Introduction to the Geometric Theory of Defects*

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Abstract

We describe defects – dislocations and disclinations – in the framework of Riemann–Cartan geometry. Curvature and torsion tensors are interpreted as surface densities of Frank and Burgers vectors, respectively. Equations of nonlinear elasticity theory are used to fix the coordinate system. The Lorentz gauge yields equations for the principal chiral $\mathbb{SO}(3)$-field. In the absence of defects the geometric model reduces to the elasticity theory for the displacement vector field and to the principal chiral $\mathbb{SO}(3)$-field model for the spin structure.

1. Introduction

Many solids have a crystalline structure. However, ideal crystals are absent in Nature, and most of their physical properties, such as plasticity, melting, growth, etc., are defined by defects of the crystalline structure. Therefore a study of defects is the actual scientific problem important for applications in the first place. A broad experimental and theoretical investigations of defects in crystals started in the thirties of the last century and are continued to nowadays. At present a fundamental theory of defects is absent in spite of the existence of dozens of monographs and thousands of articles.

One of the most promising approach to the theory of defects is based on Riemann–Cartan geometry, which is given by nontrivial metric and torsion. In this approach a crystal is considered as continuous elastic media with a spin structure. If a displacement vector field is a smooth function then there are only elastic stresses corresponding to diffeomorphisms of the Euclidean space. If a displacement vector field have discontinuities then we are saying that there are defects in the elastic structure. Defects in the elastic structure are called dislocations and lead to the appearance of nontrivial geometry. Precisely, they correspond to nonzero torsion tensor which is equal to the surface density of the Burgers vector.

The idea to relate torsion to dislocations appeared in the fifties [1]. This approach is being successfully developed up to now (note reviews [2]) and

* Work supported by grants RFBR-02-01-01084 and NSH-00-15-96073.
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called often the gauge theory of dislocations. A similar approach is developed also in gravity [3]. It is interesting to note that E. Cartan introduced torsion in geometry [4] having analogy with mechanics of elastic media.

The gauge approach to the theory of defects is developed successfully, and interesting results are obtained in this way [5] Let us note in this connection two respects in which the approach proposed below is essentially different. In the gauge models of dislocations based on the translational group or on the semidirect product of rotational group on translations one chooses usually distortion and displacement field as independent variables. It is always possible to fix the invariance with respect to local translations in such a way that the displacement field becomes zero because the displacement field moves by simple translation under the action of the translational group. In this sense the displacement field is the gauge parameter of local translations, and physical observables do not depend on it in the gauge invariant models.

The other disadvantage of the gauge approach is the equations of equilibrium. One considers usually equations of Einstein type for distortion or vielbein with the right hand side depending on the stress tensor. This appears unacceptable from the physical point of view because of the following reason. Consider, for example, one straight edge dislocation. In this case the elastic stress field differs from zero everywhere. Then the torsion tensor (or curvature) is also nontrivial due to the equations of equilibrium. This is wrong from our point of view. Really, consider a domain of media outside the cutting surface and look at the process of creation of the edge dislocation. The chosen domain was the part of the Euclidean space with identically zero torsion and curvature before the defect creation. It is clear that torsion and curvature remain zero because the process of dislocation formation is a diffeomorphism for the considered domain. Besides, the cutting surface may be chosen arbitrary for the defect formation leaving the axis of dislocation unchanged. We deduce from this that torsion and curvature must be zero everywhere except the axis of dislocation. In other words the elasticity stress tensor can not be the source of dislocations. To avoid the appearing contradiction we propose a cardinal way out: we do not use the displacement field as an independent variable at all. It does not mean that the displacement field does not exist in real crystals. In the proposed approach the displacement field exists in those regions of media which do not contain cores of dislocations, and it can be computed. In this case it satisfies the equations of nonlinear elasticity theory.

The proposed geometric approach allows one to include into consideration other defects which do not relate directly to defects in elastic structure. The intensive investigations of these defects were conducted in parallel with the study of dislocations. The point is that many solids do not only have elastic properties but possess a spin structure. For example, there are ferromagnets, liquid crystals, spin glasses, etc. In this case there are defects in the spin structure which are called disclinations [6]. They arise when the director field has discontinuities. The presence of disclinations is also connected to nontrivial geometry. Namely, the curvature tensor equals to the surface density of the Frank vector. The gauge approach based on the rotational group $\mathbb{SO}(3)$ was also used for describing disclinations [7]. $\mathbb{SO}(3)$-gauge models of spin glasses with defects were considered in [8].

