Reductions and Orbifolds: Construction of self-dual 4-metrics

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1 Introduction

This is a lecture note from "The II Summer School in Modern Mathematical Physics", held at Kopaonik, 1-12.09.2002. The main goal is to introduce the notions of reductions and orbifolds, important in geometry and in physics as well. In this note examples will be used to introduce different notions and to illustrate the possibilities of the reduction method.

The symplectic reduction was introduced by Marsden and Weinstein, [14]. It is very important and useful technique for understanding the symmetry. Essentially, by this method manifold with isometry and some additional structure (such as Kähler, hyper-Kähler, quaternionic-Kähler, ...) is reduced to the manifold with isometry devide out, but still to preserve that additional structure.

Here is a brief outline to this paper. In §2, we deal with the reduction and orbifolds and in §3 we define self-duality. The next section is dedicated to the para-quaternionic numbers $\tilde{\mathbb{H}}$ (Clifford algebra $C(1,1) = C(0,2)$), to the associated projective plane $\tilde{\mathbb{H}}P^2$ and to the geometry of split signature. In §5 and §6 we study explicitly the orbit space introduced in [6] including its curvature. The action of the group $G = \mathbb{R}$ on $\tilde{\mathbb{H}}P^2$ and on the algebraic submanifold $\mathcal{K}_0$ of $\tilde{\mathbb{H}}P^2$ is defined. We shall also study the basic geometry of the orbit space $\mathcal{K}_0/G$ which are equipped with the metric $g_{p,q}$ induced by the submersion $\mathcal{K}_0 \to \mathcal{K}_0/G$. We will compute the sectional curvature and the Jacobi curvature operator of the metric $g_{p,q}$ in terms

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of the para-quaternions and prove that $g_{p,q}$ is a self-dual, Einstein metric which is pointwise Osserman. Moreover, this metric is not even locally homogeneous.

Finally, in §7, we use approach from [6] to study the curvature of the positive definite self-dual orbifolds constructed by Galicki and Lawson ([8, 9]).

It is a pleasant task to thank Srdjan Vukmirović for interest and help during the preparation of this note.

2 Basic examples of reductions and orbifolds.

Let us consider Riemannian manifold (M, g) and let G be a subgroup of the full isometry group $\mathrm{Isom}\,(M, g)$. The corresponding orbit space is an important object, but it is not necessary a manifold. Here is an elementary example.

Example 2.1 *Let M be a complex plane $\mathbb{C}$ and Z_n a cyclic isometry group of order n, $Z_n = \{\exp 2\pi i(k/n) \mid 0 \leq k \leq n-1\}$. The orbit space, quotient $\mathbb{C}/Z_n$, is a cone and* it is not a manifold. *Cone is the simplest type of singularities that locally characterize orbifolds. Notice that this action is not free.*

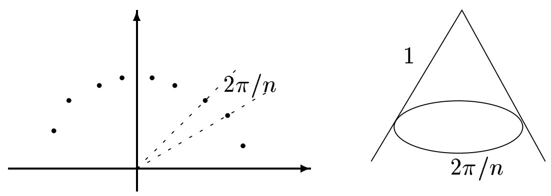

Figure: 1. Cone

Orbifolds is locally determined as a manifold (M, g) divided by a finite group of isometries of (M, g). We may think on orbifold as an manifold with cusps (attached connes) (Fig. 2).

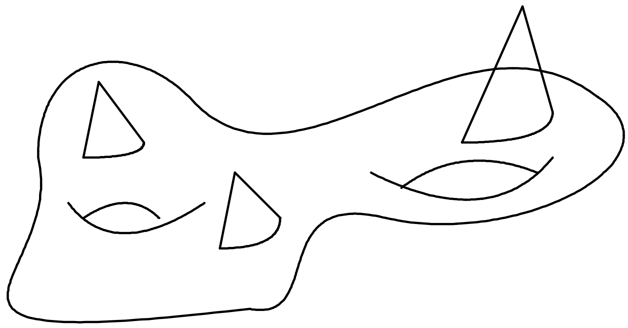

Figure: 2. Orbifold

Let us give a few more examples of quotient spacers. The next example is classical projective space.

