Noncommutative Quantum Mechanics with Path Integral*

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Abstract

We consider classical and quantum mechanics related to an additional noncommutativity, symmetric in position and momentum coordinates. We show that such mechanical system can be transformed to the corresponding one which allows employment of the usual formalism. In particular, we found explicit connections between quadratic Hamiltonians and Lagrangians, in their commutative and noncommutative regimes. In the quantum case we give general procedure how to compute Feynman's path integral in this noncommutative phase space with quadratic Lagrangians (Hamiltonians). This approach is applied to a charged particle in the noncommutative plane exposed to constant homogeneous electric and magnetic fields.

1. Introduction

Looking for an approach to solve the problem of ultraviolet divergences, already in the 1930s Heisenberg conjectured that position coordinates might be noncommutative (NC). Snyder [1] was the first who started systematically to develop this idea in 1947. An intensive interest to NC quantum theories emerged after observation of noncommutativity in string theory with D-branes in 1998. Most of the research has been devoted to noncommutative (NC) field theory (for reviews, see e.g. [2] and [3]). NC quantum mechanics (NCQM) has been also actively investigated. It enables construction of simple NC models which have relevance to concrete phenomenological systems and can be regarded as the corresponding one-particle nonrelativistic sector of NC quantum field theory.

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We consider here D-dimensional NCQM which is based on the following algebra:

$$[\hat{x_a}, \hat{p_b}] = i\,\hbar\,(\delta_{ab} - \frac{1}{4}\,\theta_{ac}\,\sigma_{cb}), \quad [\hat{x_a}, \hat{x_b}] = i\,\hbar\,\theta_{ab}, \quad [\hat{p_a}, \hat{p_b}] = i\,\hbar\,\sigma_{ab}, \quad (1)$$

where $(\theta_{ab}) = \Theta$ and $(\sigma_{ab}) = \Sigma$ are the antisymmetric matrices with constant elements. This kind of unusual noncommutativity maintains a symmetry between canonical variables of the phase space. It also allows simple reduction to the usual algebra

$$[\hat{q_a}, \hat{k_b}] = i\,\hbar\,\delta_{ab}, \qquad [\hat{q_a}, \hat{q_b}] = 0, \qquad [\hat{k_a}, \hat{k_b}] = 0, \quad (2)$$

using the following linear transformations:

$$\hat{x_a} = \hat{q_a} - \frac{\theta_{ab}\,\hat{k_b}}{2}\,, \qquad \hat{p_a} = \hat{k_a} + \frac{\sigma_{ab}\,\hat{q_b}}{2}\,, \quad (3)$$

where summation over repeated indices is understood. In the sequel we often take $\theta_{ab} = \theta\,\varepsilon_{ab}$ and $\sigma_{ab} = \sigma\,\varepsilon_{ab}$, where

$$\varepsilon_{ab} = \begin{cases} 1\,, & a < b \\ 0\,, & a = b \\ -1\,, & a > b\,. \end{cases} \quad (4)$$

The standard Feynman path integral [4]

$$\mathcal{K}(x'', t''; x', t') = \int_{x'}^{x''} \exp\left(\frac{i}{\hbar}\int_{t'}^{t''} L(\dot{q}, q, t)\, dt\right) \mathcal{D}q\,, \quad (5)$$

where $\mathcal{K}(x'', t''; x', t')$ is the kernel of the unitary evolution operator $U(t)$ and $x'' = q(t'')$, $x' = q(t')$ are end points, we generalize to the above NCQM with quadratic Lagrangians. Such Lagrangians contain many important and exactly solvable quantum-mechanical systems. In ordinary quantum mechanics (OQM), Feynman's path integral for quadratic Lagrangians can be evaluated analytically and obtained result has the form [5]

$$\mathcal{K}(x'', t''; x', t') = \frac{1}{(ih)^{\frac{D}{2}}}\sqrt{\det\left(-\frac{\partial^2 \bar{S}}{\partial x''_a \partial x'_b}\right)} \exp\left(\frac{2\pi i}{h}\,\bar{S}(x'', t''; x', t')\right), \quad (6)$$

where $\bar{S}(x'', t''; x', t')$ is the action for the classical trajectory. According to (1), (2) and (3), NCQM related to the quantum phase space $(\hat{p},\, \hat{x})$ can be regarded as an OQM on the standard phase space $(\hat{k},\, \hat{q})$ and one can apply usual path integral formalism.

