Time dependent solutions in SFT *

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Abstract

We present exact time–dependent solutions in vacuum string field theory. We show that they have many characteristics of the rolling tachyon solutions. We discuss the consequences of introducing a background electric field and, in connection with it, the existence of fundamental string solutions in vacuum string field theory.

1. Introduction

One of the most important problems in nowadays string/brane theory is the description of how unstable branes decay. Of course string theory contains many example of stable branes, but this stability involves supersymmetry. In real world supersymmetry is broken and branes become unstable. Therefore instability is a generic property for real world branes. There exists already a considerable literature concerning brane decay. When the latter is described by means of a classical solution in a (super)gravity theory, it is referred to as an S–brane. Instability in D–branes is signalled by the presence of a tachyon and described in terms of an effective tachyon theory. In this case the solution describing the brane decay is called rolling tachyon. More effectively rolling tachyons have been formulated in terms of boundary conformal field theories. Altogether these formulations allow us to, at least qualitatively, describe the final products of brane decays, which consist of very massive closed string states and, possibly, of macroscopic fundamental strings.

Making progress in the description of brane decay is very important, not only because we would like to know the fate of the branes that inhabit the string landscape, but also because the decay of a brane could provide a very

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appealing prototype of cosmological evolution: the bouncing off a tachyon potential contains suggestions both for dark matter and dark energy, not to speak of inflationary and pre–inflationary evolution. To have access to a more effective description one must however start from a more fundamental point of view than the ones mentioned so far: in particular we need a framework in which to be able to describe loop corrections. In this sense the most general vantage point is offered at present by string field theory. In this paper we would like to review the progress made in this direction.

In recent years, the study of tachyon condensation has driven a renewed attention upon String Field Theory (SFT), where this typically off-shell phenomenon can be most appropriately approached. Among the various formulations of string field theory Witten's Open String Field Theory, [13], sticks out for its elegance and completeness, and in this review we will limit ourselves to it, for reviews see [2, 3, 4, 5, 6]. It is worth, to start with, to summarize the historical developments that eventually lead us to the time dependent solutions that form the subject of this paper. The initial input is represented by A.Sen's conjectures, [1], which can be summarized as follows. Bosonic open string theory in D=26 dimensions is quantized on an unstable vacuum. This instability manifests itself through the existence of the open string tachyon. The effective tachyonic potential has, beside the local maximum where the theory is quantized, a local minimum. Sen's conjectures concern the nature of the theory around this local minimum. First of all, the energy density difference between the maximum and the minimum should exactly compensate for the D25–brane tension characterizing the unstable vacuum: this is a condition for the (relative) stability of the theory at the minimum. Therefore the theory around the minimum should not contain any quantum fluctuation pertaining to the original (unstable) theory. The minimum should therefore correspond to an entirely new theory, which can only be the bosonic closed string theory. If so, in the new theory one should be able to find in particular all the classical solutions characteristic of closed string theory, the D25–brane as well as all the lower dimensional D–branes.

A lot of evidence in favor of these conjectures has been collected over the last few years (see below) although it does not have a uniform degree of accuracy and reliability. However it is sufficient to conclude that they provide a correct description of tachyon condensation in SFT. Particularly elegant is the proof of the existence of solitonic solutions in Vacuum String Field Theory (VSFT), the SFT version that is believed to represent the theory near the minimum.

The D25–brane and its lower dimensional companions are unstable, because there is no conserved charge (like in the corresponding supersymmetric theories) associated with them. They are bound to decay. Therefore SFT must contain also time–dependent solutions that describe their decay as well as their decay products. This represents a basic problem in tachyon condensation, for the decay of the space filling D25–brane represents the evolution from the maximum of the tachyon potential to the (local) minimum. That such a solution, the rolling tachyon, exists has been argued in many ways, [25], see also [27, 28, 29] using effective field theories or boundary conformal field theories. As far as the SFT proper is concerned, although there have been some attempts [7], no analytical control has been achieved so far.

In contrast with the situation in SFT proper, encouraging exact results have been obtained in the framework of vacuum SFT (VSFT). Let us recall that VSFT, [9], is a version of Witten's open SFT, [13], that is supposed to describe the theory at the minimum of the tachyonic potential. As mentioned above, there is evidence that at this minimum the negative tachyonic potential exactly compensates for the D25–brane tension. No open string mode is expected to be excited, so that the BRST cohomology must be trivial. This state can only correspond to the closed string vacuum.

Thanks to these properties VSFT is a simplified version of SFT. The BRST operator $\mathcal{Q}$ takes a very simple form in terms of ghost oscillators alone. It is clearly simpler to work in such a framework than in the original SFT. In fact many classical solutions have been shown to exist, which are candidates for representing D–branes (the sliver, the butterfly, etc.), and other classical solutions have been found (lump solutions) which may represent lower dimensional D–branes [8, 10, 11, 12, 14, 15, 24].

In [37] it was shown that VSFT contains exact solitonic solutions that are localized in time: the time profile of such solutions is dominated for large t by a factor $\exp(-at^2)$ with positive constant a. At time $t = 0$ the solution takes the form of a deformed sliver (D25-brane) and at infinite future (and infinite past) time it becomes 0, i.e. it flows into the stable vacuum. If the initial configuration happens to coincide exactly with the sliver (no deformation present) there cannot be any time evolution. Therefore an initial deformation away from the sliver is essential for true time evolution. All this is strongly reminiscent of Sen's rolling tachyon solution, [25] or of an S–brane, [36], i.e of a bouncing state finely tuned at $t = -\infty$ to be poised at time $t = 0$ near the top of the tachyon potential and let free to evolve. This result has been subsequently confirmed in [19], where the same kind of solutions has been studied in the presence of a background electric field. In this background one expects that among the final products of the brane decay also fundamental strings be present. As a necessary condition for this, in the last part of this review, we show that in VSFT there exist in fact fundamental strings solutions.

This paper is organized as follows. In the next section we briefly review rolling tachyon solutions obtained with boundary conformal field theory methods. In section 3 we collect general results and formulas in VSFT which are necessary in the sequel. In section 4 we present our method to obtain rolling tachyon–type time–localized solutions in VSFT and illustrate their connection with the rolling tachyon. In section 5 we briefly discuss the consequences of introducing a background electric field and show the existence of fundamental strings solutions in vacuum string field theory.

2. The rolling tachyon in effective field theory and BCFT

In this section, for later reference, we briefly review the main points concerning rolling tachyon solution and brane decay (for a complete account see the original literature, [25, 27, 28, 29] or the review [26]). It is well–known that in bosonic open string theory the tachyon potential has a local maximum and a local minimum. The rolling tachyon is a solution describing the system prepared near the maximum of the potential and let free to evolve toward the minimum and eventually stop there. One could think that the system, starting from the local minimum at time $-\infty$, was pushed toward the local maximum and tuned so that at $t = 0$ it is poised near the maximum and free to evolve back toward the minimum. As pointed out by Sen, a classical tachyon field solution in accord with these conditions would be

$$T(x^0) = \lambda \cosh(x^0) \tag{1}$$

at least for small x^0. Here λ is small and represents the distance from the potential maximum. In general one doesn't go too far with a solution like (1). However Sen remarked that this is just the classical limit of an exact conformal field theory. In fact if we consider the action (involving only the x^0 mode, the other string modes are dropped)

$$-\frac{1}{2\pi}\int d^2z\, \partial_z X^0\, \partial_{\bar z} X^0 \; + \tilde\lambda \int dt \cosh X^0(t) \tag{2}$$

where t parametrizes the world–sheet boundary, and Wick–rotate the field $X^0 = iX$, we end up with a solvable conformal field theory, since the boundary perturbation $\tilde\lambda \int dt \cos X(t)$ turns out to be exactly marginal. The (positive) coupling $\tilde\lambda$ coincides with λ for small λ. By expressing the solution in terms of the associated boundary state in closed string theory, one is able to extract information concerning in particular the energy momentum tensor. For a generic Dk–brane the result, after inverse–Wick–rotating it, is expressed in terms of the function

$$\tilde f(x^0) = \frac{1}{1 + e^{x^0}\sin(\tilde\lambda\pi)} + \frac{1}{1 + e^{-x^0}\sin(\tilde\lambda\pi)} - 1 \tag{3}$$

The total energy and pressure are

$$T_{00} = \mathcal{T}_k \cos^2(\pi\tilde\lambda)\delta(x_\perp) \tag{4}$$

$$p(x^0) = -\mathcal{T}_k\, \tilde f(x^0)\delta(x_\perp) \tag{5}$$

where $\mathcal{T}_k$ is the tension of the Dk–brane, $x_\perp$ denotes the transverse direction to it. For $0 < \tilde\lambda < \frac{1}{2}$, it follows that $\tilde f(x^0) \to 0$ as $x^0 \to \infty$. Therefore, while the energy is conserved, the pressure tends to 0. This pressureless matter (or dust) is referred to as tachyon matter.