Example 2.2 *Let us consider $G = S^1 = U(1)$ action on the unit sphere S^{2n+1} in $\mathbb{C}^{n+1}$, defined by $\phi_t(z) = \exp^{2\pi it} z$ for t in S^1. Here the action is* free *and the resulting orbit space S^{2n+1}/S^1 is a manifold, the complex projective space $\mathbb{C}P^n$.*

Here is an important example of the quotient space when the action of the isometry group G is just locally free (not free).

Example 2.3 (Weighted projective space.) *For mutually coprime integers p, q and r, we consider $G = S^1 = U(1)$-action on the unit sphere S^5 in $\mathbb{C}^3$, defined by $\Phi_{p,q,r}\tau(z_0, z_1, z_2) = (\tau^p z_0, \tau^q z_1, \tau^r z_2)$ for t in S^1. The resulting orbifold is* weighted projective space $\mathbb{C}P^2_{p,q,r} = S^5/\Phi_{p,q,r}$.

In a general case we have the following. Assume that M is a smooth manifold and G is a compact Lie group *acting freely* on M. Then $N = M/G$ can be given the structure of manifold, such that the projection map $\pi : M \longrightarrow M/G$ is a smooth submersion.

Moreover, since the action of the group G on M is free, non-zero vector fields on M generated by G have no zeroes. Then, there is a metric h on the orbit space $N = M/G$ naturally defined (by decomposing the tangent space on M as a direct sum of vertical and the horizontal component).

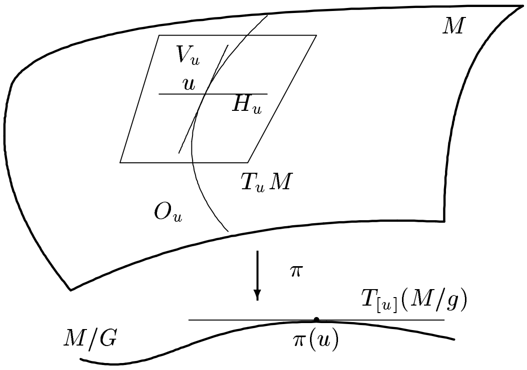

Figure: 3. Orbit space

In the next step we assume that there is an additional structure on the manifold M, for example let M admits a symplectic structure with the symplectic form ω. Let us recall that ω is closed non-degenerate 2-form. Clearly, it is not possible to preserve the symplectic structure by the ordinary quotient construction. Thus, we have the well known modifications, the symplectic quotient construction, Marsden-Weinstein reductions.

Assume that M^{2n} is a symplectic manifold and G is a compact Lie group of symplectic transformations acting freely on M. Let X be a vector field on M generated by the Lie algebra $\mathfrak{g}$ of the group G. If the first cohomology group $H^1(M, \mathbb{R})$ of the manifold M vanishes, a *Hamilton function* $\mu^X : M \longrightarrow \mathbb{R}$, defined by

$$d\mu^X = i(X)\omega = \omega(X, \cdot),$$

is uniquely determined up to a constant. The map $\mu : M \longrightarrow \mathfrak{g}^*$, defined by $\mu(m)(\xi) = \mu^X(m)$, $m \in M$, $\xi \in \mathfrak{g}$ is called *moment map* if μ is equivariant (i.e. μ commutes with the action of G on M and the coadjoint action of G on $\mathfrak{g}^*$).

The Marsden-Weinstein reduction is now defined in the following way. Let $N = \mu^{-1}(0)$ be the submanifold of M. One can show that G acts on N (because μ is equivariant) so there is quotient $P = N/G$. Then P is a new symplectic manifold, the dimension of which is $\dim M - 2\dim G$.

The Marsden-Weinstein reductions is an powerfull method. Its modification, the hyper-Kähler quotient construction is quite universal and was used to constuct many examples of hyper-Kähler (Ricci flat) manifolds. For example, using hyper-Kähler reductions of the flat space, the following well known metrics can be obtained: Multi-Eguchi-Hanson gravitational instanton, Multi-Taub-NUT gravitation instanton, Calabi metric on $T^*(\mathbb{C}P^{n-1})$, Linström-Roček metric (new generalization with Taubian infinity), Multi-Calabi spaces, Lindström-Roček spaces (hyper-Kähler metrics on cotangent bundles of complex Grassmanians) (see [8]).

The quaternionic-Kähler geometry, is also interesting but the reduction which will preserve the quaternionic-Kähler structure is more complicated (we may not have a symplectic structure there). The quaternionic reduction of any quaternionic-Kähler manifold with isometries to another quaternionic-Kähler manifold in which the isometries are divided out, is obtained by Galicki and Lawson [8, 9] and also by Swan [20].

