On symmetry properties of nonlinear Schrödinger equations with potentials

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Abstract

As an example of using the notion of normalized classes to investigate symmetry properties of differential equations, we completely solve the group classification problem in the class of (1+1)-dimensional nonlinear Schrödinger equations $i\psi_t + \psi_{xx} + |\psi|^\gamma\psi + V(t,x)\psi = 0$, where γ is a real non-zero constant and V is an arbitrary complex-valued potential depending on t and x.

1. Introduction

In physical models differential equations often contain parameters or functions which have been found experimentally and so are not strictly fixed, and additionally there often exist a priori requirements to symmetry groups that follow from physical laws (in particular, from Galilei or relativistic theory). At the same time these models have to be simple enough to analyze and solve them. One can formulate the criteria of simplicity that modelling differential equations have to admit a group with certain properties or the most extensive symmetry group from the possible ones.

Group classification is one of symmetry tools used to choose physically relevant models from parametric classes of systems of (partial or ordinary) differential equations. The parameters can be constants or functions of independent variables, unknown functions and their derivatives. Exhaustive

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consideration of the problem of group classification for a parametric class $\mathcal{L}$ of systems of differential equations includes the following steps:

1. Finding the group $G^{\ker}$ (the kernel of maximal invariance groups) of local transformations that are symmetries for all systems from $\mathcal{L}$.
2. Construction of the group $G^{\sim}$ (the equivalence group) of local transformations which transform $\mathcal{L}$ into itself.
3. Description of all possible $G^{\sim}$-inequivalent values of parameters that admit maximal invariance groups wider than $G^{\ker}$.

Following S. Lie, one usually considers infinitesimal transformations instead of finite ones. This approach essentially simplifies the problem of group classification, reducing it to a problem for Lie algebras of vector fields.

Group classification for Schrödinger equations was first performed by S. Lie. More precisely, his classification [2] of all the linear equations with two independent complex variables contains, in an implicit form, solving the classification problem for the linear (1+1)-dimensional Schrödinger equations with an arbitrary potential. Group properties of more complicated classes of Schrödinger equations were studied, in particular, in [1, 3, 5, 6].

In this paper we investigate a class of NSchEs of the form

$$i\psi_t + \psi_{xx} + |\psi|^\gamma\psi + V\psi = 0, \tag{1}$$

where the potential $V = V(t,x)$ is an arbitrary complex-valued smooth function of the variables t and x, γ is a real non-zero constant. (Here and below subscripts of functions denote differentiation with respect to the respective variables.)

2. Normalized classes of differential equations

Let $\mathcal{L}(\theta)$ be a system $L(x,u_{(\rho)},\theta(x,u_{(\rho)})) = 0$ of l differential equations for m unknown functions $u = (u^1,\dots,u^m)$ of n independent variables $x = (x_1,\dots,x_n)$. Here $u_{(\rho)}$ denotes the set of all the derivatives of the functions u with respect to x of order no greater than ρ, including u as the derivatives of the zero order. $L = (L^1,\dots,L^l)$ is a tuple of l fixed functions depending on x, $u_{(\rho)}$ and θ. θ denotes the tuple of arbitrary (parametric) functions $\theta(x,u_{(n)}) = (\theta^1(x,u_{(\rho)}),\dots,\theta^k(x,u_{(\rho)}))$ running the set $\mathcal{S}$ of solutions of the system $S(x,u_{(\rho)},\theta_{(q)}(x,u_{(\rho)})) = 0$. This system consists of differential equations with respect to θ, where x and $u_{(\rho)}$ play the role of independent variables and $\theta_{(q)}$ stands for the set of all the partial derivatives of θ of order no greater than q. In what follows we call the functions θ as arbitrary elements. Denote the class of systems $\mathcal{L}(\theta)$ with the arbitrary elements θ running $\mathcal{S}$ as $\mathcal{L}|_{\mathcal{S}}$.

For θ, $\tilde\theta \in \mathcal{S}$ we will call the set of point transformations which maps the system $\mathcal{L}(\theta)$ into the system $\mathcal{L}(\tilde\theta)$ as the set of admissible transformations from $\mathcal{L}(\theta)$ into $\mathcal{L}(\tilde\theta)$ and will denote it $\mathrm{T}(\theta,\tilde\theta)$. The set

$$\mathrm{T}(\mathcal{L}|_{\mathcal{S}}) = \{\,(\theta,\tilde\theta,\varphi) \mid \theta,\tilde\theta\in\mathcal{S},\ \mathrm{T}(\theta,\tilde\theta)\neq\varnothing,\ \varphi\in\mathrm{T}(\theta,\tilde\theta)\,\}$$

is called as the *set of admissible transformations in* $\mathcal{L}|_{\mathcal{S}}$.

