Self-duality, conformal geometry and curvature characterizations[*†‡]

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Abstract

The self-duality is important in physics as well as in mathematics. We present basic notions and some recent results. The main goal is to discuss characterizations of self-duality in terms of symmetries of the curvature and the conformal curvature. We are interested in Riemanian, and Kleinian geometry, i.e. in positive definite $(++++)$ and of split signature $(--++)$ metrics. Also, families of examples of homogeneous, left invariant self-dual metrics (admitting additional symmetries) are studied. The geometry of the left-invariant metrics on these Lie groups is very reach.

1. Notions and terminology

Let (M, g) be a pseudo-Riemannian manifold of dimension n and signature (p, q), $p+q = n$. The Levi-Civita connection on the tangent bundle TM and the corresponding curvature tensor are denoted by ∇ and R respectively.

Let us recall the basic facts concerning conformal changes of metric. Metrics g and $\tilde{g}$ are *conformally related* if there exist a smooth function ϕ such that $\tilde{g} = \exp(2\phi)g$. The Weyl curvature tensor is defined by

$$\begin{aligned} \mathcal{W}(x,y) := \mathcal{R}(x,y) \quad &+ \quad \frac{\tau}{(n-1)(n-2)}\mathcal{R}_g^0(x,y) \\ &+ \quad \frac{1}{n-2}\mathcal{L}(x,y) \end{aligned}$$

where τ is the scalar curvature, where ρ is the Ricci operator, where

$$\mathcal{R}_g^0(x,y)z := g(y,z)x - g(x,z)y, \quad \text{and where}$$
$$\mathcal{L}(x,y)z := g(\rho y,z)x - g(\rho x,z)y + g(y,z)\rho x - g(x,z)\rho y,$$

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and $n > 2$.

It is well known that Weyl tensor is a conformal invariant, i.e. $\mathcal{W}_{\tilde{g}} = \mathcal{W}_g$ for conformally related metrics.

Assume now that M is an oriented manifold. The Hodge $*$ operator $* : \Lambda^k \longrightarrow \Lambda^{(n-k)}$ is defined as a unique $(n-k)$-form $*\beta$ which satisfies

$$\alpha \wedge (*\beta) = g(\alpha, \beta)\omega_g,$$

for all $(n-k)$-forms α, where ω_g is the volume form and g also denotes the induced scalar product on the space of $(n-k)$-forms.

Dimension four is a very special one. For example, $* : \Lambda^2 \longrightarrow \Lambda^2$ and $*^2 = 1$ if (M, g) is of definite or split signature. The eigenvectors of 1 and -1 are *self-dual* and *anti self-dual* 2-forms and the corresponding eigenspaces are denoted by Λ^2_+ and Λ^2_- respectively. Let $e^{ij} := e^i \wedge e^j$ where $\{e^i\}$ is a base of 1-forms. Then, in the Riemannian case,

$$f_1^{\pm} = e^{12} \pm e^{34},\ f_2^{+} = e^{13} \mp e^{24},\ f_3^{+} = e^{14} \pm e^{23}$$

form basses for $\Lambda^2_{\pm}$. Also, the decomposition $\Lambda^2 = \Lambda^2_+ \oplus \Lambda^2_-$ holds.

In the Lorentzian case, $*^2 = -1$ and we do not have the corresponding notion of real self-dual forms.

This is an appropriate place to discuss principal bundles. Particularly we would like to compare terminology used by physicist and mathematicians, i.e. here will be presented a vocabular of basic notions to increase understanding between mathematicians and physicist. For a Lie group G let P be a G-principle bundle parametrized by the points of manifold M.

VOCABULARY [1]

Mathematics		Physics
total space of fibre bundle	P	phase space[2]
base space	M	spacetime
structure group	G	gauge group
local sections of the principal bundle	f	local gauge
connection on P	ω	connection, gauge potential
curvature	Ω	curvature, gauge field strength
action of G in P	ψ	gauge transformation

[1]We thank to M. Blagojević for help in preparing this vocabulary.
[2]typical interpretation

2. Self-duality

Let us recall the Maxwel electromagnetic equations. The electric field E and the magnetic field B satisfy the following set of equations:

$$\begin{array}{ll} \operatorname{div} B = 0 & \operatorname{div} E = 0 \\ \operatorname{curl} E + \frac{\partial B}{\partial t} = 0 & \operatorname{curl} B - \frac{\partial E}{\partial t} = 0. \end{array}$$