3 Self-dual Einstein metrics

Let us give some physics motivation for studying self-duality, quaternionic-Kähler geometry and Einstein metrics.

Let (M, g) be a 4-dimensional space-time and F the Faradey electromagnetic tensor. Then by the Maxwell equation in the vacuum

$$dF = 0, \qquad d \star F = 0,$$

where $\star$ is Hodge star, $\star : \Lambda^2 M \longrightarrow \Lambda^2 M$. In geometric terminology, F is a curvature form (gauge field) of a connection (gauge potential) of the corresponding bundle. Then, the first equation is automatically satisfied since it represents the first Bianchi identity. The second equation motivates us to consider connections (gauge potentials) which curvature (gauge fields) satisfy $\star F = \pm F$. Such connection satisfies both Maxwell equations and is called a *(anti-)self-dual connection*. If we consider the bundle $\Lambda^2 T^* M$ of two forms on M, the curvature R induced by the Levi-Civita connection of the Einstein metric g is (anti-)self dual ($\star R = \pm R$) if and only if it is half-conformally flat, i.e. self-dual or anti-self-dual part of the Weyl curvature tensor W vanish $W_- = 0$ ($W_+ = 0$). Such metric g is also called *(anti-)self-dual metric* (see [22]).

This is very important notion since the self-dual metrics are strongly related to the holomorphic geometry. More precisely, with a four dimensional manifold (M, g) one can consider $\mathcal{Z}$, the bundle whose fibres are spaces of compatible almost complex structures on the tangent space $T_p M$. Moreover, it caries a naturally defined complex coordinates if the metric g is (anti-)self-dual.

Quaternionic-Kähler manifolds are characterized by $Sp_1 \cdot Sp_n$ holonomy. Recently, there has been growth of interest in quaternionic-Kähler manifolds because of the following reason. The classical σ-model, which consists of the energy functional for maps $f : \Sigma \to X$ of a compact Riemannian 4-manifold into a Riemannian manifold X, admits locally an $N = 2$ sypersymmetric extension only when X is a quaternionic-Kähler space with non-zero scalar curvature.

In four dimensions, self-dual manifolds are important not only in Riemannian geometry, i.e. signature $(0, 4)$, but also in Kleinian (or neutral) geometry, i.e. in signature $(2, 2)$. We refer to [12] where examples of Einstein, self-dual neutral signature metrics are given. However, non-trivial examples of non-Ricci flat, Einstein self-dual manifolds were not known previously. We refer to [8] and [9] where families of such examples were constructed in the Riemannian setting. In the present note, motivated by the examples cited above, we shall construct a family of neutral signature manifolds $M_{p,q}$ (p and q are coprime positive integers) which are not locally homogenous.

4 Neutral (split) signature

The quaternions (the Clifford algebra $C(2)$) provide a natural setting for the Riemannian (positive definite) quaternionic-Kähler geometry. Similarly, the para-quaternions (the Clifford algebra $C(1, 1)$) are a natural language to study para-quaternionic-Kähler geometry of split signature $(2n, 2n)$.

The importance of four dimensional metrics of signature $(0, 4)$ and $(2, 2)$ are coming from the existence of decompositions

$$SO(4) = SO(3) \times SO(3) \quad \text{and}$$

$$SO(2, 2) = SO(1, 2) \times SO(1, 2) = SL(2) \times SL(2).$$

During the recent years, interest in neutral geometry is growing [1, 3, 10, 11, 13, 17].

The para-quaternions are the real unital Clifford algebra $\tilde{\mathbb{H}} := C(1, 1) = C(0, 2)$ which is generated by elements i, j, k subject to the relations:

$$i^2 = -\epsilon_1 = -1,\ \ j^2 = -\epsilon_2 = 1,\ \ k^2 = -\epsilon_3 = 1,\ \ ij = k = -ji.$$

Let $q := x + yi + zj + wk$. The conjugate $\bar{q}$, the real part $\Re q$, the imaginary part $\Im q$, and the square norm $|\cdot|^2$ are then defined by:

$$\begin{aligned}
&\bar{q} := x - yi - zj - wk,\\
&\Re q := x, \quad \Im q := yi + zj + wk,\\
&|q|^2 := q\bar{q} = x^2 + y^2 - z^2 - w^2\,.
\end{aligned}$$