The conclusion of this analysis is that the products of the Dk–brane decay are made of pressureless matter which stays close to the brane plane. When $\tilde\lambda = \frac{1}{2}$ something peculiar happens, for $\tilde f(x^0) = 0$ and the energy momentum tensor identically vanishes. One can interpret this static solution as the vacuum corresponding to the minimum of the potential. However this conclusion is only apparently true. The solution identified by $\tilde\lambda = \frac{1}{2}$

has a simple geometric interpretation: it corresponds to an infinite periodic array of D–branes located at imaginary time $x = -ix^0 = (2n+1)\pi$. According to the analysis of [28], this background actually corresponds to a closed string background. For instance, if one considers a disk amplitude with two closed string tachyon insertions in the presence of this background and inverse–Wick–rotate it, one finds that the contribution from the amplitude comes from a region of the moduli space that corresponds to the disk extending to cover all of the complex plane, i.e. becoming a sphere with one puncture (at infinity). The resulting amplitude is a sphere amplitude with two tachyon insertions and a closed string state inserted at the puncture. This extra state contains the full tower of massive closed string excitations. This analysis extends to more general amplitudes and leads to the conclusion that the point $\tilde{\lambda} = \frac{1}{2}$ represents a purely closed string background, composed of massive closed string states with characteristics very similar to the tachyon matter. In other words this background seems to represent the final result of the brane decay.

In the presence of a background electric field the above picture does not essentially change except for the presence in the final state of (macroscopic) fundamental strings coupled to the electric field, [17].

Our purpose in the sequel is to try to reproduce the results we have summarized in this section in the context of VSFT.

3. Projectors in VSFT

VSFT, [9], is a version of Witten's open SFT, [13], that is supposed to describe the theory at the minimum of the tachyonic potential. The action has functionally the same form as the original SFT, but we do not know the precise form of the BRST charge. Some heuristic argument is necessary to complement our ignorance. Relying on the evidence that, at the minimum, the negative tachyonic potential exactly compensates for the D25–brane tension, one can conclude that no open string mode should be excited. So that the BRST cohomology must be trivial. The possible BRST operators that satisfy this condition are of course manifold. However it is possible to find evidence that the correct form is given by eq.(7) below, [18].

In short the formulas relevant to VSFT are as follows. The action is

$$\mathcal{S}(\Psi) = -\frac{1}{g_0^2}\left(\frac{1}{2}\langle\Psi|\mathcal{Q}|\Psi\rangle + \frac{1}{3}\langle\Psi|\Psi * \Psi\rangle\right) \tag{6}$$

where

$$\mathcal{Q} = c_0 + \sum_{n>0}(-1)^n\left(c_{2n} + c_{-2n}\right) \tag{7}$$

The equation of motion is

$$\mathcal{Q}\Psi = -\Psi * \Psi \tag{8}$$

Our aim below is to find solutions of this equation that represent both D–branes and time–dependent solutions representing D–brane decays. The ansatz for nonperturbative solutions we constantly refer to has the factorized form

$$\Psi = \Psi_m \otimes \Psi_g \tag{9}$$

where Ψ_g and Ψ_m depend purely on ghost and matter degrees of freedom, respectively. Then eq.(8) splits into

$$\begin{aligned} \mathcal{Q}\Psi_g &= -\Psi_g *_g \Psi_g &(10)\\ \Psi_m &= \Psi_m *_m \Psi_m &(11)\end{aligned}$$

where $*_g$ and $*_m$ refers to the star product involving only the ghost and matter part, respectively. The action for this type of solution becomes

$$\mathcal{S}(\Psi) = -\frac{1}{6g_0^2}\langle\Psi_g|\mathcal{Q}|\Psi_g\rangle\langle\Psi_m|\Psi_m\rangle \tag{12}$$

$\langle\Psi_m|\Psi_m\rangle$ is the ordinary inner product product, $\langle\Psi_m|$ being the *bpz* conjugate of $|\Psi_m\rangle$.

It it well–known how to solve the ghost equation of motion (10), [18]. We give this solution for granted and ignore it throughout. Henceforth we concentrate on the matter part, eq.(11). The solutions are projectors of the $*_m$ algebra. The $*_m$ product is defined as follows

$${}_{123}\langle V_3|\Psi_1\rangle_1|\Psi_2\rangle_2 =_3 \langle\Psi_1 *_m \Psi_2| \tag{13}$$

where the three strings vertex V_3 is given by

$$|V_3\rangle = \int d^{26}p_{(1)}d^{26}p_{(2)}d^{26}p_{(3)}\delta^{26}(p_{(1)}+p_{(2)}+p_{(3)})\exp(-E)\,|0,p\rangle_{123} \tag{14}$$

where

$$\begin{aligned} E \;=\; \sum_{r,s=1}^{3}\Bigg(\frac{1}{2}\sum_{m,n\geq 1}\eta_{\mu\nu}a_m^{(r)\mu\dagger}V_{mn}^{rs}a_n^{(s)\nu\dagger} &&(15)\\ +\sum_{n\geq 1}\eta_{\mu\nu}p_{(r)}^{\mu}V_{0n}^{rs}a_n^{(s)\nu\dagger}+\frac{1}{2}\eta_{\mu\nu}p_{(r)}^{\mu}V_{00}^{rs}p_{(s)}^{\nu}\Bigg)\end{aligned}$$

Summation over the Lorentz indices $\mu,\nu = 0,\ldots,25$ is understood and η denotes the flat Lorentz metric. The operators $a_m^{(r)\mu}, a_m^{(r)\mu\dagger}$ denote the non–zero modes matter oscillators of the r–th string, which satisfy

$$[a_m^{(r)\mu}, a_n^{(s)\nu\dagger}] = \eta^{\mu\nu}\delta_{mn}\delta^{rs}, \qquad m,n\geq 1 \tag{16}$$

$p_{(r)}$ is the momentum of the r–th string and $|0,p\rangle_{123} \equiv |p_{(1)}\rangle\otimes|p_{(2)}\rangle\otimes|p_{(3)}\rangle$ is the tensor product of the Fock vacuum states relative to the three strings.

$|p_{(r)}\rangle$ is annihilated by the annihilation operators $a_m^{(r)\mu}$ and it is eigenstate of the momentum operator $\hat{p}^\mu_{(r)}$ with eigenvalue $p^\mu_{(r)}$. The normalization is

$$\langle p_{(r)}|\,p'_{(s)}\rangle = \delta_{rs}\delta^{26}(p+p') \tag{17}$$

The symbols $V^{rs}_{nm}, V^{rs}_{0m}, V^{rs}_{00}$ will denote the coefficients computed in [30]. We will use them in the notation of Appendix A and B of [15]. The notation V^{rs}_{MN} for them will also be used at times (with $M(N)$ denoting the couple $\{0,m\}$ $(\{0,n\})$) .