A systematic path integral approach to NCQM with quadratic Lagrangians (Hamiltonians) has been investigated during the last few years in [6], [7] and [8]. In [6] and [7], general connections between quadratic Lagrangians and Hamiltonians for standard and $\theta \neq 0$, $\sigma = 0$ noncommutativity are established, and this formalism was applied to the two-dimensional NC particle in a constant field and to the NC harmonic oscillator. Paper [8] presents generalization of papers [6] and [7] on noncommutativity (1). The present article reviews general formalism of [8] and contains its application to a charged particle in NC plane exposed to homogeneous electric and magnetic fields.

2. Quadratic Lagrangians and Hamiltonians Related to Noncommutative Phase Space

Let the most general quadratic Lagrangian for a D-dimensional system with position coordinates $x^T = (x_1, x_2, \cdots, x_D)$ be

$$L(\dot{x}, x, t) = \frac{1}{2}\,\left(\dot{x}^T \alpha\, \dot{x} + \dot{x}^T \beta\, x + x^T \beta^T \dot{x} + x^T \gamma\, x\right) + \delta^T \dot{x} + \eta^T x + \phi\,, \quad (7)$$

where coefficients of the $D \times D$ matrices $\alpha = ((1 + \delta_{ab})\, \alpha_{ab}(t))$, $\beta = (\beta_{ab}(t))$, $\gamma = ((1 + \delta_{ab})\, \gamma_{ab}(t))$, D-dimensional vectors $\delta = (\delta_a(t))$, $\eta = (\eta_a(t))$ and a scalar $\phi = \phi(t)$ are some analytic functions of the time t. Matrices α and γ are symmetric, α is nonsingular ($\det \alpha \neq 0$) and index ^{T} denotes transposition.

The Lagrangian (7) can be rewritten in the more compact form:

$$L(X, t) = \frac{1}{2}\, X^T\, M\, X + N^T\, X + \phi, \quad (8)$$

where $2D \times 2D$ matrix M and $2D$-dimensional vectors X, N are defined as

$$M = \begin{pmatrix} \alpha & \beta \\ \beta^T & \gamma \end{pmatrix}, \qquad X^T = (\dot{x}^T\,,\, x^T)\,, \qquad N^T = (\delta^T\,,\, \eta^T). \quad (9)$$

Using the equations $p_a = \frac{\partial L}{\partial \dot{x}_a}$, one finds $\dot{x} = \alpha^{-1}\,(p - \beta\, x - \delta)$. Since the function $\dot{x}$ is linear in p and x, the corresponding classical Hamiltonian $H(p, x, t) = p^T\, \dot{x} - L(\dot{x}, x, t)$ becomes also quadratic, i.e.

$$H(p, x, t) = \frac{1}{2}\,\left(p^T A p + p^T B x + x^T B^T p + x^T C x\right) + D^T p + E^T x + F, \quad (10)$$

where:

$$\begin{aligned} &A = \alpha^{-1}, && B = -\,\alpha^{-1}\,\beta, && C = \beta^T\,\alpha^{-1}\,\beta - \gamma, \\ &D = -\,\alpha^{-1}\,\delta, && E = \beta^T\,\alpha^{-1}\,\delta - \eta, && F = \frac{1}{2}\,\delta^T\,\alpha^{-1}\delta\, - \phi\,. \end{aligned} \quad (11)$$

Due to the symmetry of matrices α and γ one can easily see that matrices $A = ((1 + \delta_{ab})\, A_{ab}(t))$ and $C = ((1 + \delta_{ab})\, C_{ab}(t))$ are also symmetric

($A^T = A,\ C^T = C$). The nonsingular ($\det\alpha \neq 0$) Lagrangian $L(\dot{x}, x, t)$ implies nonsingular ($\det A \neq 0$) Hamiltonian $H(p, x, t)$. Note that the inverse map, i.e. $H \to L$, is given by the same relations (11).

The Hamiltonian (10) can be also presented in the compact form

$$H(\Pi, t) = \frac{1}{2}\, \Pi^T \mathcal{M}\, \Pi + \mathcal{N}^T\, \Pi + F, \tag{12}$$

where matrix $\mathcal{M}$ and vectors Π, $\mathcal{N}$ are

$$\mathcal{M} = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix}, \qquad \Pi^T = (p^T\,,\, x^T)\,, \qquad \mathcal{N}^T = (D^T\,,\, E^T)\,. \tag{13}$$