The geometric theory of static distribution of defects which describes both types of defects – dislocation and disclinations – from a single point of view was proposed in [9]. In contrast to other approaches we have vielbein and $\mathbb{SO}(3)$-connection as the only independent variables. Torsion and curvature tensors have direct physical meaning as the surface densities of dislocations and disclinations, respectively. Covariant equations of equilibrium for vielbein and $\mathbb{SO}(3)$-connection similar to those in a gravity model with torsion are postulated. To define the solution uniquely we must fix the coordinate system (fix the gauge) because any solution of the equations of equilibrium is defined up to general coordinate transformations and local $\mathbb{SO}(3)$-rotations. The elastic gauge for the vielbein [10] and Lorentz gauge for the $\mathbb{SO}(3)$-connection [11] were proposed recently. We stress that the notions of displacement vector and rotational angle are absent in our approach at all. These notions can be introduced only in those domains where defects are absent. In this case equations for vielbein and $\mathbb{SO}(3)$-connection are satisfied identically, the elastic gauge reduces to the equations of nonlinear elasticity theory for the displacement vector, and the Lorentz gauge yields equations for the principal chiral $\mathbb{SO}(3)$-field. In other words, to fix the coordinate system we choose two fundamental models: elasticity theory and the principal chiral field model.

2. Elastic deformations

We consider infinite three dimensional elastic media. Suppose that undeformed media in the absence of defects is invariant under translations and rotations in some coordinate system. Then the media in this coordinate system y^i, $i = 1, 2, 3$, is described by the Euclidean metric $\delta_{ij} = \mathsf{diag}\,(+ + +)$, and the system of coordinates is called Cartesian. Thus in the undeformed state we have the Euclidean space $\mathbb{R}^3$ with a given Cartesian coordinate system. We assume also that torsion in the media equals zero.

Let a point of media has coordinates y^i in the ground state. After deformation this point will have coordinates, see Fig. 1,

$$y^i \to x^i(y) = y^i + u^i(x) \tag{1}$$

in the initial coordinate system. The inverse notations are used in the elasticity theory. One writes usually $x^i \to y^i = x^i + u^i(x)$. These are equivalent writings because both coordinate systems x^i and y^i cover the whole $\mathbb{R}^3$. However in the theory of defects considered in the next sections the situation is different. In a general case the elastic media fills the whole Euclidean space only in the final state. Here and in what follows we assume that fields depend on coordinates x which are coordinates of points of media after the deformation and cover the whole Euclidean space $\mathbb{R}^3$ by assumption. In the presence of dislocations the coordinates y^i do not cover the whole $\mathbb{R}^3$ in a general case because part of the media may be removed or, inversely, added. Therefore the system of coordinates related to points of the media after an elastic deformation and defect creation is more preferable.

In the linear elasticity theory relative deformations are assumed to be small $\partial_j u^i \ll 1$. Then the functions $u^i(x) = u^i(y(x))$ are components of a vector field which is called the displacement vector field and is the basic variable in the elasticity theory.

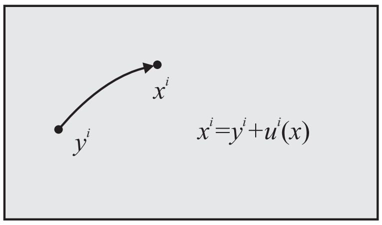


Figure 1: Elastic deformations

In the absence of defects we assume that the displacement field is a smooth vector field in the Euclidean space $\mathbb{R}^3$. The presence of discontinuities and singularities in displacement field is interpreted as the presence of defects.

We shall consider only static deformations in what follows when displacement field u^i does not depend on time. Then the basic equations of equilibrium for small deformations are (see, for example, [13])

$$\partial_j \sigma^{ji} + f^i = 0, \tag{2}$$

$$\sigma^{ij} = \lambda\, \delta^{ij}\, \epsilon_k{}^k + 2\, \mu\, \epsilon^{ij}, \tag{3}$$

where σ^{ji} is the stress tensor which is assumed to be symmetric. The tensor of small deformations ϵ_{ij} is given by the symmetrized partial derivative of the displacement vector

$$\epsilon_{ij} = \frac{1}{2}(\partial_i u_j + \partial_j u_i)\,. \tag{4}$$

Lowering and raising of the Latin indices is performed with the Euclidean metric δ_{ij} and its inverse δ^{ij}. The letters λ and μ denote constants characterizing elastic properties of media and are called Lame coefficients. Functions $f^i(x)$ describe total density of nonelastic forces inside the media. We assume that such forces are absent in what follows, $f^j(x) = 0$. Equation (2) is Newton's law, and Eq. (3) is Hook's law relating stresses with deformations.

In Cartesian coordinate system for small deformations the difference between upper and lower indices disappears because raising and lowering of indices is performed with the help of the Euclidean metric. One usually forgets about this difference due to this reason, and this is fully justified. But in the presence of defects the notion of Cartesian coordinate system and Euclidean metric is absent, and raising and lowering of indices are performed with the help of Riemannian metric. Therefore we distinguish upper and lower indices as it is accepted in differential geometry having in mind the following transition to elastic media with defects.

The main problem in the linear elasticity theory is the solution of the second order equations for displacement vector which arise after substitution of (3) into (2) with some boundary conditions. Many known solutions are in good agreement with experiment. Therefore one may say that equations (3), (2) have a solid experimental background.