An important ingredient in the following are the *bpz* transformation properties of the oscillators

$$bpz(a_n^{(r)\mu}) = (-1)^{n+1}a_{-n}^{(r)\mu} \tag{18}$$

The solutions to eq.(11) are projectors of the $*_m$ algebra. The simplest one is the sliver. Let us recall its main features. It is translationally invariant. As a consequence all momenta can be set to zero. The integration over the momenta can be dropped and the only surviving part in E will be the one involving V^{rs}_{nm}. The sliver is defined by

$$|\Xi\rangle = \mathcal{N}e^{-\frac{1}{2}a^\dagger S a^\dagger}|0\rangle, \qquad a^\dagger S a^\dagger = \sum_{n,m=1}^{\infty} a_n^{\mu\dagger}S_{nm}a_m^{\nu\dagger}\eta_{\mu\nu} \tag{19}$$

This state satisfies eq.(11) provided the matrix S satisfies the equation

$$S = V^{11} + (V^{12},V^{21})(1-\Sigma\mathcal{V})^{-1}\Sigma\begin{pmatrix} V^{21}\\ V^{12}\end{pmatrix} \tag{20}$$

where

$$\Sigma = \begin{pmatrix} S & 0\\ 0 & S\end{pmatrix}, \qquad \mathcal{V} = \begin{pmatrix} V^{11} & V^{12}\\ V^{21} & V^{22}\end{pmatrix}, \tag{21}$$

The proof of this fact is well–known, [14]. First one expresses eq.(21) in terms of the twisted matrices $X = CV^{11}, X_+ = CV^{12}$ and $X_- = CV^{21}$, together with $T = CS = SC$, where $C_{nm} = (-1)^n\delta_{nm}$ is the twist matrix. The matrices X, X_+, X_- are mutually commuting. Then, requiring T to commute with them as well, one can show that eq.(21) reduces to the algebraic equation

$$(1-T)(XT^2 - (1+X)T + X) = 0 \tag{22}$$

Apart form the identity solution, the significant solution is the sliver

$$T = \frac{1}{2X}(1 + X - \sqrt{(1+3X)(1-X)}) \tag{23}$$

which evidently commutes with X, X_+, X_-.

The normalization constant $\mathcal{N}$ is

$$\mathcal{N} = (\det(1 - \Sigma\mathcal{V}))^{\frac{D}{2}} \tag{24}$$

where $D = 26$. The contribution of the sliver to the matter part of the action (see (12)) is given by

$$\langle \Xi | \Xi \rangle = \frac{\mathcal{N}^2}{(\det(1 - S^2))^{\frac{D}{2}}} \tag{25}$$

Both eq.(24) and (25) are ill–defined and need to be regularized. Some proposals in this sense have been put forward in [22, 23].

The sliver solution represents the space–filling D25–brane. In order to find D–brane solution of lower dimensions we have to define transverse directions, i.e. directions along which the solutions are not translational invariant. The lump solutions are engineered to represent a lower dimensional brane, therefore they are characterized by the breaking of translational invariance along a subset of directions. Accordingly we split the three strings vertex into the tensor product of the perpendicular part and the parallel part

$$|V_3\rangle = |V_{3,\perp}\rangle \otimes |V_{3,_\|}\rangle \tag{26}$$

and the exponent E, accordingly, as $E = E_\| + E_\perp$. The parallel part is the same as in the sliver case while the perpendicular part is modified as follows. Following [15], we denote by x^α, p^α, $\alpha = 1, ..., k$ the coordinates and momenta in the transverse directions and introduce the zero mode combinations

$$a_0^{(r)\alpha} = \frac{1}{2}\sqrt{b}\hat{p}^{(r)\alpha} - i\frac{1}{\sqrt{b}}\hat{x}^{(r)\alpha}, \qquad a_0^{(r)\alpha\dagger} = \frac{1}{2}\sqrt{b}\hat{p}^{(r)\alpha} + i\frac{1}{\sqrt{b}}\hat{x}^{(r)\alpha}, \tag{27}$$

where $\hat{p}^{(r)\alpha}, \hat{x}^{(r)\alpha}$ are the zero momentum and position operator of the r–th string, and we have introduced the parameter b as in [15]. It follows

$$[a_0^{(r)\alpha}, a_0^{(s)\beta\dagger}] = \eta^{\alpha\beta}\delta^{rs} \tag{28}$$

Denoting by $|\Omega_b\rangle$ the oscillator vacuum ($a_0^\alpha|\Omega_b\rangle = 0$), the relation between the momentum basis and the oscillator basis is defined by

$$|\{p^\alpha\}\rangle = \left(\frac{b}{2\pi}\right)^{\frac{k}{4}} \exp\left[\sum_{r=1}^{3}\left(-\frac{b}{4}p_\alpha^{(r)}\eta^{\alpha\beta}p_\beta^{(r)} + \sqrt{b}a_0^{(r)\alpha\dagger}p_\alpha^{(r)} - \frac{1}{2}a_0^{(r)\alpha\dagger}\eta_{\alpha\beta}a_0^{(r)\beta\dagger}\right)\right]|\Omega_b\rangle$$

Next we insert this equation inside $E'_\perp$ and eliminate the momenta along the perpendicular directions by integrating them out. The overall result of this operation is that, while $|V_{3,\|}\rangle$ is the same as in the ordinary case,

$$|V_{3,\perp}\rangle' = K\, e^{-E'}|\Omega_b\rangle \tag{29}$$

with

$$K = \left(\frac{\sqrt{2\pi b^3}}{3(V_{00}+b/2)^2}\right)^{\frac{k}{2}}, \qquad E' = \frac{1}{2}\sum_{r,s=1}^{3}\sum_{M,N\geq 0} a_M^{(r)\alpha\dagger} V'^{rs}_{MN} a_N^{(s)\beta\dagger}\eta_{\alpha\beta} \quad (30)$$

The coefficients V'^{rs}_{MN} are given in Appendix B of [15]. The new Neumann coefficients matrices V'^{rs} satisfy the same relations as the V^{rs} ones. In particular one can introduce the matrices $X'^{rs} = CV'^{rs}$, where $C_{NM} = (-1)^N\delta_{NM}$, which turn out to commute with one another. All the relations valid for $X, X_\pm$ hold with primed quantities as well. We can therefore repeat word by word the derivation of the sliver from eq.(19) through eq.(25). The new solution, denoted $|\Xi'_k\rangle$ will have the form (19) with S along the parallel directions and S replaced by S' along the perpendicular ones. In turn S' is obtained as a solution to eq.(20) where all the quantities are replaced by primed ones. This amounts to solving eq.(22) with primed quantities. Therefore in the transverse directions S is replaced by S', given by

$$S' = CT', \qquad T' = \frac{1}{2X'}(1+X'-\sqrt{(1+3X')(1-X')}) \quad (31)$$

In a similar way we have to adapt the normalization and energy formulas (24,25). Once this is done, one can compute the energy density, which, for a static solution, corresponds to the negative of the action calculated via (12) divided by the volume. The absolute value of this energy is not well defined (see below), but one can at least compute the ratio for the tensions of two lumps of contiguous dimensions, [15],

$$\frac{\mathcal{T}_k}{2\pi\mathcal{T}_{k+1}} = \frac{3}{\sqrt{2}\pi b^3}(V_{00}+\frac{b}{2})^2\left(\frac{\det(1-X')^3\det(1+3X')}{\det(1-X)^3\det(1+3X)}\right)^{\frac{1}{4}} \quad (32)$$

This ratio has been proven both numerically, [15], and analytically, [32], to be 1. In this way we find the expected value of the ratio of tension of D–branes (in $\alpha' = 1$ units). Another confirmation of the D–brane interpretation of a lump comes from the space profile, which can be calculated by contracting the lump solution with the coordinate eigenstate $|x^\alpha\rangle$ along the transverse directions. After regularization, [8], or after introduction of a constant background B field, [33, 34], this profile turns out to be a Gaussian centered at the transverse coordinate origin and thus represents a space–localized solution.

Finally we remark that there is another solution beside (31), for which T' is replaced by $1/T'$. The solution obtained in this way is referred to as *inverse lump*. It is not clear whether such solutions make sense at all. However below we will use this suggestion to invert only the discrete spectrum of T'. This makes sense and will allow us to find interesting time–dependent solutions.