One can easily show that

$$\mathcal{M} = \sum_{i=1}^{3} \Upsilon_i^T(M)\, M\, \Upsilon_i(M), \tag{14}$$

where

$$\Upsilon_1(M) = \begin{pmatrix} \alpha^{-1} & 0 \\ 0 & -I \end{pmatrix}, \qquad \Upsilon_2(M) = \begin{pmatrix} 0 & \alpha^{-1}\beta \\ 0 & 0 \end{pmatrix}, \qquad \Upsilon_3(M) = \begin{pmatrix} 0 & 0 \\ 0 & i\sqrt{2}\, I \end{pmatrix}, \tag{15}$$

and I is $D \times D$ unit matrix. One has also $\mathcal{N} = Y(M)\, N$, where

$$Y(M) = \begin{pmatrix} -\,\alpha^{-1} & 0 \\ \beta^T\, \alpha^{-1} & -I \end{pmatrix} = -\Upsilon_1(M) + \Upsilon_2^T(M) + i\,\sqrt{2}\ \Upsilon_3(M)\,, \tag{16}$$

and $F = N^T\, Z(M)\, N - \phi$, where

$$Z(M) = \begin{pmatrix} \frac{1}{2}\,\alpha^{-1} & 0 \\ 0 & 0 \end{pmatrix} = \frac{1}{2}\,\Upsilon_1(M) - \frac{i}{2\sqrt{2}}\,\Upsilon_3(M)\,. \tag{17}$$

We have shown that Hamiltonian compact quantities $\mathcal{M}, \mathcal{N}$ and F can be related to the corresponding Lagrangian ones M, N and ϕ using auxiliary matrices $\Upsilon_1(M), \Upsilon_2(M)$ and $\Upsilon_3(M)$.

Eqs. (3) can be rewritten in the compact form as

$$\hat{\Pi} = \Xi\, \hat{K}, \qquad \Xi = \begin{pmatrix} I & \frac{1}{2}\,\Sigma \\ -\frac{1}{2}\,\Theta & I \end{pmatrix}, \qquad \hat{K} = \begin{pmatrix} \hat{k} \\ \hat{q} \end{pmatrix}. \tag{18}$$

Since Hamiltonians depend on canonical variables, the transformation (18) leads to the transformation of Hamiltonians (10) and (12). To this end, let us quantize the Hamiltonian (10) and it easily becomes

$$H(\hat{p}, \hat{x}, t) = \frac{1}{2}\,(\hat{p}^T A\,\hat{p} + \hat{p}^T B\,\hat{x} + \hat{x}^T B^T \hat{p} + \hat{x}^T C\,\hat{x}) + D^T\hat{p} + E^T\hat{x} + F \tag{19}$$

because (10) is already written in the Weyl symmetric form.

Performing linear transformations (3) in the above Hamiltonian we again obtain quadratic quantum Hamiltonian

$$\begin{aligned} H_{\theta\sigma}(\hat{k}, \hat{q}, t) = \frac{1}{2}\, &\Big(\hat{k}^T A_{\theta\sigma}\,\hat{k} + \hat{k}^T B_{\theta\sigma}\,\hat{q} + \hat{q}^T B_{\theta\sigma}^T\,\hat{k} + \hat{q}^T C_{\theta\sigma}\,\hat{q}\Big) \\ &+ D_{\theta\sigma}^T\,\hat{k} + E_{\theta\sigma}^T\,\hat{q} + F_{\theta\sigma}\,, \end{aligned} \tag{20}$$

where

$$\begin{aligned} A_{\theta\sigma} &= A - \frac{1}{2}\,B\,\Theta + \frac{1}{2}\,\Theta\,B^T - \frac{1}{4}\,\Theta\,C\,\Theta, \qquad & D_{\theta\sigma} &= D + \frac{1}{2}\,\Theta\,E, \\ B_{\theta\sigma} &= B + \frac{1}{2}\,\Theta\,C + \frac{1}{2}\,A\,\Sigma + \frac{1}{4}\,\Theta\,B^T\,\Sigma, \qquad & E_{\theta\sigma} &= E - \frac{1}{2}\,\Sigma\,D, \\ C_{\theta\sigma} &= C - \frac{1}{2}\,\Sigma\,B + \frac{1}{2}\,B^T\,\Sigma - \frac{1}{4}\,\Sigma\,A\,\Sigma, \qquad & F_{\theta\sigma} &= F. \end{aligned} \tag{21}$$