Let us look at the elastic deformations from the point of view of differential geometry (see, for example, [12]). From mathematical standpoint the map (1) by itself is the diffeomorphism of the Euclidean space $\mathbb{R}^3$. In this case the Euclidean metric δ_{ij} is induced by the map $y^i \to x^i$. It means that in the deformed state the metric in the linear approximation has the form

$$g_{ij}(x) = \frac{\partial y^k}{\partial x^i}\,\frac{\partial y^l}{\partial x^j}\,\delta_{kl} \approx \delta_{ij} - \partial_i u_j - \partial_j u_i = \delta_{ij} - 2\,\epsilon_{ij}\,, \tag{5}$$

i.e. it is defined by the tensor of small deformations (4). Note that in the linear approximation $\epsilon_{ij}(x) = \epsilon_{ij}(y)$ and $\partial u_j/\partial x^i = \partial u_j/\partial y^i$.

In Riemann geometry the metric defines uniquely the Levi–Civita connection $\widetilde{\Gamma}_{ij}{}^k(x)$ (Christoffel's symbols)

$$\widetilde{\Gamma}_{ijk} = \frac{1}{2}\left(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij}\right). \tag{6}$$

We can compute the curvature tensor

$$\widetilde{R}_{ijk}{}^l = \partial_i \widetilde{\Gamma}_{jk}{}^l - \widetilde{\Gamma}_{ik}{}^m \widetilde{\Gamma}_{jm}{}^l - (i \leftrightarrow j), \tag{7}$$

for these symbols. This tensor equals identically zero, $\widetilde{R}_{ijk}{}^l(x) = 0$, because curvature of the Euclidean space is zero, and the map $y^i \to x^i$ is a diffeomorphism. The torsion tensor equals zero for the same reason. Thus an elastic deformation of media corresponds to trivial Riemann–Cartan geometry because curvature and torsion tensors are equal to zero.

The physical interpretation of the metric (5) is the following. External observer fixes Cartesian coordinate system corresponding to the ground undeformed state of media. The media is deformed afterwards, and external observer discovers that the metric in this coordinate system becomes nontrivial. If we assume that elastic perturbations in media (phonons) propagate along extremals $x^k(t)$ (lines of minimal length), then in the deformed media their trajectories will be defined by equations

$$\ddot{x}^k = -\widetilde{\Gamma}_{ij}{}^k\,\dot{x}^i\,\dot{x}^j\,,$$

where dots denote differentiation with respect to a canonical parameter t. Trajectories of phonons will be not now straight lines because Christoffel's symbols are nontrivial, $\widetilde{\Gamma}_{ij}{}^k \neq 0$. In this sense the metric (5) is observable. Here we see the essential role of the Cartesian coordinate system y^i defined by the undeformed state with which the measurement process is connected.

Assume that the metric $g_{ij}(x)$ given in the Cartesian coordinates corresponds to some state of elastic media without defects. In this case the displacement vector is defined by the system of equations (5), and its integrability conditions are the equality of the curvature tensor to zero. In the linear approximation these conditions are known in the elasticity theory as the Saint–Venant integrability conditions.

Let us make the remark important for the following consideration. For appropriate boundary conditions the solution of the elasticity theory equations (2), (3) is unique. From the geometric viewpoint it means that elasticity theory fixes diffeomorphisms. This fact will be used in the geometric theory of defects. Equations of nonlinear elasticity theory written in terms of metric or vielbein will be used for fixing the coordinate system.

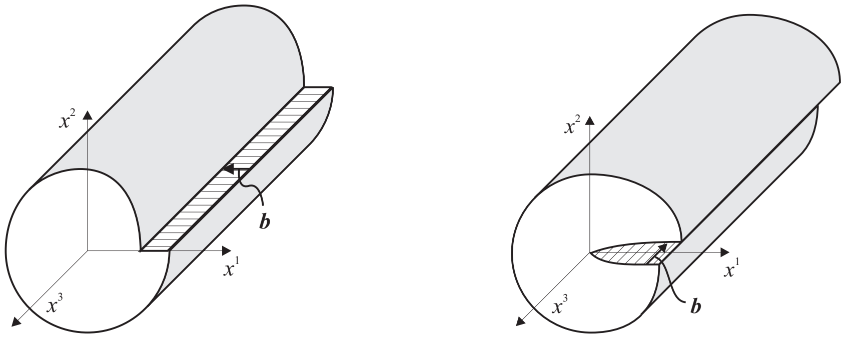
3. Dislocations

We start with description of linear dislocations in elastic media (see, for example, [13, 14]). The simplest and widely spreaded examples of linear dislocations are shown in Fig. 2. Cut the media along the half plane $x^2 = 0$, $x^1 > 0$. Move the upper part of the media located over the cut $x^2 > 0$, $x^1 > 0$ on the vector $\boldsymbol{b}$ towards the dislocation axis x^3, and glue the cutting surfaces. The vector $\boldsymbol{b}$ is called the Burgers vector. In a general case the Burgers vector may not be constant on the cut. For the edge dislocation it varies from zero to some constant value $\boldsymbol{b}$ as it moves from the dislocation axis. After the gluing the media comes to the equilibrium state which is called the edge dislocation shown in Fig. 2,*a*. If the Burgers vector is parallel to the dislocation line then it is called the screw dislocation, Fig. 2,*b*.

