Floer Homology in Classical Mechanics and Quantum Field Theory*

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Abstract

In this article, we discuss several applications of symplectic Floer theory related to Classical and Quantum Physics.

1. Introduction

In the series of papers [4]-[11] A. Floer applied Witten's formulation of finite dimensional Morse theory [29] to certain infinite dimensional situation. Although his original motivation was rooted in a problem arising in Classical Mechanics, the main role of his homology theory in Mathematical Physics today is related to the Quantum Field Theory. Floer homology can be considered as 1+1 dimensional quantum field theory. Structure on the Floer homology ring are related to the ring structure on the Quantum cohomology, given by Gromov-Witten invariants. Furthermore, Floer homology for Lagrangian submanifolds is the symplectic mirror part for the category of coherent sheaves in Mirror Symmetry. In this article, we will review some aspects of Floer homology and sketch some recent developments.

2. Topological Quantum Field Theory

Topological quantum field theory in dimension d is a functor Z from the category of d–dimensional compact oriented smooth manifolds to the category of complex vector spaces, that satisfies the following axioms:

(A0) $Z(\emptyset) = \mathbf{C}$ – the complex numbers,

(A1) $Z(\Sigma^*) = Z(\Sigma)^*$, where the star on the left denotes the change of orientation, and on the right the dual vector space,

(A2) $Z(\Sigma_1 \cup \Sigma_2) = Z(\Sigma_1) \otimes Z(\Sigma_2) \cong Hom(Z(\Sigma_1)^*, Z(\Sigma_2))$

If Σ is the boundary of $(d+1)$–dimensional manifold X ($\Sigma = \partial X$), then $Z(X)$ is a vector in $Z(\Sigma)$ and

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(A3) If $\partial X_1 = \Sigma_1 \cup \Sigma_2$, $\partial X_2 = \Sigma_2 \cup \Sigma_3$ are two (d_1)–dimensional manifolds with common part of the boundary Σ_2, and $X_1 \cup_{\Sigma_2} X_2$ is the manifold obtained by gluing X_1 and X_2 along Σ_2, then

$$Z(X_1 \cup_{\Sigma_2} X_2) = Z(X_2) \circ Z(X_1),$$

(A4) $Z(\Sigma \times [0,1]) = Id : Z(\Sigma) \to Z(\Sigma)$.

(see [1, 30] for the physical background and motivation). We will consider the case $d = 1$. Since the only one–dimensional compact manifold is a circle $\mathbf{S}^1$, we have to deal with only one complex vector space $H := Z(\mathbf{S}^1)$. Let X be a compact 2–dimensional manifold with boundary diffeomorphic to the union of circles. Every 2–dimensional compact manifold can be decomposed to the union of pants, caps and cylinders. Therefore, in order to understand the homomorphism $Z(X)$ it is enough to understand it on these three model cases. The first one gives rise to the multiplication

$$H \otimes H \to H, \quad a \otimes b \mapsto a * b,$$

the second to the one–form $\theta \in H^*$, i.e. the linear mapping

$$\theta : H \to \mathbf{C},$$

and the third one to the scalar product

$$H \otimes H \to \mathbf{C}, \quad a \otimes b \mapsto \langle a, b \rangle.$$

Using (A3) we can easily see that these three operations are related by

$$\langle a, b \rangle = \theta(a * b).$$

This algebraic structure is called *Frobenius algebra* (see [3, 13]).

3. Two examples

Let M be a compact symplectic manifold.

3.1. Quantum Cohomology

For homology classes $\alpha, \beta, \gamma \in H_*(M)$ and $A \in H_2(M)$, *Gromov – Witten invariant* $\Phi_A(\alpha, \beta, \gamma)$ is the number of holomorphic spheres in class A that touch the representatives of the classes α, β and γ. *The quantum cohomology ring* is the ring

$$QH^*(M) := H^*(M) \otimes \mathbf{Z}[q, q^{-1}]$$

with multiplication defined in the following way. For

$$a = \sum a_i q^i, \quad b = \sum b_j q^j, \quad c = \sum c_k q^k,$$

we define

$$\langle a * b, c \rangle := \sum_{i,j,k} \sum_{A} \Phi_A(D(a_i), D(b_j), D(c_k)),$$

where D denotes the Poincaré dual and $\langle \cdot, \cdot \rangle$ the scalar product induced by intersection form in (co)homology (see [16]).