In the above notions, the maximal Lie symmetry group $G^{\max}(\theta)$ of the system $\mathcal{L}(\theta)$ coincides with $\mathrm{T}(\theta,\theta)$ and the equivalence group $G^{\sim}(\mathcal{L}|_{\mathcal{S}})$ of the class $\mathcal{L}|_{\mathcal{S}}$ is the set of admissible transformations which can be applied to any $\theta\in\mathcal{S}$.

The class $\mathcal{L}|_{\mathcal{S}}$ is called *normalized* if any admissible transformation in this class belongs to $G^{\sim}(\mathcal{L}|_{\mathcal{S}})$ and is called *strongly normalized* if additionally the projection $G^{\sim}(\mathcal{L}|_{\mathcal{S}})|_{(x,u)}$ of the equivalence group $G^{\sim}(\mathcal{L}|_{\mathcal{S}})$ on the space of the variables (x,u) is generated by elements from $G^{\max}(\theta)$, $\theta\in\mathcal{S}$.

Lemma 1. *Let G^i, $i=1,2$, be local groups of point transformations in the space of (x,u), for which $\mathcal{S}^i=\{\theta\in\mathcal{S}\mid G^{\max}(\theta)=G^i\}\neq\varnothing$. Then $\mathcal{S}^1\sim\mathcal{S}^2 \bmod G^{\sim}(\mathcal{L}|_{\mathcal{S}})$ iff $G^1\sim G^2 \bmod G^{\sim}(\mathcal{L}|_{\mathcal{S}})$.*

Lemma 2. *If $\mathcal{L}|_{\mathcal{S}}$ is normalized then for any $\theta^0\in\mathcal{S}$ the Lie symmetry group $G^{\max}(\theta^0)$ coincides with restriction, on the space of (x,u), of the subgroup of $G^{\sim}(\mathcal{L}|_{\mathcal{S}})$ preserving the value $\theta=\theta^0(x,u_{(\rho)})$.*

Lemma 3. *Two systems from a normalized class are pointwise equivalent iff they are equivalent with respect to the equivalence group of this class.*

In view of the above propositions, group classification problem in any normalized class of differential equations is reduced to subgroup analysis of the corresponding equivalence group. The property of strong normalization allows us to hope that essential part of subgroups will be Lie symmetry groups of systems from the class under consideration.

All the adduced notions and results can be reformulated in the infinitesimal terms by means of using the notions of operators, Lie algebras instead of transformations, Lie groups etc.

3. Group classification of class (1)

Theorem 1. *Class* (1) *is strongly normalized. The equivalence group $G^{\sim}$ of the class* (1) *is generated by the family of continuous transformations*

$$\tilde t=T,\quad \tilde x=x\sqrt{T_t}+X,\quad \tilde\psi=\psi\frac{1}{(T_t)^{1/\gamma}}\exp\left(\frac{i}{8}\frac{T_{tt}}{T_t}x^2+\frac{i}{2}\frac{X_t}{\sqrt{T_t}}x+i\Psi\right),$$
$$\tilde V=\frac{1}{T_t}\left(V+\left(\frac{T_{tt}}{T_t}\right)_t\frac{x^2}{8}+\left(\frac{X_t}{\sqrt{T_t}}\right)_t\frac{x}{2}+i\frac{\widehat\gamma}{4}\frac{T_{tt}}{T_t}-\left(\frac{T_{tt}}{T_t}\frac{x}{4}+\frac{1}{2}\frac{X_t}{\sqrt{T_t}}\right)^2+\Psi_t\right),$$
$$\tilde\gamma=\gamma, \tag{2}$$

and two discrete transformations: the space reflection I_x ($\tilde t=t$, $\tilde x=-x$, $\tilde\psi=\psi$, $\tilde\gamma=\gamma$, $\tilde V=V$) and the Wigner time reflection I_t ($\tilde t=-t$, $\tilde x=x$, $\tilde\psi=\psi^*$, $\tilde\gamma=\gamma$, $\tilde V=V^*$). Here T, X and Ψ are arbitrary smooth functions of t, $T_t>0$, $\widehat\gamma=\gamma^{-1}(4-\gamma)$.

Note 1. It follows from Theorem 1 that there exist no point local transformations changing γ. Therefore, we can assume that γ *is fixed* in our consideration below and omit it from notations of the maximal Lie invariance algebras of an equation of form (1) etc.