One and the same dislocation can be made in different ways. For example, if the Burgers vector is perpendicular to the cutting plane and directed from it in the considered cases then the produced cavity must be filled with media before the gluing. One can easily imagine that the edge dislocation is also obtained as the result but rotated by the angle $\pi/2$ around the axis x^3. This example shows that a dislocation is characterized not by the cutting surface but by the dislocation line and the Burgers vector.

From topological point of view the media containing several dislocations or even the infinite number of them represents itself the Euclidean space $\mathbb{R}^3$. In contrast to elastic deformations the displacement vector in the presence of dislocations is no longer a smooth function because of the presence of cutting surfaces. At the same time we assume that partial derivatives of the displacement vector $\partial_j u^i$ (the distortion tensor) are smooth functions on the cutting surface. This assumption is justified physically because these derivatives define the deformation tensor (4). In its turn partial derivatives of deformation tensor must exist and be smooth functions in the equilibrium state everywhere except, possibly, the axis of dislocation because otherwise the equations of equilibrium (2) do not have meaning. We assume that the metric and vielbein are smooth functions everywhere in $\mathbb{R}^3$ except, may be, dislocation axes because the deformation tensor defines the induced metric (5).

The main idea of the geometric approach reduces to the following. To describe single dislocations in the framework of elasticity theory we must



(a) The edge dislocation. The Burgers vector $\boldsymbol{b}$ is perpendicular to the dislocation line.

(b) The screw dislocation. The Burgers vector $\boldsymbol{b}$ is parallel to the dislocation line.

Figure 2: Straight linear dislocations.

solve equations for the displacement vector with some boundary conditions on the cuts. For small number of dislocations this is possible. However with increasing number of dislocations the boundary conditions become so complicated that the solution of the problem becomes unreal. Besides, one and the same dislocation may be produced by different cuts which lead to ambiguity in the displacement vector field. Another shortcoming of this approach is that it can not be applied for description of continuous distribution of dislocations because in this case the displacement vector field does not exist at all for the reason that it must have discontinuities at every point. In the geometric approach the basic variable is the vielbein which is a smooth function everywhere except, possibly, dislocation axes by assumption. We postulate new equations for the vielbein (see [9]). In the geometric approach the transition from finite number of dislocations to their continuous distribution is simple and natural. In that way the smoothing of singularities occur on dislocation axes in analogy with smoothing of mass distribution for point particles when going to continuous media.

Let us start to build the formalism of the geometric approach. In a general case in the presence of defects we do not have a preferred Cartesian coordinates frame in the equilibrium state because there is no symmetry. Therefore we consider arbitrary coordinates x^μ, $\mu = 1, 2, 3$, in $\mathbb{R}^3$. Now we are using Greek letters to enumerate coordinates admitting arbitrary coordinate changes. Then the Burgers vector can be expressed as the integral of the displacement vector

$$\oint_C dx^\mu \, \partial_\mu u^i(x) = -\oint_C dx^\mu \, \partial_\mu y^i(x) = -b^i \,, \tag{8}$$

where C is a closed contour surrounding the dislocation axis, Fig. 3.

This integral is invariant with respect to arbitrary coordinate transformations $x^\mu \to x^{\mu'}(x)$ and covariant under global $\mathbb{SO}(3)$-rotations of y^i. Here components of the displacement vector field $u^i(x)$ are considered with respect to the orthonormal basis in the tangent space $u = u^i e_i$. If components of the displacement vector field were considered with respect to the coordinate basis $u = u^\mu \partial_\mu$, then the invariance of the integral (8) under general coordinate changes is violated.

In the geometric approach we introduce new independent variable – the vielbein – instead of partial derivatives $\partial_\mu u^i$

$$e_\mu{}^i(x) = \begin{cases} \partial_\mu y^i, & \text{outside the cut,} \\ \lim \partial_\mu y^i, & \text{on the cut.} \end{cases} \tag{9}$$

The vielbein is a smooth function on the cut by construction. Note that if the vielbein was simply defined as partial derivative $\partial_\mu y^i$, then it would have the δ-function singularity on the cut because functions $y^i(x)$ have a jump. Then the Burgers vector can be expressed through the integral over a surface S having contour C as the boundary

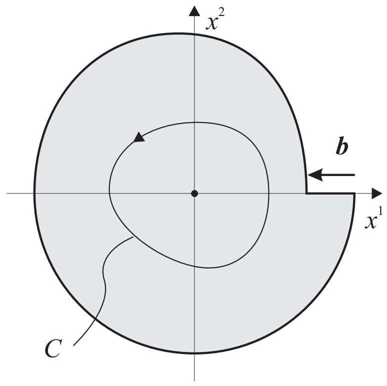


Figure 3: The section of the media with the edge dislocation. C is the integration contour for the Burgers vector $\boldsymbol{b}$.

$$\oint_C dx^\mu e_\mu{}^i = \iint_S dx^\mu \wedge dx^\nu (\partial_\mu e_\nu{}^i - \partial_\nu e_\mu{}^i) = b^i \,, \tag{10}$$

where $dx^\mu \wedge dx^\nu$ is the surface element. From the definition of the vielbein (9) we see that the integrand equals zero everywhere except the axis of dislocation. For the edge dislocation with constant Burgers vector the integrand has δ-function singularity at the origin. The criterion for the presence of dislocation is a violation of integrability conditions for the system of equations $\partial_\mu y^i = e_\mu{}^i$:

$$\partial_\mu e_\nu{}^i - \partial_\nu e_\mu{}^i \neq 0. \tag{11}$$

If dislocations are absent then functions $y^i(x)$ exist and define transformation to the Cartesian coordinates frame.