3.1. Spectroscopy and diagonal representation

A good deal of information about the sliver and lump solutions can be obtained by diagonalizing the $X, X_\pm$ matrices. The diagonalization program was carried out in [31], while the same analysis for X' was accomplished in [21] and [20]. Here, for later use, we summarize the results of these references. The eigenvalues of $X = X^{11}, X_+ = X^{12}, X_- = X^{21}$ and T are given, respectively, by

$$\mu^{rs}(k) = \frac{1 - 2\,\delta_{r,s} + e^{\frac{\pi k}{2}}\,\delta_{r+1,s} + e^{-\frac{\pi k}{2}}\,\delta_{r,s+1}}{1 + 2\cosh\frac{\pi k}{2}} \tag{33}$$

$$t(k) = -e^{-\frac{\pi|k|}{2}} \tag{34}$$

where $-\infty < k < \infty$. The generating function for the eigenvectors is

$$f^{(k)}(z) = \sum_{n=1}^{\infty} v_n^{(k)} \frac{z^n}{\sqrt{n}} = \frac{1}{k}(1 - e^{-k \arctan z}) \tag{35}$$

The completeness and orthonormality equations for the eigenfunctions are as follows

$$\sum_{n=1}^{\infty} v_n^{(k)} v_n^{(k')} = \mathcal{N}(k)\,\delta(k-k')\,, \qquad \mathcal{N}(k) = \frac{2}{k}\sinh\frac{\pi k}{2}\,,$$
$$\int_{-\infty}^{\infty} dk\, \frac{v_n^{(k)} v_m^{(k)}}{\mathcal{N}(k)} = \delta_{nm} \tag{36}$$

The spectrum of X is continuous and lies in the interval $[-1/3, 0)$. It is doubly degenerate except at 0. The continuous spectrum of X' lies in the same interval, but X' in addition has a discrete spectrum. To describe it we follow [20]. We consider the decomposition, see [5],

$$X'^{rs} = \frac{1}{3}(1 + \alpha^{s-r} CU' + \alpha^{r-s} U'C) \tag{37}$$

It is convenient to express everything in terms of CU' eigenvalues and eigenvectors. The discrete eigenvalues are denoted by ξ and $\bar{\xi}$. Since CU' is unitary they lie on the unit circle. The discrete eigenvalues ξ and $\bar{\xi}$ are determined as follows. Let

$$\xi = -\frac{2 - \cosh\eta - i\sqrt{3}\sinh\eta}{1 - 2\cosh\eta} \tag{38}$$

and

$$F(\eta) = \psi\left(\frac{1}{2} + \frac{\eta}{2\pi i}\right) - \psi\left(\frac{1}{2}\right), \qquad \psi(z) = \frac{d\log\Gamma(z)}{dz} \tag{39}$$

Then the eigenvalues ξ and $\bar{\xi}$ are solutions to

$$\Re F(\eta) = \frac{b}{4} \tag{40}$$

To each value of b there corresponds a couple of values of η with opposite sign (except for $b = 0$ which implies $\eta = 0$).

The eigenvectors $V_n^{(\xi)}$ are defined via the generating function

$$\begin{aligned} F^{(\xi)}(z) = & \sum_{n=1}^{\infty} V_n^{(\xi)} \frac{z^n}{\sqrt{n}} = -\sqrt{\frac{2}{b}} V_0^{(\xi)} \left[\frac{b}{4} + \frac{\pi}{2\sqrt{3}} \frac{\xi - 1}{\xi + 1} + \log iz \right. \\ & \left. + e^{-2i(1+\frac{\eta}{\pi i})\arctan z} \Phi(e^{-4i \arctan z}, 1, \frac{1}{2} + \frac{\eta}{2\pi i}) \right] \end{aligned} \tag{41}$$

where $\Phi(x, 1, y) = \frac{1}{y} {}_2F_1(1, y; y+1; x)$, while

$$V_0^{(\xi)} = \left(\sinh \eta \frac{\partial}{\partial \eta} \left[\log \Re F(\eta) \right] \right)^{-\frac{1}{2}} \tag{42}$$

As for the continuous spectrum, it is spanned by the variable k, $-\infty < k < \infty$. The eigenvalues of CU' are given by

$$\nu(k) = -\frac{2 + \cosh \frac{\pi k}{2} + i\sqrt{3} \sinh \frac{\pi k}{2}}{1 + 2 \cosh \frac{\pi k}{2}}$$

The generating function for the eigenvectors is

$$\begin{aligned} & F_c^{(k)}(z) = \sum_{n=1}^{\infty} V_n^{(k)} \frac{z^n}{\sqrt{n}} = \\ & V_0^{(k)} \sqrt{\frac{2}{b}} \left[-\frac{b}{4} - \left(\Re F_c(k) - \frac{b}{4} \right) e^{-k \arctan z} - \log iz - \left(\frac{\pi}{2\sqrt{3}} \frac{\nu(k) - 1}{\nu(k) + 1} + \frac{2i}{k} \right) \right. \\ & \left. + 2i\, f^{(k)}(z) - \Phi(e^{-4i \arctan z}, 1, 1 + \frac{k}{4i})\, e^{-4i \arctan z}\, e^{-k \arctan z} \right] \end{aligned} \tag{43}$$

where

$$F_c(k) = \psi(1 + \frac{k}{4\pi i}) - \psi(\frac{1}{2}),$$

while

$$V_0^{(k)} = \sqrt{\frac{b}{2\mathcal{N}(k)}} \left[4 + k^2 \left(\Re F_c(k) - \frac{b}{4} \right)^2 \right]^{-\frac{1}{2}} \tag{44}$$

The continuous eigenvalues of X', X'_-, X'_- and T' (for the conventional lump) are given by same formulas as for the X, X_+, X_- and T case, eqs.(33,34). As for the discrete eigenvalues, they are given by the formulas

$$\mu_\xi^{rs} = \frac{1 - 2\,\delta_{r,s} - e^{\eta}\,\delta_{r+1,s} - e^{-\eta}\,\delta_{r,s+1}}{1 - 2\cosh\eta}$$
$$t_\xi = e^{-|\eta|} \tag{45}$$

The set of continuous and discrete eigenvectors form a complete basis, the completeness relation being

$$\int_{-\infty}^{\infty} dk\, V_N^{(k)} V_M^{(k)} + V_N^{(\xi)} V_M^{(\xi)} + V_N^{(\bar\xi)} V_M^{(\bar\xi)} = \delta_{NM} \tag{46}$$

It is convenient to expand all the relevant quantities in VSFT by means of this basis. To this end we define

$$a_k = \sum_{N=0}^{\infty} V_N^{(k)} a_N, \qquad a_\xi = \sum_{N=0}^{\infty} V_N^{(\xi)} a_N, \qquad a_{\bar\xi} = \sum_{N=0}^{\infty} V_N^{(\bar\xi)} a_N$$
$$a_N = \int_{-\infty}^{\infty} dk\, V_N^{(k)}\, a_k + V_N^{(\xi)} a_\xi + V_N^{(\bar\xi)} a_{\bar\xi} \tag{47}$$

and introduce the even and odd twist combinations

$$e_k = \frac{a_k + C a_k}{\sqrt{2}}, \quad e_\eta = \frac{a_\xi + C a_\xi}{\sqrt{2}}, \quad o_k = \frac{a_k - C a_k}{i\sqrt{2}}, \quad o_\eta = \frac{a_\xi - C a_\xi}{i\sqrt{2}}. \tag{48}$$

The commutation relations among them are

$$[e_k, e_{k'}^\dagger] = \delta_{k,k'}, \qquad [e_\eta, e_\eta^\dagger] = 1, \qquad [o_k, o_{k'}^\dagger] = \delta_{k,k'}, \qquad [o_\eta, o_\eta^\dagger] = 1, \tag{49}$$

while all the other commutators vanish. The twist properties are defined by

$$C a_k = a_{-k}, \qquad C a_\xi = a_{\bar\xi},$$

Using these combinations the three–strings vertex can be cast in diagonal form and, for instance, the exponent of the lump state can be written

$$\begin{aligned} a^\dagger S' a^\dagger &= \int_{-\infty}^{\infty} dk\, t(k)\, (a_k^\dagger, C a_k^\dagger) + 2\, t_\xi\, (a_\xi^\dagger, C a_\xi^\dagger) \\ &= \frac{1}{2}\int_{-\infty}^{\infty} dk\, t(k)\, (e_k^\dagger e_k^\dagger + o_k^\dagger o_k^\dagger) + \; t_\eta\, (e_\eta^\dagger e_\eta^\dagger + o_\eta^\dagger o_\eta^\dagger) \end{aligned} \tag{50}$$

where $t_\eta \equiv t_\xi = e^{-|\eta|}$.

4. Time–localized solutions

As discussed in the introduction, D–branes of bosonic theories, in particular the D25–brane, are unstable. This piece of information should be stored in SFT and also in VSFT, if the latter is, as we suppose, a self-contained theory. Therefore the theory should contain a description of how these branes decay. We expect therefore that among the classical solutions of the equations of motion there exist time–dependent ones which precisely describe brane decays. What we expect to find is something like the rolling tachyon solution[25, 27, 28, 29] or an S–brane–like solution[36]. A state having a time profile dominated by a factor $\exp(-at^2)$ with positive constant a would do.