Consider an operator $Q = \xi^t\partial_t + \xi^x\partial_x + \eta\,\partial_\psi + \eta^*\partial_{\psi^*}$ from the maximal Lie invariance algebra $A^{\max}(V)$ of equation (1) with a potential V. Here ξ^t, ξ^x and η are smooth functions of t, x, ψ and ψ^*. The infinitesimal invariance condition [4] of equation (1) with respect to the operator Q implies the linear overdetermined system on the coefficients of Q. Solving this system, we obtain the following theorem.

Theorem 2. *Any operator Q from $A^{\max}(V)$ of equation (1) with arbitrary potential V lies in the linear span of operators of the form*

$$D(\xi) = \xi\partial_t + \frac{1}{2}\,\xi_t x\partial_x + \frac{1}{8}\,\xi_{tt}x^2 M - \frac{1}{\gamma}\,\xi_t I,\ G(\chi) = \chi\partial_x + \frac{1}{2}\,\chi_t xM,\ \lambda M. \quad (3)$$

Here χ, ξ and λ are arbitrary smooth functions of t, $M = i(\psi\partial_\psi - \psi^*\partial_{\psi^*})$, $I = \psi\partial_\psi + \psi^*\partial_{\psi^*}$. Moreover, the coefficients of $Q = D(\xi) + G(\chi) + \lambda M \in A^{\max}(V)$ should satisfy the classifying condition

$$\xi V_t + \left(\frac{1}{2}\,\xi_t x + \chi\right) V_x + \xi_t V = \frac{1}{8}\,\xi_{ttt}x^2 + \frac{1}{2}\,\chi_{tt}x + \lambda_t + i\frac{\widehat{\gamma}}{4}\,\xi_{tt}. \quad (4)$$

Theorem 3. *The Lie algebra of the kernel of maximal Lie invariance groups of equations from class (1) is $A^{\ker} = \langle M\rangle$.*

The linear span of operators of the form (3) (γ is fixed!) is an (infinite-dimensional) Lie algebra $A^\cup$ under the usual Lie bracket of vector fields. Since for any $Q \in A^\cup$ where $(\xi^t, \xi^x) \neq (0,0)$ we can find V satisfying condition (4) then $A^\cup = \langle \bigcup_V A^{\max}(V) \rangle$. We use the notation $\mathrm{Aut}(A^\cup)$ for the automorphism group acting on $A^\cup$, which is generated by all the one-parameter groups corresponding to the adjoint representations of operators of $A^\cup$ into $A^\cup$ and two discrete transformations $\mathrm{Ad}\, I_x$ and $\mathrm{Ad}\, I_t$ included additionally.

Since class (1) is normalized, $G^\sim \simeq \mathrm{Aut}(A^\cup)$, and the isomorphism is determined by means of prolongation of operators from $A^\cup$ to the space (V, V^*). Let A^1 and A^2 be the maximal Lie invariance algebras of equations from class (1) for some potentials, and $\mathcal{V}^i = \{\, V \,|\, A^{\max}(V) = A^i\}$, $i = 1, 2$. Then $\mathcal{V}^1 \sim \mathcal{V}^2 \bmod G^\sim$ iff $A^1 \sim A^2 \bmod \mathrm{Aut}(A^\cup)$. Therefore, it is natural to continue the process of group classification by finding inequivalent subalgebras of $A^\cup$. A complete list of $\mathrm{Aut}\, A^\cup$-inequivalent one-dimensional subalgebras of $A^\cup$ is exhausted by the algebras $\langle\partial_t\rangle$, $\langle\partial_x\rangle$, $\langle tM\rangle$, $\langle M\rangle$.

Let us suppose that there exists an extension of Lie invariance algebra of equations (1), i.e. $A^{\max}(V) \neq A^{\ker}$. Then there exists an operator $Q = D(\xi) + G(\chi) + \lambda M \in A^{\max}(V)$ which does not belong to $\langle M\rangle$. Condition (4) implies $(\xi, \chi) \neq (0,0)$. Therefore, $\langle Q\rangle \sim \langle\partial_t\rangle$ or $\langle\partial_x\rangle$ mod $\mathrm{Aut}\, A^\cup$, i.e. $V_tV_x = 0 \bmod G^{\sim}$, and to complete the group classification, it is sufficient to study two cases: $V_x = 0$ and $V_t = 0$. In fact, we obtain the complete results of group classifications for both special cases and then unite them for the general case under consideration.