In the geometric theory of defects the field $e_\mu{}^i$ is identified with the vielbein. Next, compare the integrand in (10) with the expression for torsion in Cartan variables

$$T_{\mu\nu}{}^i = \partial_\mu e_\nu{}^i - e_\mu{}^j \, \omega_{\nu j}{}^i - (\mu \leftrightarrow \nu) \,, \tag{12}$$

They differ only by terms containing $\mathbb{SO}(3)$-connection $\omega_{\nu j}{}^i(x)$. This is the ground for the introduction of the following postulate. In the geometric theory of defects the Burgers vector corresponding to a surface S is defined by the integral of the torsion tensor

$$b^i = \iint_S dx^\mu \wedge dx^\nu T_{\mu\nu}{}^i \,.$$

This definition is invariant with respect to general coordinate transformations of x^μ and covariant with respect to global rotations. Thus the torsion tensor has straightforward physical interpretation: it equals the surface density of the Burgers vector.

Physical interpretation of the $\mathbb{SO}(3)$-connection will be given in the next section, and now we show how this definition reduces to the expression for the Burgers vector (10) obtained within the elasticity theory. If the curvature tensor for $\mathbb{SO}(3)$-connection equals zero, then the connection is locally trivial, and there exists such $\mathbb{SO}(3)$ rotation that $\omega_{\mu i}{}^j = 0$. In that case we return to expression (10).

If $\mathbb{SO}(3)$-connection is zero and vielbein is a smooth function then the Burgers vector corresponds uniquely to every contour. In this case it can be expressed as the surface integral of the torsion tensor. The surface integral depends only on the boundary contour but not on the surface due to the Stokes theorem.

We have shown that the presence of linear defects results in a nontrivial torsion tensor. In the geometric theory of defects the equality of torsion tensor to zero $T_{\mu\nu}{}^i = 0$ is naturally considered as the criterion for the absence of dislocations. Then under the name dislocation fall not only linear dislocations but, in fact, arbitrary defects in elastic media. For example, point defects: vacancies and impurities are also dislocations. In the first case we cut out a ball from the Euclidean space $\mathbb{R}^3$ and then shrink the boundary sphere to a point, Fig. 4. In the case of impurity a point of the Euclidean space is blown up to a sphere and the produced cavity is filled with the media. Point defects are characterized by the mass of the removed or added media which is also defined by the vielbein [9]

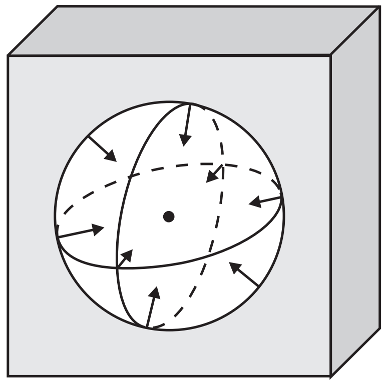


Figure 4: Point defect – vacancy – appears when a ball is cut out from the media, and the boundary sphere is shrank to a point.

$$M = \rho_0 \iiint_{\mathbb{R}^3} d^3x \left(\det e_\mu{}^i - \det \overset{\circ}{e}_\mu{}^i\right), \qquad \overset{\circ}{e}_\mu{}^i = \partial_\mu y^i \,, \tag{13}$$

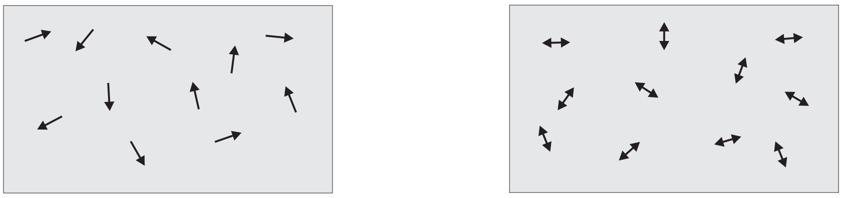
where $y^i(x)$ are the transition functions to Cartesian coordinate frame in $\mathbb{R}^3$, and ρ_0 is the density of the media which is supposed to be constant. The mass is defined by the difference of two integrals each of them being divergent separately. The first integral equals to the volume of the media with defects and the second is equal to the volume of the Euclidean space. According to the given definition the mass of an impurity is positive because the matter is added to the media, and the mass of a vacancy is negative since part of the media is removed. The torsion tensor for a vacancy or impurity equals zero everywhere except one point where it has a δ-function singularity. For point defects the notion of the Burgers vector is absent.