Note that for the nonsingular Hamiltonian $H(\hat{p}, \hat{x}, t)$ and for sufficiently small θ_{ab} the Hamiltonian $H_{\theta\sigma}(\hat{k}, \hat{q}, t)$ is also nonsingular. It is worth noting that $A_{\theta\sigma}$ and $D_{\theta\sigma}$ do not depend on σ, as well as $C_{\theta\sigma}$ and $E_{\theta\sigma}$ do not depend on θ. Classical analogue of (20) maintains the same form

$$\begin{aligned} H_{\theta\sigma}(k, q, t) = \frac{1}{2}\, &(k^T A_{\theta\sigma}\,k + k^T B_{\theta\sigma}\,q + q^T B_{\theta\sigma}^T\,k + q^T C_{\theta\sigma}\,q) \\ &+ D_{\theta\sigma}^T\,k + E_{\theta\sigma}^T\,q + F_{\theta\sigma}\,. \end{aligned}$$

In the more compact form, Hamiltonian (20) is

$$\hat{H}_{\theta\sigma}(\hat{K}, t) = \frac{1}{2}\,\hat{K}^T\,\mathcal{M}_{\theta\sigma}\,\hat{K} + \mathcal{N}_{\theta\sigma}^T\,\hat{K} + F_{\theta\sigma}, \tag{22}$$

where $2D \times 2D$ matrix $\mathcal{M}_{\theta\sigma}$ and $2D$-dimensional vectors $\hat{K}$, $\mathcal{N}_{\theta\sigma}$ are

$$\mathcal{M}_{\theta\sigma} = \begin{pmatrix} A_{\theta\sigma} & B_{\theta\sigma} \\ B_{\theta\sigma}^T & C_{\theta\sigma} \end{pmatrix}, \qquad \hat{K}^T = (\hat{k}^T\,,\,\hat{q}^T), \qquad \mathcal{N}_{\theta\sigma}^T = (D_{\theta\sigma}^T\,,\,E_{\theta\sigma}^T). \tag{23}$$

From (12), (18) and (22) one can find connections between $\mathcal{M}_{\theta\sigma}$, $\mathcal{N}_{\theta\sigma}$, $F_{\theta\sigma}$ and $\mathcal{M}$, $\mathcal{N}$, F, which are given by the following relations:

$$\mathcal{M}_{\theta\sigma} = \Xi^T\,\mathcal{M}\,\Xi\,, \qquad \mathcal{N}_{\theta\sigma} = \Xi^T\,\mathcal{N}, \qquad F_{\theta\sigma} = F. \tag{24}$$

The corresponding Schrödinger equation is

$$i\,\hbar \frac{\partial \Psi(q,t)}{\partial t} = H_{\theta\sigma}(\hat{k},q,t)\,\Psi(q,t) \tag{25}$$

and this approach has been mainly exploited to analyze dynamical evolution in NCQM .

To compute a path integral, which is a basic instrument in Feynman's approach to quantum mechanics, one can start from its Hamiltonian formulation on the phase space. However, such path integral on a phase space can be reduced to the Lagrangian path integral on configuration space whenever Hamiltonian is a quadratic polynomial with respect to momentum k (see, e.g. [7]). Hence, we would like to have the corresponding classical Lagrangians related to the Hamiltonians (20) and (22). Using equations $\dot{q}_a = \frac{\partial H_{\theta\sigma}}{\partial p_a}$ which give $k = A_{\theta\sigma}^{-1}\,(\dot{q} - B_{\theta\sigma}\,q - D_{\theta\sigma})$ we can pass from Hamiltonian (20) to the corresponding Lagrangian by relation $L_{\theta\sigma}(\dot{q},q,t) = k^T\dot{q} - H_{\theta\sigma}(k,q,t)$. Note that coordinates q_a and x_a coincide when $\theta = \sigma = 0$. Performing necessary computations we obtain

$$\begin{aligned} L_{\theta\sigma}(\dot{q},q,t) = \frac{1}{2}\,&\left(\dot{q}^T\alpha_{\theta\sigma}\,\dot{q} + \dot{q}^T\beta_{\theta\sigma}\,q + q^T\beta_{\theta\sigma}^T\,\dot{q} + q^T\gamma_{\theta\sigma}\,q\right)\\ &+ \delta_{\theta\sigma}^T\,\dot{q} + \eta_{\theta\sigma}^T\,q + \phi_{\theta\sigma}, \end{aligned} \tag{26}$$

or in the compact form:

$$L_{\theta\sigma}(Q,t) = \frac{1}{2}\,Q^T\,M_{\theta\sigma}\,Q + N_{\theta\sigma}^T\,Q + \phi_{\theta\sigma}, \tag{27}$$

where

$$M_{\theta\sigma} = \begin{pmatrix} \alpha_{\theta\sigma} & \beta_{\theta\sigma} \\ \beta_{\theta\sigma}^T & \gamma_{\theta\sigma} \end{pmatrix}, \quad Q^T = (\dot{q}^T\,,\,q^T)\,, \quad N^T = (\delta_{\theta\sigma}^T\,,\,\eta_{\theta\sigma}^T). \tag{28}$$