The technique to produce such a solution is based on double Wick–rotation. Our reference solution is obtained by picking a Euclidean lump solution with one transverse space direction (a D24-brane) and then performing an inverse Wick–rotation along such a direction. However the important ingredient is that our lump solution is an unconventional one. Since the spectrum of the twisted Neumann coefficient matrices of the three strings vertex nicely split into a continuous and a discrete part, we define a new solution in which the squeezed state matrix is made of a continuous part which is the same as for the conventional lump, and a discrete part which is *inverted* with respect to the ordinary lump. After inverse–Wick–rotating such solution we get the desired time behavior.

4.1. Why ordinary lumps don't work

Let us see first why we have to start from an unconventional lump solution, and not from the ordinary one (see section 3 from eq.(26) through eq.(31)). In fact, for those who are familiar with lump solutions, it would be natural to start from a lump with one transverse space direction (which represents a D24-brane) and inverse–Wick–rotate it. For simplicity we denote the transverse direction coordinate, momenta and oscillators simply by x, p and a_N, dropping the Lorentz label. The solution is written as follows:

$$\begin{aligned} |\Psi'\rangle &= |\Xi_{25}\rangle \otimes |\Lambda'\rangle \\ |\Lambda'\rangle &= \mathcal{N}' \exp\left[-\frac{1}{2}\sum_{N,M\geq 0} a_N^\dagger S'_{NM} a_M^\dagger\right]|\Omega_b\rangle \end{aligned} \tag{51}$$

where $|\Xi_{25}\rangle$ is the usual sliver along the longitudinal 25 directions. For normalization constant $\mathcal{N}'$, see [37].

We need to know the space profile of this solution in the transverse direction. To this end we contract it with the x_0–coordinate eigenstate

$$|x_0\rangle = \left(\frac{2}{b\pi}\right)^{\frac{1}{4}} \exp\left[-\frac{1}{b}x_0^2 - \frac{2}{\sqrt{b}} i a_0^\dagger x_0 + \frac{1}{2}(a_0^\dagger)^2\right]|\Omega_b\rangle \tag{52}$$

The result is

$$\langle x_0|\Lambda'\rangle = \left(\frac{2}{b\pi}\right)^{\frac{1}{4}} \frac{\mathcal{N}'}{1+s'} \exp\left[\frac{1}{b}\frac{s'-1}{s'+1}x_0^2 - \frac{2i}{\sqrt{b}}\frac{x_0 f_0}{1+s'} - \frac{1}{2}a^\dagger W' a^\dagger\right] \tag{53}$$

with the following condensed notation

$$f_0 = \sum_{n=1} S'_{0n} a_n^\dagger, \quad a^\dagger W' a^\dagger = \sum_{n,m=1} a_n^\dagger W'_{nm} a_m^\dagger, \quad W'_{nm} = S'_{nm} - \frac{S'_{0n} S'_{0m}}{1+s'} \tag{54}$$

and

$$s' = S'_{00} \tag{55}$$

After an inverse Wick rotation $x_0 \to i\mathrm{x}_0, a_n^\dagger \to i a_n^\dagger$ (53) becomes

$$\langle \mathrm{x}_0 | \Lambda' \rangle = \left(\frac{2}{b\pi}\right)^{\frac{1}{4}} \frac{\mathcal{N}'}{1+s'} \exp\left[\frac{1}{b}\frac{1-s'}{1+s'}\mathrm{x}_0^2 + \frac{2i}{\sqrt{b}}\frac{\mathrm{x}_0 f_0}{1+s'} + \frac{1}{2} a^\dagger W' a^\dagger\right] \tag{56}$$

We are interested in solutions which are localized in time. The second term in the exponent gives rise to time oscillations. Only the first term there can give rise to time localization. This happens precisely when $|s'| > 1$. However such a condition can never be achieved within the present scheme in which ordinary lump solutions are utilized. For it is possible to show that for such solutions $|s'| \leq 1$. Let us see this in detail. Using the basis of the previous section we can write

$$s' \equiv S'_{00} = \int_{-\infty}^{\infty} dk\, V_0^{(k)} (-e^{-\frac{\pi|k|}{2}}) V_0^{(k)} + V_0^{(\xi)} e^{-|\eta|} V_0^{(\xi)} + V_0^{(\bar{\xi})} e^{-|\eta|} V_0^{(\bar{\xi})} \tag{57}$$

We evaluate first this in the extreme limit $b \to 0$ and $b \to \infty$. To this end we need the behavior of the eigenvectors near these values of b. Near $b = 0$ we have

$$\begin{aligned} &b \approx 0, \qquad \eta \approx 0, \qquad \xi \approx 1 \\ &V_0^{(\xi)} = \frac{1}{\sqrt{2}} + \mathcal{O}(\eta), \qquad V_n^{(\xi)} = \mathcal{O}(\eta^2) \end{aligned} \tag{58}$$

The same behavior holds for the $V^{(\bar{\xi})}$ basis.

When $b \to \infty$ we have instead

$$\begin{aligned} &b \to \infty, \qquad b \approx 4 \log \eta, \qquad \xi \approx -e^{\frac{\pi i}{3}} \\ &V_0^{(\xi)} \approx e^{-\frac{\eta}{2}} \sqrt{2\eta \log \eta}, \qquad V_n^{(\xi)} \sim e^{-\frac{\eta}{2}} \sqrt{\eta} \end{aligned} \tag{59}$$

and the same for $V^{(\bar{\xi})}$.

As for the asymptotic expansions of the $V^{(k)}$ basis, we have to be more careful. The point is that the expression $(V_0^{(k)})^2$, see (44), would superficially seem to vanish in the limit $\eta \to \infty$, but it is in fact a representation of the Dirac delta function $\delta(k)$. Therefore the result of taking the $\eta \to \infty$ limit in an integral containing $(V_0^{(k)})^2$ is to concentrate it at the point $k = 0$. This renders the use of the generating function (43) very tricky and, consequently, such integrals as $\int dk\, V_n^{(k)} V_m^{(k)} f(k)$ difficult to handle in this limit.

Anyway whenever we meet such an integral we can bypass the problem by an indirect evaluation, [37]. The result is as follows.

The first term in the RHS of (57) does not contribute in the limit $b \to 0$ (i.e. $\eta \to 0$) and using the approximants (58) we immediately see that the remaining two terms add up to 1. Therefore when $b \to 0$, $s' \to 1$. On the other hand, in the limit $b \to \infty$, using (59) we see that the last two terms in the RHS of (57) do not contribute, while the first term contributes exactly -1.

For generic values of b we cannot calculate s' analytically but it is easy to evaluate it numerically and to show that it is a monotonically decreasing function of b for $0 \leq b < \infty$.

Our conclusion is therefore that we cannot obtain a time–localized solution by inverse–Wick–rotating an ordinary lump solution. Some drastic change has to be made in order to produce a localized time–dependent solution.

4.2. A Rolling Tachyon–like Solution

The above unsuccessful attempt tells us how we should proceed. It is not hard to realize that should we replace $e^{-|\eta|}$ with $e^{|\eta|}$ in eq.(57) we would reverse the conclusion at the end of the previous section. In fact, as we shall see in a moment, we would have $|s'| \geq 1$. In section 3 we saw that, if in the lump solution we replace T' by $1/T'$, formally, we still have a projector. This is true also if we make this replacement for the discrete spectrum alone. Therefore we define an *unconventional lump*, by replacing $|\Lambda'\rangle$ in (51) with

$$|\check{\Lambda}'\rangle = \check{\mathcal{N}}' \exp(-\frac{1}{2} a^\dagger C \check{T}' a^\dagger)|\Omega_b\rangle \tag{60}$$

where

$$\check{T}'_{NM} = -\int_{-\infty}^{\infty} dk\, V_N^{(k)}\, V_M^{(k)} \exp\left(-\frac{\pi|k|}{2}\right) + \left(V_N^{(\xi)}\, V_M^{(\xi)} + V_N^{(\bar{\xi})}\, V_M^{(\bar{\xi})}\right) \exp|\eta| \tag{61}$$

Due to the fact that the star product is split into eigenspaces of the Neumann coefficients matrices X', X'_+, X'_-, the projector equation splits accordingly into the continuous and discrete spectrum part, [24]. Therefore (60) is still a projector, as one can on the other hand easily verify by direct calculation. This is the solution we set out to examine. We will see that it is localized in time.