In three dimensional space surface defects may also exist along with point and line dislocations. In the geometric approach all of them are called dislocations because they correspond to nontrivial torsion.

4. Disclinations

We relate dislocations to nontrivial torsion tensor in the preceeding section. To this end we introduced the $\mathbb{SO}(3)$-connection. In the present section we show that the curvature tensor for the $\mathbb{SO}(3)$-connection defines the surface density of the Frank vector characterizing other well known defects – disclinations in the spin structure [13].

Let the unit vector field $n^i(x)$, $(n^i n_i = 1)$, be given in all points of media. For example, n^i has the meaning of magnetic moment located at each point of the media for ferromagnets, Fig. 5,a. For nematic liquid crystals the unit vector field n^i with the equivalence relation $n^i \sim -n^i$ describes the director field, Fig. 5, b.



(a) Ferromagnet (b) Liquid crystal

Figure 5: Examples of media with the spin structure

Let us fix some direction in the media n^i_0. Then the field $n^i(x)$ at a point x can be uniquely defined by the field $\omega^{ij}(x) = -\omega^{ji}(x)$ taking values in the rotation algebra $\mathfrak{so}(3)$ (the angle of rotation):

$$n^i = n^j_0\, S_j{}^i(\omega),$$

where $S_j{}^i \in \mathbb{SO}(3)$ is the rotation matrix corresponding to the algebra element ω^{ij}. Here we use the following parameterization of the rotation group $\mathbb{SO}(3)$ by elements of its algebra (see, for example, [11])

$$S_i{}^j = (e^{(\omega\,\varepsilon)})_i{}^j = \cos\omega\,\delta^j_i + \frac{(\omega\,\varepsilon)_i{}^j}{\omega}\,\sin\omega + \frac{\omega_i\,\omega^j}{\omega^2}\,(1-\cos\omega)\ \in \mathbb{SO}(3), \quad (14)$$

where $(\omega\,\varepsilon)_i{}^j = \omega^k\,\varepsilon_{ki}{}^j$ and $\omega = \sqrt{\omega^i\,\omega_i}$ is the modulus of the vector ω^i. The pseudovector $\omega^k = \frac{1}{2}\,\omega_{ij}\,\varepsilon^{ijk}$, where ε^{ijk} is the totally antisymmetric third rank tensor, $\varepsilon^{123} = 1$, is directed along the rotational axis and its length equals to the angle of rotation. We shall call the field ω^{ij} a spin structure of the media.

If the media possesses a spin structure then it may have defects called disclinations. For linear disclinations parallel to the x^3 axis the vector field n lies in the perpendicular plain x^1, x^2. The simplest examples of linear disclinations are shown in Fig. 6. Every linear disclination is characterized

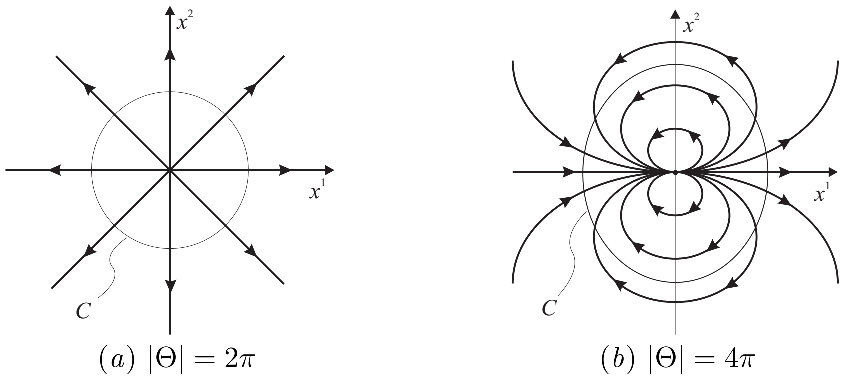


Figure 6: The vector field distributions in the plane x^1, x^2 for the linear dislocations parallel to the x^3 axis.

by the Frank vector

$$\Theta_i = \epsilon_{ijk}\,\Omega^{jk}, \tag{15}$$

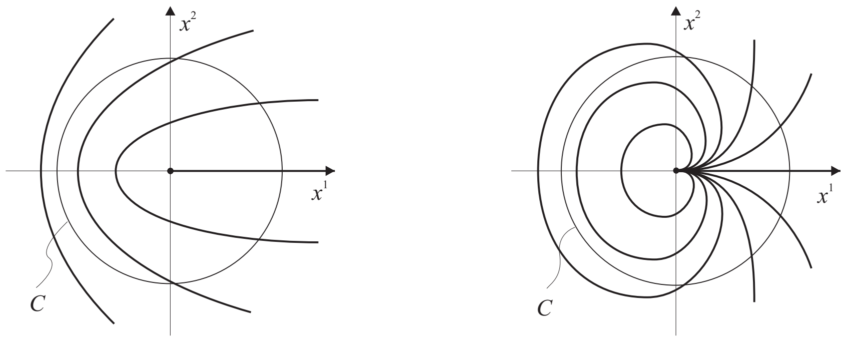
where

$$\Omega^{ij} = \oint_C dx^\mu\,\partial_\mu\omega^{ij}, \tag{16}$$

and the integral is taken along closed contour C surrounding the disclination axis. The length of the Frank vector is equal to the total angle of rotation of the field n^i as we go around the disclination.