Then the connection between $M_{\theta\sigma}$, $N_{\theta\sigma}$, $\phi_{\theta\sigma}$ and M, N, ϕ are given by the following relations:

$$\begin{aligned} M_{\theta\sigma} &= \sum_{i,j=1}^{3} \Xi_{ij}^T\;M\;\Xi_{ij}, & \Xi_{ij} &= \Upsilon_i(M)\;\Xi\;\Upsilon_j(\mathcal{M}_{\theta\sigma}),\\ N_{\theta\sigma} &= Y(\mathcal{M}_{\theta\sigma})\,\Xi^T\,Y(M)\,N, & \phi_{\theta\sigma} &= \mathcal{N}_{\theta\sigma}^T\,Z(\mathcal{M}_{\theta\sigma})\,\mathcal{N}_{\theta\sigma} - F\,. \end{aligned} \tag{29}$$

In more detail, the connection between coefficients of the Lagrangians $L_{\theta\sigma}$

and L is given by the relations:

$$\begin{aligned}
\alpha_{\theta\sigma} &= [\,\alpha^{-1} - \tfrac{1}{2}\,(\Theta\,\beta^T\,\alpha^{-1} - \alpha^{-1}\,\beta\,\Theta) - \tfrac{1}{4}\,\Theta\,(\beta^T\,\alpha^{-1}\,\beta - \gamma)\,\Theta\,]^{-1}\,,\\
\beta_{\theta\sigma} &= \alpha_{\theta\sigma}\,(\alpha^{-1}\,\beta - \tfrac{1}{2}\,(\alpha^{-1}\,\Sigma - \Theta\,\gamma + \Theta\,\beta^T\,\alpha^{-1}\,\beta) + \tfrac{1}{4}\,\Theta\,\beta^T\,\alpha^{-1}\,\Sigma)\,,\\
\gamma_{\theta\sigma} &= \gamma + \beta_{\theta\sigma}^T\,\alpha_{\theta\sigma}^{-1}\,\beta_{\theta\sigma} - \beta^T\,\alpha^{-1}\,\beta + \tfrac{1}{4}\,\Sigma\,\alpha^{-1}\,\Sigma\\
&\quad - \tfrac{1}{2}\,(\Sigma\,\alpha^{-1}\,\beta - \beta^T\,\alpha^{-1}\,\Sigma)\,, \qquad (30)\\
\delta_{\theta\sigma} &= \alpha_{\theta\sigma}\,(\alpha^{-1}\,\delta + \tfrac{1}{2}\,(\Theta\,\eta - \Theta\,\beta^T\,\alpha^{-1}\,\delta))\,,\\
\eta_{\theta\sigma} &= \eta + \beta_{\theta\sigma}^T\,\alpha_{\theta\sigma}^{-1}\,\delta_{\theta\sigma} - \beta^T\,\alpha^{-1}\,\delta - \tfrac{1}{2}\,\Sigma\,\alpha^{-1}\,\delta\,,\\
\phi_{\theta\sigma} &= \phi + \tfrac{1}{2}\,\delta_{\theta\sigma}^T\,\alpha_{\theta\sigma}^{-1}\,\delta_{\theta\sigma} - \tfrac{1}{2}\,\delta^T\,\alpha^{-1}\,\delta\,.
\end{aligned}$$

Note that $\alpha_{\theta\sigma}$, $\delta_{\theta\sigma}$ and $\phi_{\theta\sigma}$ do not depend on σ.

3. Noncommutative Path Integral

If we know Lagrangian (7) and algebra (1) we can obtain the corresponding effective Lagrangian (26) suitable for the path integral in NCQM. Exploiting the Euler-Lagrange equations

$$\frac{\partial L_{\theta\sigma}}{\partial q_a} - \frac{d}{dt}\frac{\partial L_{\theta\sigma}}{\partial \dot{q}_a} = 0$$

one can obtain the classical trajectory $q_a = q_a(t)$ connecting end points $x' = q(t')$ and $x'' = q(t'')$, and the corresponding action

$$\bar{S}_{\theta\sigma}(x'', t''; x', t') = \int_{t'}^{t''} L_{\theta\sigma}(\dot{q}, q, t)\, dt.$$