A few comments are in order. Passing from a squeezed state solution with a matrix T' to another characterized by the inverse matrix $1/T'$ may lead in general to unacceptable features of the state. However in the case at hand, in which one inverts only the discrete spectrum, such unpleasant aspects are not present. First of all the matrix $\check{T}'$ is well defined both in the oscillator and in the continuous basis. Second, such expression as $\sqrt{\det(1-\check{T}')}$ are well–defined. This is due to the fact that, if we are allowed to factorize the discrete and continuous spectrum contributions, the former can be written as $\det(1-\check{T}')^{(d)} = (1-\exp|\eta|)^2$, so that the possible dangerous - sign under the square root disappears due to the double multiplicity of the discrete eigenvalue. Thirdly, the energy density of the (Euclidean) solution (60) is well–defined and equals the energy density of the ordinary lump. In fact, using the formulas similar to (32), the ratio between the energy densities of the two solutions reduces to

$$\sqrt{\frac{\det(1+\check{T}')}{\det(1-\check{T}')}}\Big/\sqrt{\frac{\det(1+T')}{\det(1-T')}} = \sqrt{\frac{(1+e^{\eta})^2}{(1-e^{\eta})^2}}\Big/\sqrt{\frac{(1+e^{-\eta})^2}{(1-e^{-\eta})^2}} = 1 \qquad (62)$$

after factorization of the discrete and continuous parts of the spectrum.

After these important remarks it remains for us to show that this solution has the appropriate features to represent a rolling tachyon–type solution. To see if this is true we have to represent it in a more explicit way. In particular we have to extract the explicit time dependence (that is, we have to extract the space dependence and then inverse–Wick–rotate it). To do so, we choose a (coordinate) basis on which to project (60). The most obvious choice is the ordinary (open string) time, i.e. the time defined by the center of mass of the string, and analyze the corresponding time profile. Despite the fact that this coordinate is not diagonal for the $*$–product we can still have complete control on the corresponding profile. The center of mass position operator is given by

$$\hat{x}_0 = \frac{i}{\sqrt{b}}(a_0 - a_0^\dagger) \qquad (63)$$

The center of mass position eigenstate is

$$|x_0\rangle = \left(\frac{2}{b\pi}\right)^{\frac{1}{4}} \exp\left(-\frac{1}{b}x_0x_0 - \frac{2}{\sqrt{b}}ia_0^\dagger x_0 + \frac{1}{2}a_0^\dagger a_0^\dagger\right)|\Omega_b\rangle \qquad (64)$$

Let us compute the center of mass time profile. After inverse–Wick–rotating x_0, it turns out to be

$$|\check{\Lambda}'(\mathrm{x}_0)\rangle = \langle \mathrm{x}_0|\check{\Lambda}'\rangle = \left(\frac{2}{b\pi}\right)^{\frac{1}{4}} \frac{\check{\mathcal{N}}'}{\sqrt{1+\check{T}'_{00}}} \qquad (65)$$

$$\times \exp\left(\frac{1}{b}\frac{1-\check{T}'_{00}}{1+\check{T}'_{00}}\mathrm{x}_0^2 + \frac{2i}{\sqrt{b}(1+\check{T}'_{00})}\mathrm{x}_0\check{T}'_{0n}a_n^\dagger + \frac{1}{2}a_n^\dagger \check{W}'_{nm}a_m^\dagger\right)|\Omega_b\rangle$$

$$\check{W}'_{nm} = \check{S}'_{nm} - \frac{1}{1+\check{T}'_{00}}\check{S}'_{0n}\check{S}'_{0m} \qquad (66)$$

The quantities $\check{S}'_{0n}$ and $\check{S}'_{nm}$ can be computed in the diagonal basis

$$\check{S}'_{0n} = \check{T}'_{0n} \tag{67}$$
$$= (1 + (-1)^n)\left(-\int_0^\infty V_0(k)\, V_n^{(k)} \exp\left(-\frac{\pi k}{2}\right) + V_0^{(\xi)}\, V_n^{(\xi)} \exp|\eta|\right)$$
$$\check{S}'_{nm} = (-1)^n \check{T}'_{nm} = \tag{68}$$
$$= ((-1)^n + (-1)^m)\left(-\int_0^\infty V_n^{(k)}\, V_m^{(k)} \exp\left(-\frac{\pi k}{2}\right) + V_n^{(\xi)}\, V_m^{(\xi)} \exp|\eta|\right)$$

It is evident that the leading time dependence is contained in $\exp\left(\frac{1}{b}\frac{1-\check{T}'_{00}}{1+\check{T}'_{00}}\mathrm{x}_0^2\right)$. The number $\check{T}'_{00}$ is $b(\eta)$–dependent and can be computed via

$$\check{T}'_{00}(\eta) = -2\int_0^\infty dk\left(V_0^{(k)}(b(\eta))\right)^2 \exp\left(-\frac{\pi k}{2}\right) + 2(V_0^{(\xi)})^2 \exp|\eta| \tag{69}$$

This is the crucial quantity as far as the time profile is concerned. An analytic evaluation of it is beyond our reach. However it was shown in [37] that

$$\lim_{\eta\to 0}\check{T}'_{00} = 1 \tag{70}$$
$$\lim_{\eta\to\infty}\check{T}'_{00} = \infty \tag{71}$$

A numerical analysis shows that this quantity is a function monotonically increasing with η within such limits.

This means that the quantity $\frac{1-\check{T}'_{00}}{1+\check{T}'_{00}}$ is always *negative* (it lies in the interval $[-1, 0]$) and so the profile is always localized in the center of mass time, except in the extreme case $\eta \to 0$, which corresponds to the tensionless limit (see below). This has to be compared with the usual lump solution (see previous subsection) for which the corresponding quantity is always positive and takes values in the interval $(0, \infty)$, allowing for localized *space* profiles but divergent along a time-like direction.

Actually, for reasons explained in [37], one is obliged to introduce another free parameter, which we denote by y. y is the eigenvalue of the operator

$$\hat{y}_\eta = \frac{i}{\sqrt{2}}(o_\eta - o_\eta^\dagger). \tag{72}$$

corresponding to the eigenstate

$$|y\rangle = \frac{1}{\sqrt{\pi}}\exp\left(-\tfrac{1}{2}y^2 - \sqrt{2}io_\eta^\dagger y + \tfrac{1}{2}o_\eta^\dagger o_\eta^\dagger\right)|\Omega_{\eta_o}\rangle, \tag{73}$$
$$o_\eta|\Omega_{\eta_o}\rangle = 0,$$
$$\hat{y}_\eta|y\rangle = y|y\rangle.$$

The operator $\hat{y}_\eta$ is the twist odd combination of the string oscillators projected along the discrete basis.

We contract our solution with the eigenvector (73). This operation can be done before or after projecting along the center of mass coordinate and does not interfere with it because $\hat{y}$ does not contain the zero mode. This leads to the following representation of our solution (inverse Wick–rotation is included)

$$|\check{\Lambda}'(\mathrm{x}_0, y)\rangle \;=\; \langle \mathrm{x}_0, \mathrm{y}|\check{\Lambda}'\rangle = \left(\frac{2}{b\pi}\right)^{\frac{1}{4}} \frac{\check{\mathcal{N}}'}{\sqrt{2\pi(1+e^\eta)}} \exp\left(\frac{1-e^{|\eta|}}{1+e^{|\eta|}}\mathrm{y}^2\right) \qquad (74)$$
$$\frac{1}{\sqrt{1+\check{T}'_{00}}} \exp\left(\frac{1}{b}\frac{1-\check{T}'_{00}}{1+\check{T}'_{00}}\mathrm{x}_0^2 + \frac{2i}{\sqrt{b}(1+\check{T}'_{00})}\mathrm{x}_0 \check{T}'_{0n} a_n^\dagger + \frac{1}{2} a_n^\dagger \check{W}''_{nm} a_m^\dagger\right)|0\rangle$$

The quantities $\check{T}'_{00}$ and $\check{T}'_{0n}$ are the same as in (69, 67) since the momentum $\hat{y}_\eta$ is twist–odd. Some changes occur instead in $\check{W}''_{nm}$, see [37].