The Faradey tensor is defined as 2-form

$$\begin{aligned} F = & - E_x dt \wedge dx - E_y dt \wedge dy - E_z dt \wedge dz \\ & + B_x dy \wedge dz + B_y dz \wedge dx + B_z dx \wedge dy, \end{aligned}$$

where $E = (E_x, E_y, E_z)$, $B = (B_x, B_y, B_z)$ and (t, x, y, z) are coordinates on $\mathbb{R}^4$. Then the Maxwel's equations simplify to

$$dF = 0 \qquad d * F = 0.$$

For an arbitrary fiber bundle P over 4-dimensional compact oriented manifold M the space of connections is denoted by $\mathcal{A}$. The Yang-Mills functional $\mathcal{JM} : \mathcal{A} \longrightarrow \mathbb{R}$ is defined as

$$\mathcal{JM}(A) = \int_M F_A \wedge *F_A.$$

The critical points of this functional are determined by the conditions

$$dF_A = 0 \qquad d * F_A = 0,$$

so we may consider Yang-Mills equations (Yang-Mills connections) as a generalization of Maxwell's equations in the curved space in a naturally defined geometric setting. The only independent equation is the second one, $d * F_A = 0$, since the first equation, $dF_A = 0$, represents the 1^{St} Bianchi identity. Moreover, *self-dual connections*, defined by the condition

$$*F_A = F_A,$$

are Yang-Mills connections. The case when P is a tangent bundle TM of the base space and g is a metric of definite or split signature is studied in more details. We say that g is *self-dual metric* if $*\mathcal{W} = \mathcal{W}$, where $\mathcal{W}$ is the Weyl curvature 2-form on M. Similarly, we define anti self-dual metric by $*\mathcal{W} = -\mathcal{W}$.

The moduli space of self-dual metrics is rich (see [2]). Some of the known examples of self-dual metrics are $K3$ surfaces, Eguchi-Hanson (hyperKähler) metric, Taub-NUT metrics. Much more examples are known and constructing new examples is still widely open problem (for some details see [7]and [8]).

Self-dual metrics play a very important role in physics. It provides a base for twistor theory. Also, it is related to the existence of local $N = 2$

supersymmetric extension of the standard σ-model $f : \Sigma \longrightarrow X$, X is a 4 dimensional manifold.

Here we offer some additional arguments to illustrate the significance of the self-dual metrics. Self-dual Einstein manifolds could be understood as a generalization of quaternionic-Kähler manifolds which are usually defined for $4k$-dimensional manifolds, $k > 1$. Using higher-dimensional generalizations of self-duality and geometric symmetries by reduction method is possible to construct lower dimensional self-dual metrics. Some self-dual metrics on $\mathbb{R}^{2,2}$ are used to construct $(2+1)$-chiral model by the use of Bäcklund transformation.

3. Symmetries, conformal geometry and self-duality

In this section we will give a survey of some results where self-duality is characterized in terms of the curvature of the manifolds and its symmetry.

With a curvature tensor R of (M, g) and a unit vector $x \in T_pM$ we associate *Jacobi operator* $J_x : T_pM \longrightarrow T_pM$ as $J_x(y) = R(y, x)x$. We say that (M, g) is *Osserman manifold* if the eigenvalues of J_x are independent on the choice of a unit direction x at a point. Notice that the eigenvalues may vary from point to point. Let's recall that the variational vector field y along a geodesic γ_0 (tidal force) satisfies the Jacobi equation

$$y'' = -R(y, \gamma')\gamma' = -J_{\gamma'}(y).$$

Thus, we may say that all directions in a given point of an Osserman manifold defines the same local dynamics (dynamically isotropic manifolds).

The basic examples are 2-point homogeneous spaces but in $(2, 2)$ signature there are some additional examples ([3]). Self-dual manifolds can be characterized in the terms of curvature as follows.

Theorem. *Let (M, g) be a pseudo-Riemannian 4-dimensional manifolds of signature $(+ + ++)$ or $(- - ++)$. Then (M, g) is self-dual, Einstein manifold if and only if (M, g) is Osserman i.e. if and only if it is dynamically homogeneous.*

For proof and details see [1].

The modification of the Jacobi operator play also important role in the conformal geometry. *Conformal Jacobi* operator $J_x^{conf} : T_pM \longrightarrow T_pM$ is defined by $J_x^{conf}(y) = W(y, x)x$ where W is the Weyl curvature tensor of the metric g. We say that (M, g) is *conformally Osserman* manifold if the eigenvalues of the conformal Jacobi operator J_x^{conf} are independent on unit direction $x \in T_pM$. It is a conformaly invariant notion. The examples are given by the conformal classes of 2-point homeogeneous metrics.