Path integral in NCQM is a direct analogue of (5) and its exact expression in the form of quadratic actions $\bar{S}_{\theta\sigma}(x'', t''; x', t')$ is

$$\mathcal{K}_{\theta\sigma}(x'', t''; x', t') = \frac{1}{(ih)^{\frac{D}{2}}}\sqrt{\det\left(-\frac{\partial^2 \bar{S}_{\theta\sigma}}{\partial x''_a\,\partial x'_b}\right)}\exp\left(\frac{2\,\pi\,i}{h}\,\bar{S}_{\theta\sigma}(x'', t''; x', t')\right). \qquad (31)$$

4. A charged particle in a noncommutative plane exposed to homogeneous electric and magnetic fields

As an example to illustrate many features of the above formalism we consider a particle of a charge $e > 0$ moving in a plane (x_1, x_2) with noncommutativity parameters θ and σ. Let this particle be also under the influence of a constant electric field $\mathcal{E}$ along coordinate x_1 and a constant magnetic field $\mathcal{B}$ perpendicular to the plane and oriented along the axis $-x_3$. It is suitable to start from the nonrelativistic Hamiltonian

$$H(p,x) = \frac{1}{2\,m}\Big[(p_1 - e\,\mathcal{A}_1)^2 + (p_2 - e\,\mathcal{A}_2)^2\Big] + e\,\varphi, \tag{32}$$

where $\mathcal{A}_1 = \frac{\mathcal{B}}{2}\,x_2$, $\mathcal{A}_2 = -\frac{\mathcal{B}}{2}\,x_1$ and $\varphi = -\mathcal{E}\,x_1$. Using the inverse map of (11) we get the corresponding Lagrangian

$$L(\dot{x},x) = \frac{m}{2}\,(\dot{x}_1^2 + \dot{x}_2^2) + \frac{e\,\mathcal{B}}{2}\,(\dot{x}_1 x_2 - \dot{x}_2 x_1) + e\,\mathcal{E}\,x_1. \tag{33}$$

Employing formulas (21) and (30) one obtains Hamiltonian

$$\begin{aligned} H_{\theta\sigma}(k,q) = {}& \frac{1}{2\mu}\,(k_1^2 + k_2^2) - \frac{\lambda}{2\mu}\,(k_1\,q_2 - k_2\,q_1) + \frac{\lambda^2}{8\mu}\,(q_1^2 + q_2^2) \\ & + \frac{\theta\,e\,\mathcal{E}}{2}\,k_2 - e\,\mathcal{E}\,q_1 \end{aligned} \tag{34}$$

and Lagrangian

$$L_{\theta\sigma}(\dot{q},q) = \frac{\mu}{2}\,(\dot{q}_1^2 + \dot{q}_2^2) + \frac{\lambda}{2}\,(\dot{q}_1\,q_2 - \dot{q}_2\,q_1) - \frac{\mu\theta\,e\,\mathcal{E}}{2}\,\dot{q}_2 + \nu_0\,q_1 + \frac{\mu\,\theta^2 e^2\,\mathcal{E}^2}{8}\,, \tag{35}$$

where

$$\mu = \frac{m}{(1 - \frac{\theta\,e\,\mathcal{B}}{4})^2}\,, \qquad \lambda = \frac{e\,\mathcal{B} - \sigma}{1 - \frac{\theta\,e\,\mathcal{B}}{4}}\,, \qquad \nu_0 = e\,\mathcal{E}\left(1 + \frac{\theta\,\lambda}{4}\right). \tag{36}$$

The above Hamiltonian and Lagrangian are related to the dynamics in noncommutative phase space, where noncommutativity is characterized by parameters θ and σ.

The Lagrangian given by (35) implies the Euler-Lagrange equations,

$$\mu\,\ddot{q}_1 + \lambda\,\dot{q}_2 = \nu_0, \qquad \mu\,\ddot{q}_2 - \lambda\,\dot{q}_1 = 0\,. \tag{37}$$

One can transform the system (37) to

$$\mu^2\,q_1^{(3)} + \lambda^2\,q_1^{(1)} = 0, \qquad \mu^2\,q_2^{(3)} + \lambda^2\,q_2^{(1)} - \lambda\,\nu_0 = 0\,. \tag{38}$$