Eq.(74) provides the complete time profile of our solution (60). It represents a solution localized in x_0, with the desired profile. It depends on two free parameters y and η (or b). These are all desirable features. But we can make a closer comparison of the representation (74) with the rolling tachyon solution. This can be done by considering the limit $b \to \infty$, which can be derived from the eqs.(59). $b \to \infty$ means $\eta \to \infty$ and $\check{T}'_{00}$ is given by

$$\check{T}'_{00} \approx 2\,\eta\, \log\, \eta\,(1 - \frac{\log(2\pi)}{\log\, \eta} + \ldots) \qquad (75)$$

where dots denote higher order terms and, for simplicity, we take η positive. Therefore we see that in this limit any time dependence in (74) disappears. Moreover, using a result of [37], we also have that $\check{W}''_{nm} \to S_{nm}$. In other words, in the limit $b \to \infty$ we obtain a static solution corresponding to the initial sliver (notice the change of sign in the quadratic oscillator term in (74) with respect to the initial sliver; this is as it should be after inverse Wick–rotation). From this we understand that the parameter $1/b$, for large b, plays a role similar to Sen's parameter $\tilde{\lambda}$ near 0, see section 2.

A second remark concerns the limit $\mathrm{y} \to \infty$. In this case the first exponential factor in the RHS of (74) suppresses everything, so that the limit is the 0 state. In other words, we can consider this value of the parameter y as identifying the stable vacuum state.

4.3. Low energy limit

It is instructive to study the solution (60) in a regime we are more familiar with, the low energy regime $\alpha' \to 0$, in which the solution considerably simplifies and an analytic treatment is possible. What we would like to see more closely is whether, for sufficiently small values of the parameters, the solution at time 0 is close enough to the sliver configuration (19), whose decay the solution is expected to describe.

The Neumann coefficients V'^{rs}_{NM} explicitly depend on the b parameter. In the low energy limit the three–strings vertex can be expanded by means of a parameter ϵ (a dimensionless parameter which represents the size of α') , see [8]. This translates into an expansion for V'^{rs}_{NM} whose dominant terms are given by the following rescalings

$$\begin{aligned} &V^{(rs)}_{mn} \to V^{rs}_{mn} \\ &V^{rs}_{m0} \to \sqrt{\epsilon} V^{rs}_{m0} \\ &V_{00} \to \epsilon V_{00} \end{aligned} \tag{76}$$

Let us analyze in detail what is the limit of the various quantities appearing in (74). Throughout this analysis the parameter b is kept constant. First of all we have

$$\lim_{\eta\to\infty} \frac{1-\check{T}'_{00}}{1+\check{T}'_{00}} = -1 \tag{77}$$

This follows from (59) and from the property of $(V_0^{(k)})^2$ of approximating $\delta(k)$ in the limit we are considering, which implies that $\check{T}'_{00} \to \infty$ in the same limit. For the oscillating term we have[37]

$$\lim_{\eta\to\infty} \frac{\check{T}'_{0n}}{1+\check{T}'_{00}} = \lim_{\eta\to\infty} \frac{1}{\sqrt{\log\,\eta}} = 0 \tag{78}$$

Thus the oscillating part decouples completely from the time dependent part. It remains for us to consider the limit of the quadratic form $\check{W}''_{nm}$, (74). This analysis, which is more complicated, is carried out in [37]. The result is

$$\check{W}''_{nm} = S_{nm} + \mathcal{O}(\epsilon/b) \tag{79}$$

Going back to equation (74) we see that, if we ignore for a moment the normalization factor, we obtain

$$\lim_{\alpha'\to 0} |\check{\Lambda}'(\mathrm{x}_0,\mathrm{y})\rangle = \check{\mathcal{N}}''(\mathrm{y})\, e^{-\frac{\mathrm{x}_0^2}{b}}\, |\Xi\rangle \tag{80}$$

where $|\Xi\rangle$ is the zero momentum sliver state. This result can be phrased as follows: in the low energy limit the solution takes the form of a time–Gaussian multiplying a deformed sliver, the deformation being parametrized by ϵ/b.

It is also interesting to analyze the the opposite limit, that is $\alpha' \to \infty$ (tensionless limit). As in the previous case this is formally achieved by taking the $\eta \to 0$ limit in all the quantities which are related to the Neumann coefficients, but keeping b as a free parameter. One easily gets

$$\lim_{\eta\to 0} \frac{1-\check{T}'_{00}}{1+\check{T}'_{00}} = 0 \tag{81}$$

This result implies that the Gaussian representing time dependence in (74) is actually completely flat: time dependence has disappeared! We believe this is related to the fact that all strings modes become massless in this limit [38], so there are no modes the brane can decay into.

In conclusion the solution represented by the time profile (74) enjoys many properties of the rolling tachyon: it is localized in time, therefore it is a good candidate to represent a solution bouncing off the tachyon potential; it depends on two parameters b and y; at time $t = 0$ it represents a deformation of the sliver; in the limit in which the parameter $b \to \infty$ the solution becomes exactly the sliver, but no time evolution is possible for this value of b; instead if b is large but finite, $1/b$ seems to play the role of the $\tilde{\lambda}$ parameter in the boundary conformal field theory approach; finally in low energy limit the solution takes the form of a time–Gaussian multiplying a deformed sliver.

5. Background electric field and fundamental strings

In [17] the rolling tachyon solution was studied in the presence of a background electric field. The solution does not change qualitatively except for the fact that the final products of the brane decay include, beside massive closed string states, also fundamental strings (the evidence for the latter being indirect).

The analysis of the same problem in VSFT was carried out in [19]. The solution depends on two parameter, b and E, the electric field. After projecting on the center of mass time coordinate it has been shown that in the extreme cases $b = 0$ and $b = \infty$ time dependence is lost and we are left respectively with the 0 state (VSFT vacuum) and the sliver state (VSFT D–brane), confirming in this case too these two cases are likely to reproduce the $\tilde{\lambda} = \frac{1}{2}$ and $\tilde{\lambda} = 0$ of Sen's BCFT, section 2. The E–field is another free parameter in the range $0 \leq E \leq E_c = \frac{1}{2\pi\alpha'}$. For E going to its critical value, time dependence is lost and we get a flat (non zero) profile in time (effective tensionless regime), this result still persists in the $b \to 0$ limit if we keep the ratio $\frac{1-(2\pi\alpha' E)^2}{b}$ finite. This case corresponds to the tachyon vacuum with a background of fundamental strings, prevented to decay by their polarization due to critical electric field. Of course this is an indirect way to see these fundamental strings arising in a classical solution of VSFT. A more direct construction of such objects is given in the following.

Since fundamental strings are expected to be among the brane decay products, the first thing one has to verify is that they are indeed solutions of VSFT. This was done in [39]. Macroscopic fundamental strings are particular solutions to (11). They are condensates of D0–branes.

Let us briefly review their construction. We start from a lump solution representing a D0–brane as introduced in section 3, let us denote it $|\Xi_0'\rangle$: it has a Gaussian profile in all space directions. Let us pick a particular space direction, say the α–th. For simplicity, in the following we will drop the corresponding label from the coordinate $\hat{x}^\alpha$, momenta $\hat{p}^\alpha$ and oscillators a^α along this direction. Next we need the same solution displaced by an amount s in the positive x direction (x being the eigenvalue of $\hat{x}$). The appropriate solution was constructed by Rastelli, Sen and Zwiebach, [16]:

$$|\Xi_0'(s)\rangle = e^{-is\hat{p}}|\Xi_0'\rangle \tag{82}$$

It satisfies $|\Xi_0'(s)\rangle * |\Xi_0'(s)\rangle = |\Xi_0'(s)\rangle$. Eq.(82) can be written explicitly as

$$\begin{aligned} |\Xi_0'(s)\rangle \;&=\; \mathcal{N}' e^{-\frac{s^2}{2b}(1-S_{00}')} \\ &\times \exp\left(-\frac{is}{\sqrt{b}}((1-S')\cdot a^\dagger)_0\right) \exp\left(-\frac{1}{2}a^\dagger\cdot S'\cdot a^\dagger\right)|\Omega_b\rangle \end{aligned} \tag{83}$$

where

$((1-S_{00}')\cdot a^\dagger)_0 = \sum_{N=0}^{\infty}((1-S')_{0N}a_N^\dagger)$ and

$a^\dagger\cdot S'\cdot a^\dagger = \sum_{N,M=0}^{\infty} a_N^\dagger S_{NM}' a_M^\dagger$;

$\mathcal{N}'$ is the $|\Xi_0'\rangle$ normalization constant. One can show that

$$\langle\Xi_0'(s)|\Xi_0'(s)\rangle = \langle\Xi_0'|\Xi_0'\rangle \tag{84}$$

The meaning of this solution is better understood if we make its space profile explicit by contracting it with the coordinate eigenfunction

$$\begin{aligned} \langle\hat{x}|\Xi_0'(s)\rangle \;&=\; \left(\frac{2}{\pi b}\right)^{\frac{1}{4}} \frac{\mathcal{N}'}{\sqrt{1+S_{00}'}} \\ &\times e^{-\frac{1-S_{00}'}{1+S_{00}'}\frac{(x-s)^2}{b}} e^{-\frac{2i}{\sqrt{b}}\frac{x-s}{1+S_{00}'}S_{0m}'a_m^\dagger} e^{-\frac{1}{2}a_n^\dagger W_{nm}' a_m^\dagger}|0\rangle \end{aligned} \tag{85}$$

where $W_{nm}' = S_{nm}' - \frac{S_{n0}'S_{0m}'}{1+S_{00}'}$. It is clear that (85) represents the same Gaussian profile as $|\Xi_0'\rangle = |\Xi_0'(0)\rangle$ shifted away from the origin by s.