The self-dual metrics are naturally related to the conformal Weyl curvature as it is recently shown by Blažić and Gilkey.

Theorem.([4], 2004) *Let $(M, [g])$ be a 4-dimensional conformal manifold of definite signature. This geometry is self-dual if and only if $(M, [g])$ conformally Osserman.*

4. Lie group examples

Finally we will illustrate how rich is the family of self-dual metrics by the study of homogeneous geometry. The examples of left-invariant self-dual metrics on Lie group G are presented.

Theorem. *The only self-dual, not conformally flat, left invariant positive-definite Lie groups are given by the 1-parametar family of Lie algebra* $\mathfrak{g}_\lambda$. *The commutator relations are:*

$$\begin{array}{rclcl} [f_1, f_2] & = & f_2 - \lambda f_3, & & [f_2, f_3] = -f_4, \\ [f_1, f_3] & = & \lambda f_2 + f_3, & & [f_1, f_4] = 2f_4. \end{array}$$

The case of $(2,2)$ signature is more complicated and there exist much more examples. In [5] Cliff(1,1) structure on Lie groups was studied which is also known as para-hypercomplex structure or complex product structure or anti-quaternionic structure. It is defined by the anti-commuting integrable endomorphisms J_1 and J_2 of the tangent space such that $J_1^2 = -1$ and $J_2^2 = 1$. The Cliff(1,1) structure implies the existence of left invariant metric of split signature on the Lie group G which is always self-dual with respect to the appropriate orientation. These Lie groups also provide examples for a lot interesting geometric properties [5].

Theorem.([5]) *The Lie groups which admit left invariant Cliff(1,1) structures are:* $\mathbb{R} \oplus \mathfrak{r}_{3,1}$, $\mathbb{R}^2 \oplus \mathfrak{aff}(\mathbb{R})$, $\mathfrak{d}_4$, $\mathfrak{t}_{4,1,c}$, $c \neq 1$, $\mathfrak{r}_{4,1}$, $\mathfrak{d}_{4,1/2}$. *Among them* $\mathfrak{d}_{4,1/2}$ *is not scalar flat. The left invariant metrics naturally induced by these Cliff(1,1) structures are self-dual. These are the only non conformally flat Cliff(1,1) structures.*

Theorem.*([5])* The only Lie algebras admitting a Ricci flat, but not flat, Cliff(1,1)-structure are: $\mathfrak{r}_{4,1}$, $\mathfrak{t}_{4,1,-1}$, $\mathfrak{d}_{4,2}$.

The Lie algebras mentioned in the previous theorems are defined by the following commutator relations:

$$\begin{array}{l} \mathbb{R} \oplus \mathfrak{r}_{3,1}: \ [e_1, e_2] = e_2, \ [e_1, e_4] = e_4, \\ \mathbb{R}^2 \oplus \mathfrak{aff}(\mathbb{R}): \ [e_1, e_2] = e_1, \\ \mathfrak{d}_4: \ [e_1, e_4] = e_3, \ [e_1, e_2] = e_1, \ [e_2, e_4] = e_4, \\ \mathfrak{d}_{4,\lambda}: \ [e_4, e_3] = e_3, \ [e_1, e_2] = e_3, \ [e_4, e_1] = \lambda e_1, \\ \qquad [e_4, e_2] = (1-\lambda) e_2 \\ \mathfrak{t}_{4,1,\lambda}: \ [e_4, e_1] = e_1, \ [e_4, e_2] = e_2, \ [e_4, e_3] = \lambda e_3, \\ \mathfrak{h}_4: \ [e_4, e_3] = e_3, \ [e_1, e_2] = e_3, \ [e_4, e_2] = \dfrac{1}{2} e_2, \\ \qquad [e_4, e_1] = e_2 + \dfrac{1}{2} e_1, \\ \mathfrak{r}_{4,\lambda}: \ [e_1, e_2] = e_2, \ [e_1, e_3] = \lambda e_3, \ [e_1, e_4] = e_3 + \lambda e_4. \end{array}$$

In the nonhomogeneous case, construction of self-dual metric is often based on the use of some symmetries to reduce the problem to some *known classical integrable differential equations such as Tzitéica, Painleve, Kadomtsev-Petviashvilli equations. For detaisl see [9] and [6].*

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