The solution of the equations (38) has the following form

$$\begin{aligned} q_1(t) &= C_1 + C_2\cos(\eta\,t) + C_3\sin(\eta\,t)\,, \\ q_2(t) &= D_1 + D_2\,\cos(\eta\,t) + D_3\sin(\eta\,t) + \frac{\nu_0}{\lambda}\,t\,, \qquad \eta = \frac{\lambda}{\mu}\,. \end{aligned} \tag{39}$$

Imposing coupled differential equations (37) on q_1 and q_2 , we obtain the following connection between constants: $C_2 = D_3, \quad D_2 = -C_3$. The unknown constants C_1, D_1, C_3 and D_3 can be fixed from initial conditions given by

$$q_1(0) = x'_1, \qquad q_1(T) = x''_1, \qquad q_2(0) = x'_2, \qquad q_2(T) = x''_2. \tag{40}$$

The corresponding constants are :

$$\begin{aligned}
C_1 &= \frac{x'_1 + x''_1}{2} + \frac{x'_2 - x''_2}{2}\cot\Big[\frac{T\eta}{2}\Big] + \frac{T\nu_0}{2\lambda}\cot\Big[\frac{T\eta}{2}\Big],\\
C_3 &= \frac{-x'_2 + x''_2}{2} - \frac{x'_1 - x''_1}{2}\cot\Big[\frac{T\eta}{2}\Big] - \frac{T\nu_0}{2\lambda},\\
D_1 &= \frac{x'_2 + x''_2}{2} - \frac{x'_1 - x''_1}{2}\cot\Big[\frac{T\eta}{2}\Big] - \frac{T\nu_0}{2\lambda},\\
D_3 &= \frac{x'_1 - x''_1}{2} - \frac{x'_2 - x''_2}{2}\cot\Big[\frac{T\eta}{2}\Big] - \frac{T\nu_0}{2\lambda}\cot\Big[\frac{T\eta}{2}\Big].
\end{aligned} \tag{41}$$

Inserting the above expressions for constants (41) into (39) we obtain solutions of Euler–Lagrange equations (37). For these solutions and their time derivatives we find the following expression for Lagrangian (35):

$$\begin{aligned}
L_{\theta\sigma}(\dot q, q) &= \frac{1}{8\lambda^2\mu}\Big(\mu(4C_1\lambda^2\nu_0 + \mu(-2\nu_0 + \theta\lambda e\,\mathcal{E})^2) + 4\lambda^2\\
&\times\Big((C_3\lambda(D_1\lambda + \nu_0 t) - D_3(C_1\lambda^2 - 3\mu\nu_0 + \theta\lambda\mu e\,\mathcal{E}))\cos[\eta t]\\
&- ((C_1C_3 + D_1D_3)\lambda^2 + D_3\lambda\nu_0 t - 3C_3\mu\nu_0 + C_3\theta\lambda\mu e\,\mathcal{E})\sin[\eta t]\Big)\Big).
\end{aligned} \tag{42}$$

Using (42), we finally compute the corresponding action

$$\begin{aligned}
\bar S_{\theta\sigma}(x'', T; x', 0) &= \int_0^T L_{\theta\sigma}(\dot q, q)\,dt = \frac{\lambda}{2}\Big(x'_2\,x''_1 - x'_1\,x''_2\Big) + \frac{\nu_0 T}{2}\Big(x'_1 + x''_1\Big)\\
&+ \frac{\mu\nu_0}{\lambda}\Big(x''_2 - x'_2\Big) + \frac{\theta\mu e\,\mathcal{E}}{2}\Big(x'_2 - x''_2\Big) + \frac{\mu T}{8}\Big(-4\,\frac{\nu_0^2}{\lambda^2} + \theta^2 e^2\mathcal{E}^2\Big)\\
&+ \frac{(x'_1 - x''_1)^2\,\lambda^2 + ((x'_2 - x''_2)\,\lambda + T\,\nu_0)^2}{4\,\lambda}\cot\Big[\frac{\eta\,t}{2}\Big].
\end{aligned} \tag{43}$$

Accordingly we obtain

$$\det\left(-\frac{\partial^2\bar S_{\theta\sigma}}{\partial x''_a\,\partial x'_b}\right) = \frac{\lambda^2}{4\,\sin^2[\frac{\lambda T}{2}]}\ , \tag{44}$$

and finally

$$\mathcal{K}_{\theta\sigma}(x'', T; x', 0) = \frac{|\lambda|}{2\,i\,h\,|\sin[\frac{\lambda T}{2}]|}\ \exp\left(\frac{2\pi i}{h}\,\bar{S}_{\theta\sigma}(x'', T; x', 0)\right), \tag{45}$$

where $\bar{S}_{\theta\sigma}(x'', T; x', 0)$ is given by (43).