The vector field n^i defines a map of the Euclidean space into sphere $n:\ \mathbb{R}^3 \longrightarrow \mathbb{S}^2$. For linear disclinations parallel to the x^3 axis this map is restricted to a map of the plane $\mathbb{R}^2$ into a circle $\mathbb{S}^1$. In this case the total angle of rotation must be obviously a multiple of 2π.

For nematic liquid crystals we have the equivalence relation $n^i \sim -n^i$. Therefore for linear disclinations parallel to the x^3 axis the director field defines a map of the plane into the projective line, $n:\ \mathbb{R}^2 \longrightarrow \mathbb{RP}^1$. In this case the length of the Frank vector must be a multiple of π. The corresponding examples of disclinations are shown in Fig. 7.



(a) $|\Theta| = \pi$ (b) $|\Theta| = 3\pi$

Figure 7: The director field distribution in the x^1, x^2 plane for the linear disclinations parallel to the x^3 axis.

As in the case of the displacement field, the field $\omega^{ij}(x)$ taking values in the algebra $\mathfrak{so}(3)$ is not a smooth function in $\mathbb{R}^3$ in the presence of disclinations. Let us make a cut in $\mathbb{R}^3$ bounded by the disclination axis. Then the field $\omega^{ij}(x)$ may be considered as smooth in the whole space except the cut. We assume that all partial derivatives of $\omega^{ij}(x)$ have the same limit as far as it approaches the cut from both sides. Then we define a new field

$$\omega_\mu{}^{ij} = \begin{cases} \partial_\mu \omega^{ij}, & \text{outside the cut,} \\ \lim \partial_\mu \omega^{ij}, & \text{on the cut.} \end{cases} \tag{17}$$

The functions $\omega_\mu{}^{ij}$ are smooth by construction everywhere except, may be, the disclination axis. Then the Frank vector may be given by the surface integral

$$\Omega^{ij} = \oint_C dx^\mu \, \omega_\mu{}^{ij} = \iint_S dx^\mu \wedge dx^\nu (\partial_\mu \omega_\nu{}^{ij} - \partial_\nu \omega_\mu{}^{ij}) \,, \tag{18}$$

where S is an arbitrary surface having contour C as the boundary. If the field $\omega_\mu{}^{ij}$ is given then the integrability conditions for the system of equations $\partial_\mu \omega^{ij} = \omega_\mu{}^{ij}$ are the equalities

$$\partial_\mu \omega_\nu{}^{ij} - \partial_\nu \omega_\mu{}^{ij} = 0. \tag{19}$$

This noncovariant equality yields the criterion for the absence of disclinations.

In the geometric theory of defects we identify the field $\omega_\mu{}^{ij}$ with the $\mathbb{SO}(3)$-connection. In the expression for the curvature in Cartan variables

$$R_{\mu\nu j}{}^i = \partial_\mu \omega_{\nu j}{}^i - \omega_{\mu j}{}^k \, \omega_{\nu k}{}^i - (\mu \leftrightarrow \nu) \,, \tag{20}$$

the first two terms coincide with (19), therefore we postulate the covariant criterion of the absence of disclinations as the equality of curvature tensor for $\mathbb{SO}(3)$-connection to zero

$$R_{\mu\nu}{}^{ij} = 0.$$

Simultaneously, we give the physical interpretation of the curvature tensor as the surface density of the Frank vector

$$\Omega^{ij} = \iint dx^\mu \wedge dx^\nu R_{\mu\nu}{}^{ij}. \tag{21}$$

This definition reduces to the previous expression of the Frank vector (18) in the case when rotation of vector n takes place in a fixed plane. In this case rotations are restricted by the subgroup $\mathbb{SO}(2) \subset \mathbb{SO}(3)$. The quadratic terms in the expression for the curvature (20) disappear because the rotation group $\mathbb{SO}(2)$ is Abelian, and we obtain the previous expression for the Frank vector (18).

Thus we described the media with dislocations (defects of elastic media) and disclinations (defects in the spin structure) in the framework of Riemann–Cartan geometry. Here we identified torsion tensor with the surface density of dislocations and curvature tensor with the surface density of disclinations. The relations between physical and geometrical notions are summarized in the Table 1.

Elastic deformations	$R_{\mu\nu}{}^{ij} = 0$	$T_{\mu\nu}{}^{i} = 0$
Dislocations	$R_{\mu\nu}{}^{ij} = 0$	$T_{\mu\nu}{}^{i} \neq 0$
Disclinations	$R_{\mu\nu}{}^{ij} \neq 0$	$T_{\mu\nu}{}^{i} = 0$
Dislocations and disclinations	$R_{\mu\nu}{}^{ij} \neq 0$	$T_{\mu\nu}{}^{i} \neq 0$

Table 1: The relation between physical and geometrical notions in the geometric theory of defects.