The subclass of equations from class (1) with potentials satisfying the additional assumption $V_x = 0$, i.e. $V = V(t)$, is also strongly normalized. Its equivalence group is generated by I_t, I_x and transformations of form (2), where $T = (a_1t+a_0)/(b_1t+b_0)$, $X = c_1t+c_0$, Ψ is an arbitrary smooth function of t. a_i, b_i and c_i are arbitrary constants such that $a_1b_0-b_1a_0 > 0$. We can make for the real part of any x-free potential to vanish with the equivalence transformations. The more restricted subclass $\mathcal{W}$ of equations (1) with pure imaginary x-free potentials is also strongly normalized, and moreover, its equivalence group $G^{\sim}(\mathcal{W})$ is finite-parameter and is subgroup of the above equivalence group under restriction $\Psi = \text{const}$. Let us note that $G^{\sim}(\mathcal{W}) = G^{\sim}_{\text{ker}}(\mathcal{W}) \times S$. Here $G^{\sim}_{\text{ker}}(\mathcal{W})$ is a normal subgroup of $G^{\sim}(\mathcal{W})$. It is formed by transformations $G^{\sim}(\mathcal{W})$ with $T = 1$, which do not change potentials, and is isomorphic to the kernel of maximal Lie invariance groups of equations from $\mathcal{W}$. The subgroup S consists of the transformations with $X = \Phi = 1$ and isomorphic to $SL(2,\mathbb{R})$. Therefore, the problem of group classification in the case of x-free potentials is reduced to subgroup analysis of $SL(2,\mathbb{R})$.

If equation (1) with a stationary potential admits extension of Lie symmetry group then it is equivalent either a one with an x-free potential or a one with the potential $V = V(x)$ satisfying the condition

$$xV_x + 2V = c_2x^2 + c_1x + \tilde{c}_0 + ic_0, \quad \text{where} \quad c_2, c_1, \tilde{c}_0, c_0 = \text{const} \in \mathbb{R}.$$

For more detail we refer the reader to [5].

Theorem 4. *A complete set of inequivalent cases of V admitting extensions of Lie invariance algebras is exhausted by the potentials given in the table below. Here $W(t), \nu, \alpha, \beta \in \mathbb{R}$, $(\alpha, \beta) \neq (0, 0)$.*

N	Potential	Basis of $A^{\max}$
0	$V(t,x)$	M
1	$iW(t)$	$M,\ \partial_x,\ G(t)$
2	$\dfrac{i}{2}\dfrac{\widehat{\gamma}\, t+\nu}{t^2+1}$	$M,\ \partial_x,\ G(t),\ D(t^2+1)$
3	$i\nu t^{-1},\quad \nu \neq 0, \dfrac{\widehat{\gamma}}{2}$	$M,\ \partial_x,\ G(t),\ D(t)$
4	i	$M,\ \partial_x,\ G(t),\ \partial_t$
5	$0,\ \gamma \neq 4$	$M,\ \partial_x,\ G(t),\ \partial_t,\ D(t)$
	$\gamma = 4$	$M,\ \partial_x,\ G(t),\ \partial_t,\ D(t),\ D(t^2)$
6	$V(x)$	$M,\ \partial_t$
7	$(\alpha + i\beta)x^{-2},\ \gamma \neq 4$	$M,\ \partial_t,\ D(t)$
	$\gamma = 4$	$M,\ \partial_t,\ D(t),\ D(t^2)$

Note 2. We mean that the invariance algebras for Cases 0, 1, 6 are maximal if these cases are inequivalent under the corresponding equivalence group to the other, more specialized, cases from the same table.

Note 3. There exists a discrete equivalence transformation τ for the set of potentials $i\nu t^{-1}$, $\nu \in \mathbb{R}$, which has form (2) with $T = -t^{-1}$, $X = 0$, $\Psi = 0$. It transforms ν in the following way: $\nu \to \widehat{\gamma}/2 - \nu$. For the cases under consideration to be completely inequivalent, we have to assume additionally that $\nu \geq \widehat{\gamma}/4$ (or $\nu \leq \widehat{\gamma}/4$) in Case 3. Since $I_t \in G^{\sim}$ we can assume analogously $\nu \geq 0$ in Case 2 and $\beta \geq 0$ in Case 7. Moreover, τ is a discrete symmetry transformation for Case 3 ($\nu = \widehat{\gamma}/4$) and, as a limit of the continuous transformations generated by the operator $D(t^2+1)$, for Case 2.

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