5. Discussion and Concluding Remarks

Let us mention that taking $\sigma = 0,\ \theta = 0$ in the above formulas we recover expressions for the Lagrangian $L(\dot{q}, q)$, action $\bar{S}(x'', T; x', 0)$ and probability amplitude $\mathcal{K}(x'', T; x', 0)$ of the ordinary commutative case.

Note that a similar path integral approach with $\sigma = 0$ has been considered in the context of the Aharonov-Bohm effect, the Casimir effect, a quantum system in a rotating frame, and the Hall effect (references on these and some other related subjects can be found in [6, 7, 8]). Our investigation contains all quantum-mechanical systems with quadratic Hamiltonians (Lagrangians) on NC phase space given by (1).

On the basis of the expressions presented in this article, there are many possibilities to discuss noncommutative quantum-mechanical systems with respect to various values of noncommutativity parameters θ and σ. We will discuss only some aspects of the above two-dimensional model.

Since

$$[\hat{\pi}_a, \hat{\pi}_b] = -i\hbar\,(e\mathcal{B} - \sigma)\Big(1 - \frac{e\mathcal{B}\theta}{4}\Big)\,\varepsilon_{ab}, \tag{46}$$

where $\hat{\pi}_a = \hat{p}_a - e\hat{\mathcal{A}}_a$, a charged particle in a plane with the perpendicular background magnetic field $\mathcal{B}$ and phase space noncommutativity (1) lives under an extended momentum noncommutativity depending on $-(e\mathcal{B} - \sigma)(1 - \frac{e\mathcal{B}\theta}{4})$. Note that p is a canonical momentum and π is the corresponding physical one.

Using annihilation and creation operators

$$a = \frac{\hat{\pi}_1 - i\hat{\pi}_2}{\sqrt{2\hbar m\omega}}\,, \qquad a^\dagger = \frac{\hat{\pi}_1 + i\hat{\pi}_2}{\sqrt{2\hbar m\omega}}\,, \tag{47}$$

where the frequency

$$\omega = \frac{(e\mathcal{B} - \sigma)(1 - \frac{e\mathcal{B}\theta}{4})}{m}\,, \tag{48}$$

one can write the Hamiltonian (32) with NC phase space (1) and the electric field $\mathcal{E} = 0$, in the harmonic oscillator form

$$\hat{H} = \omega\hbar\Big(a^\dagger a + \frac{1}{2}\Big) \tag{49}$$

with energy levels $E_n = \omega\hbar(n+\frac{1}{2})$. This is an extended Landau problem which reduces to the standard one if $\sigma = \theta = 0$.

The frequency ω can tuned by magnetic field $\mathcal{B}$. In particular, $\omega = 0$ if $\mathcal{B} = \frac{\sigma}{e}$ or $\mathcal{B} = \frac{4}{e\theta}$. Note that $\omega_{\mathcal{B}\to 0} = -\frac{\sigma}{m}$ and $\omega_{\mathcal{B}\to\infty} = \frac{e^2\mathcal{B}^2\theta}{4m}$.

In the strong magnetic field $\mathcal{B}$ (or very small mass m), Lagrangian (33) with $\mathcal{E} = 0$ becomes

$$L = \frac{e\mathcal{B}}{2}(\dot{x}_1\, x_2 - \dot{x}_2\, x_1). \tag{50}$$

The corresponding canonical momentum is $p_1 = \frac{\partial L}{\partial \dot{x}_1} = \frac{e\mathcal{B}}{2}\, x_2$ and

$$[\hat{x}_1, \hat{p}_1] = \Big[\hat{x}_1, \frac{e\mathcal{B}}{2}\hat{x}_2\Big] = i\hbar\Big(1 + \frac{\theta\sigma}{4}\Big).$$

In this limit one emerges NC configuration space with $[\hat{x}_1, \hat{x}_2] = i\hbar\frac{2}{e\mathcal{B}}(1 + \frac{\theta\sigma}{4})$. The spacing between energy levels diverges like $-\frac{\hbar e^2\mathcal{B}^2\theta}{4m}$ and the system practically lives in the lowest level $-\frac{\hbar e^2\mathcal{B}^2\theta}{8m}$, which differs from the standard Landau lowest level $\frac{\hbar e\mathcal{B}}{2m}$.
At the end it is worth noting that there are some other commutation relations similar to (1), which by linear transformations of canonical variables can be converted to the usual Heisenberg algebra (2). This will be presented elsewhere.

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