5. Conclusion

The geometric theory of defects describes defects in elastic media (dislocations) and defects in the spin structure (disclinations) from the unique point of view. This model can be used for description of single defects as well as their continuous distribution. The geometric theory of defects is based on the Riemann-Cartan geometry. By definition torsion and curvature tensors are equal to surface densities of Burgers and Frank vectors, respectively.

We gave here only the relation between physical and geometric notions. At the moment the static geometric theory of defects is developed much further. In [15] we considered the example of the wedge dislocation both in the frameworks of ordinary elasticity theory and geometric theory of defects and compared the results. This example shows that the elasticity theory reproduces only the linear approximation of the geometric theory of defects. In contrast to the induced metric obtained within the exact solution of the linear elasticity theory, the expression for the metric obtained as the exact solution of the Einstein equations is simpler, defined on the whole space and for all deficit angles. The obtained expression for the metric can be checked experimentally.

In [15] we showed also that the equations of asymmetric elasticity theory for the Cosserat media are naturally embedded in the geometric theory of defects as the gauge conditions. We also considered there two problems as an application of the geometric theory of defects. The first is the scattering of phonons on a wedge dislocation [16]. In the eikonal approximation the problem is reduced to the analysis of extremals for the metric describing a given dislocation. Equations for extremals are integrated explicitly, and the scattering angle is found. The second of the considered problems is the construction of wave functions and energetic spectrum of impurity in the presence of a wedge dislocation. To this purpose we solved the Schrödinger equation. This problem is mathematically equivalent to solution of the Schrödinger equation for bounded states in the Aharonov–Bohm effect [17]. The explicit dependence of the spectrum from the deficit angle and elastic properties of media is found.

Equations defining the static distribution of defects are covariant and have the same form as equations of gravity models with dynamical torsion. To choose a solution uniquely, one must fix the coordinate system. To this end the elastic gauge for the vielbein and the Lorentz gauge for the $\mathbb{SO}(3)$-connection are proposed. If defects are absent then we can introduce the displacement vector field and the field of the spin structure. In this case equations of equilibrium are identically satisfied, and the gauge conditions reduce to the equations of the elasticity theory and of the principal chiral $\mathbb{SO}(3)$-field. In this way the geometric theory of defects incorporate the elasticity theory and the model of principal chiral field.

In a definite sense the elastic gauge represents the equations of nonlinear elasticity theory. Nonlinearity is introduced in elasticity theory in two ways. First, the deformation tensor is defined through the induced metric $\epsilon_{ij} = \frac{1}{2}\left(\delta_{ij} - g_{ij}\right)$ instead of its definition by the linear relation (4). Then the stress tensor is given by the infinite series in the displacement vector. Second, Hook's law can be modified assuming nonlinear dependence of the stress tensor on the deformation tensor. Hence the elastic gauge is the equations of nonlinear elasticity theory where the deformation tensor is assumed to be defined through the induced metric, and Hook's law is kept linear. A generalization to a more general case when the relation between deformation and stress tensors becomes nonlinear is obvious.

The geometric theory of static distribution of defects can be also constructed for the membranes, i.e. on a plane $\mathbb{R}^2$. To this end one has to consider the Euclidean version [18] of two-dimensional gravity with torsion [19]. This model is favored by its integrability [20].

The developed geometric construction in the theory of defects can be inverted, and we can consider the gravity interaction of masses in the Universe as the interaction of defects in elastic ether. Then point masses and cosmic strings [21] correspond to point defects (vacancies and impurities) and wedge dislocations. We have the question in this framework about the elastic gauge which has direct physical meaning in the geometric theory of defects. If we take the standpoint of the theory of defects then the elastic properties of ether correspond to some value of the Poisson ratio which can be measured experimentally.

It seems interesting and important for applications to include time in the considered static approach for describing motion of defects in the media. Such a model is absent at present. From geometric point of view this generalization can be easily performed at least in principle. It is sufficient to change the Euclidean space $\mathbb{R}^3$ to the Minkowski space $\mathbb{R}^{1,3}$ and to write a suitable Lagrangian quadratic in curvature and torsion which corresponds to the true gravity model with torsion. One of the arising problems is the physical interpretation of additional components of the vielbein and Lorentz connection which contain the time index. The physical meaning of the time component of the vielbein ${e_0}^i \to \partial_0 u^i = v^i$ is simple – this is the velocity of a point of media. This interpretation is natural from physical point of view because motion of continuously distributed dislocations means flowing of media. In fact, the liquid can be imagined as the elastic media with continuous distribution of moving dislocations. It means that the dynamical theory of defects based on Riemann–Cartan geometry must include hydrodynamics. It is not clear at present how it could be. Physical interpretation of the other components of the vielbein and the Lorentz connections with the time index remains also obscured.

The author is very grateful to I. V. Volovich for numerous discussions.

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