3.2. Floer Cohomology

Let $H : M \times [0,1] \to \mathbf{R}$ be a Hamiltonian function. *The Floer chain group* $CF^*(M,H)$ is a group generated by time–one periodic orbits of the Hamiltonian system generated by H. The "gradient lines" of the classical Hamiltonian action functional $\mathcal{A}_H$ are the perturbed Cauchy–Riemann equations $\bar{\partial}u = -\nabla H(u)$. Considered on the model case of cylinder, they give rise to the co–boundary operator

$$d : CF^*(M,H) \to CF^{*+1}(M,H),$$

and, consequently, the cohomology $HF^*(M,H)$ called *Floer homology* (see e.g. lecture notes [27] for more details). It has a product structure, defined by means of the equation $\bar{\partial}u = -\nabla H(u)$ on the pair of pants model surface (see [28] for more details on Floer cohomology operations defined by means of $\mathbf{S}^1$–cobordisms). It is remarkable fact that these two cohomologies are isomorphic:

$$HF^*(M,H) \cong QH^*(M). \tag{1}$$

In particular, the Floer cohomology is independent of the choice of Hamiltonian. We will give now some applications of this fact.

4. Two Applications to Classical Mechanics

4.1. Arnold Conjecture

It follows from (1) that the number of the periodic orbits of a Hamiltonian system on M is estimated from below by the dimension of $H^*(M)$, i.e.

$$Per(H) \geq \sum_{k=0}^{\dim M} \beta_k(M),$$

where $\beta_k(M)$ are the Betti numbers of M. The proof of this conjecture was the original motivation for Floer to develop his homology theory.

4.2. Hofer's Geometry

Following Polterovich [26], we will formulate Hofer's geometry as an answer to the following question. Let $\varphi : M \to M$ be a Hamiltonian diffeomorphism. What is the minimal amount of energy required to generate it? If $\{\varphi_t\}_{0 \leq t \leq 1}$ is any Hamiltonian path with $\varphi_1 = \varphi$, $H(x,t)$ a Hamiltonian generating it, and

$$l(\varphi_t) := \int_0^1 (\max_{x \in M} H(x,t) - \min_{x \in M} H(x,t))\, dt,$$

then the question about finding the minimal energy is the question of finding

$$E(\varphi) := \inf\, l(\varphi_t),$$

where the infimum is taken over all Hamiltonian paths φ_t with $\varphi_1 = \varphi$. It was proved by Hofer [14] and Lalonde – McDuff [15] that

$$d(\varphi, \psi) := E(\varphi \circ \psi^{-1})$$

defines a metric on the infinite dimensional Lie group of Hamiltonian diffeomorphisms. Therefore, the dynamical question posed above can be translated to the geometric one: what is the distance between φ and the identity? The question is obviously related to the description of geodesics in Hofer's geometry. They were first studied by Bialy and Polterovich in [2] by means of Hamilton – Jacobi equations. The application of the Floer theory to study of geodesics is explained in [26]. Hofer's geometry can be generalized to Lagrangian submanifolds. Lagrangian submanifolds generalize Hamiltonian diffeomorphisms and could be considered as "symplectic relations" (see e.g. [17]). Floer homology groups can also be defined by considering the Hamiltonian orbits with Lagrangian boundary conditions instead of *periodic* orbits (see [6, 21, 22, 23] or a comprehensive survey [12]). Hofer's geometry for Lagrangian submanifolds via Floer homology is studied in [24, 25], [20], [18, 19]. Floer homology there plays the pivotal role in construction of homological symplectic invariants related to the dynamics of the Hamiltonian system. For a Hamitlonian H, Floer homology groups are filtrated by the value of the action functional $\mathcal{A}_H(C) = \int_C pdq - Hdt$. Let $HF^*_\lambda(H)$ be the subgroup of $HF^*(H)$ generated by those Hamiltonian orbits C with $\mathcal{A}_H(C) \leq \lambda$. Let

$$\rho(H) := \inf\{\lambda \mid HF^*(H) \to HF^*_\lambda(H) \,\text{is nontrivial}\}$$

(see [24, 25, 20]). Let φ_t be a Hamiltonian path generated by H. Then $d(Id, \varphi) \geq \rho(H) + \rho(\bar{H})$, where $\bar{H}(x,t) = -H(\varphi_t(x), t)$. This enables us to control the behavior of Hofer's distance by the homological invariants. Some applications of this fact are given in [24, 25, 18, 19, 20].

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