1})\sum_{s=1}^{r-1} q^{\mp(r-s)}a^{+j}_{(s)}a^-_{i(s)} = (q^{\pm\frac{1}{2}}Q^{i(+)}_{j(r^*)})^*\,,\ i>j\,, \\ q^{\mp\frac{1}{2}}Q^{j(-)}_{i(r)} = -(q-q^{-1})\sum_{s=r}^{p} q^{\mp(r-s)}a^{+j}_{(s)}a^-_{i(s)} = (q^{\pm\frac{1}{2}}Q^{i(+)}_{j(r^*)})^*\,, i<j\,. \end{array} \tag{36}$$

When dealing with the parafermionic algebra $\mathfrak{p}\mathfrak{F}_q(n)$ we have the complementary relations $(a^\pm_{i(r)})^2 = 0$.

The obtained system of relations is the counterpart of the relations (25), (26) and defines the deformation of the Green ansatz.

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