It is important to remark now that two such states $|\Xi_0'(s)\rangle$ and $|\Xi_0'(s')\rangle$ are $*$–orthogonal and bpz–orthogonal provided that $s \neq s'$. An explicit calculations shows that

$$|\Xi_0'(s)\rangle * |\Xi_0'(s')\rangle = e^{-\mathcal{C}(s,s')}|\Xi_0'(s,s')\rangle \tag{86}$$

where the state $|\Xi_0'(s,s')\rangle$ becomes proportional to $|\Xi_0'(s)\rangle$ when $s = s'$ and needs not be explicitly written down otherwise; while

$$\mathcal{C}(s,s') = -\frac{1}{2b}\left[(s^2+s'^2)\left(\frac{T'(1-T')}{1+T'}\right)_{00} + ss'\left(\frac{(1-T')^2}{1+T'}\right)_{00}\right] \tag{87}$$

The quantities $\left(\frac{T'(1-T')}{1+T'}\right)_{00}$ and $\left(\frac{(1-T')^2}{1+T'}\right)_{00}$ can be evaluated by using the basis of eigenvectors of X' and T' (subsection 3.1). The analysis of [39] shows that, when $s \neq s'$, the factor $e^{-\mathcal{C}(s,s')}$ vanishes, so that (86) becomes a $*$–orthogonality relation. When $s = s'$ we get instead

$$\mathcal{C}(s,s) = -\frac{s^2}{2b}(1 - S'_{00})$$

In conclusion we can write

$$|\Xi'_0(s)\rangle * |\Xi'_0(s')\rangle = \hat{\delta}(s,s')|\Xi'_0(s)\rangle \tag{88}$$

where $\hat{\delta}$ is the Kronecker delta.

Similarly one can prove that

$$\langle\Xi'_0(s')|\Xi'_0(s)\rangle = \hat{\delta}(s,s')\,\langle\Xi'_0|\Xi'_0\rangle \tag{89}$$

After the above preliminaries, let us consider a sequence $s_1, s_2, \ldots$ of distinct real numbers and the corresponding sequence of displaced D0–branes $|\Xi'_0(s_n)\rangle$. Due to the property (88) also the string state

$$|\Lambda\rangle = \sum_{n=1}^{\infty} |\Xi'_0(s_n)\rangle \tag{90}$$

is a solution to (11): $|\Lambda\rangle * |\Lambda\rangle = |\Lambda\rangle$. To figure out what it represents we analyze its space profile. To this end we render the sequence $s_1, s_2, \ldots$ dense, say, in the positive x–axis so that we can replace the summation with an integral. The relevant integral is

$$\int_0^{\infty} ds\, e^{-\alpha(x-s)^2 - i\beta(x-s)} = \frac{\sqrt{\pi}}{2\sqrt{\alpha}}\left(e^{-\frac{\beta^2}{4\alpha}}\left(1 + \mathrm{Erf}\left(\frac{i\beta}{2\sqrt{\alpha}} + \sqrt{\alpha}x\right)\right)\right) \tag{91}$$

where Erf is the error function and

$$\alpha = \frac{1}{b}\frac{1 - S'_{00}}{1 + S'_{00}}, \qquad \beta = \frac{2}{\sqrt{b}}\frac{S'_{0m}a^{\dagger}_m}{1 + S'_{00}}$$

Of course (91) is a purely formal expression, but it becomes meaningful in the $\alpha' \to 0$ limit. As usual, [8], we parametrize this limit with a dimensionless parameter ϵ and take $\epsilon \to 0$. Using the results of [37, 8], one can see that $\alpha \sim 1/\epsilon$, $\beta \sim 1/\sqrt{\epsilon}$, so that $\beta/\sqrt{\alpha}$ tends to a finite limit. Therefore, in this limit, we can disregard the first addend in the argument of Erf. Then, up to normalization, the space profile of $|\Lambda\rangle$ is determined by

$$\frac{1}{2}\left(1 + \mathrm{Erf}(\sqrt{\alpha}x)\right) \tag{92}$$

In the limit $\epsilon \to 0$ this factor tends to a step function, valued 1 in the positive real x–axis and 0 in the negative one. Of course a similar result can be obtained numerically to any degree of accuracy. This derivation is not rigorous and mostly qualitative, but it can be made more reliable, [37]. In conclusion, we have constructed a state $|\Lambda\rangle$ which is a solution to (11) and represents, in the low energy limit, a one–dimensional object with a constant profile that extends from the origin to infinity in the x–direction. Actually the initial point can be any finite point of the x–axis, and it is possible to construct a configuration that extend from $-\infty$ to $+\infty$.

We interpret such $|\Lambda\rangle$ solutions as fundamental strings. In order to justify this claim, let us show that they are still solutions if we attach them to a D–brane. To this end let us pick $|\Lambda\rangle$ as given by (90) with $s_n > 0$ for all n's. Now let us consider a D24–brane with the only transverse direction coinciding with the x–axis and centered at $x = 0$. The corresponding lump solution has been introduced in section 3 (case $k = 24$). Let us call it $|\Xi'_{24}\rangle$. Due to the particular configuration chosen, it is easy to prove that $|\Xi'_{24}\rangle + |\Lambda\rangle$ is still a solution to (11). This is due to the fact that $|\Xi'_{24}\rangle$ is $*$-orthogonal to the states $|\Xi'_0(s_n)\rangle$ for all n's. To be even more explicit we can study the space profile of $|\Xi'_{24}\rangle + |\Lambda\rangle$, assuming the sequence s_n to become dense in the positive x–axis. Using the previous results it is not hard to see that the resulting configuration is a Gaussian centered at $x = 0$ in the x direction (the D24-brane) with an infinite prong attached to it and extending along the positive x–axis. The latter has a Gaussian profile in all space directions except x.

In [39] we have shown many other properties of the Λ solutions. For instance it is not possible to have solutions representing configurations in which the string crosses the brane by a finite amount: the string has to stop at the brane. Moreover these solutions enjoys the exchange property typical of fundamental strings. Finally it is possible to construct solutions corresponding to any curve in space.

6. Comments

This review is an account of the attempt at finding (exact) rolling tachyon solutions in SFT. We chose to search for such solutions in VSFT: this means looking for lump solutions to (11) and inverse–Wick–rotate them. In section 4 we showed that it is indeed possible to find an exact solution that has several characteristics of a rolling tachyon. However the analysis is still incomplete. The explicit analysis of the total energy conservation during the decay process is still missing, although it is implicit in the fact that the energy of the Wick rotated (Euclidean) solution is a constant. Moreover from our solution it is not clear what the products of the brane decay are going to be. Another open problem is connected with the time x_0 we have used. Our choice of x_0 as the time is not fully motivated. It is in fact the open string time, that is the time that couples to the open string metric. Other choices are possible. In [37] we suggested a different choice, which may be connected with the closed string time. More work is necessary in this direction.

In section 5 we have seen some consequences of introducing a background electric field. We have concentrated especially on the problem of fundamental strings. The Euclidean (Wick–rotated) energy of the solution depends on the background field. One infers from this that the final state must contain fundamental strings, which are the objects in the bosonic string zoo that couple to the electric field. We found that fundamental string solutions exist in VSFT both as isolated objects and attached to D–branes. This result confirms the analysis of the brane decay in the presence of a background electric field. But of course we would like to have a more direct account of the involvement of fundamental strings in D–brane decays.

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