The norm is multiplicative, i.e. it satisfies the relation

$$|q_1q_2|^2 = |q_1|^2|q_2|^2.$$

Notice that the pseudosphere of signature $(2,1)$ of all unit spacelike para-quaternions

$$S^{2,1} = \tilde{\mathbb{H}}_1 := \{q \in \mathbb{H} \mid |q|^2 = 1\}$$

is isomorphic to the Lie group $SU(1,1)$. Give $\tilde{\mathbb{H}}^3$ the pseudo-Riemannian metric of signature $(6,6)$ defined by

$$g(u,v) := \Re(\bar{u}_1 v_1 + \bar{u}_2 v_2 + \bar{u}_3 v_3) = \Re(\bar{u} \cdot v)$$

for all $u = (u_1, u_2, u_3)$, $v = (v_1, v_2, v_3)$ in $\tilde{\mathbb{H}}^3$. Let $S^{6,5}$ be the associated pseudosphere of signature $(6,5)$.

The para-quaternionic projective plane $\tilde{\mathbb{H}}P^2$ is defined in the standard way as the space of of para-quaternionic 4-planes, i.e.

$$\tilde{\mathbb{H}}P^2 := \{[u] \mid u \in S^{6,5}\}/\sim,$$

where $u = (u_0, u_1, u_2) \sim (u_0 h, u_1 h, u_2 h) = uh$, $h \in \tilde{\mathbb{H}}_1$. Let $\pi(u) := [u]$ be the natural projection from $S^{6,5}$ to $\tilde{\mathbb{H}}P^2$ and let $\mathcal{V}_u \subset T_u S^{6,5} \subset T_u \tilde{\mathbb{H}}^3 \cong \tilde{\mathbb{H}}^3$ be the vertical subspace of the submersion π at the point $u \in S^{6,5}$, generated by the vectors ui, uj, uk. Moreover, the orthogonal decomposition with respect to the metric g

$$T_u S^{6,5} = \mathcal{V}_u \oplus \mathcal{H}_u$$

into vertical and horizontal subspace is $\tilde{\mathbb{H}}_1$ invariant. We usually fix a local horizontal section $u \subset S^{6,5}$ over $\tilde{\mathbb{H}}P^2$ and identify

$$T_{[u]}\tilde{\mathbb{H}}P^2 \cong \mathcal{H}_u \quad \text{by} \quad T_{[u]}\tilde{\mathbb{H}}P^2 \ni X \leftrightarrow \tilde{X} \in \mathcal{H}_u \subset \tilde{\mathbb{H}}^3,$$

where $\tilde{X}$ is the unique horizontal lift of the vector X. Right multiplication of vectors from $\tilde{\mathbb{H}}^3$ by i, j, k preserves the horizontal subspace $\mathcal{H}_u$. The corresponding endomorphisms of the tangent space of the projective plane $\tilde{\mathbb{H}}P^2$ are J_1, J_2 and J_3 respectively. They satisfy the same relations as i, j and k, i.e. if (α, β, γ) is any cyclic permutation of $(1,2,3)$, then we have

$$J_\alpha^2 = -\epsilon_\alpha, \quad J_\alpha J_\beta = -\epsilon_\gamma J_\gamma\,. \tag{1}$$

Although the definition of J_1, J_2 and J_3 depends on the section u over $\tilde{\mathbb{H}}P^2$, the subbundle $\mathbb{R}\langle J_1, J_2, J_3\rangle$ of $End(T\tilde{\mathbb{H}}P^2)$ is globally defined and parallel with respect to the Levi-Civita connection. This is a pseudo-Riemannian analogue of the notion of a quaternionic-Kähler structure on quaternionic projective plane $\mathbb{H}P^2$. The submersion induces a pseudo-Riemannian metric

$$\langle X, Y\rangle = g(\tilde{X}, \tilde{Y})$$

of signature $(4,4)$ on $\tilde{\mathbb{H}}P^2$ from the pseudo-Riemannian metric g on $S^{6,5}$. One can easily check that the endomorphism J_1 is an isometry and J_2, J_3 are anti-isometries.

We denote the curvature tensors of the Levi-Civita connection on $S^{6,5}$ and $\tilde{\mathbb{H}}P^2$ by $\bar{R}$ and $\bar{K}$, respectively. Note that the sectional curvature of the quaternionic and the para-quaternionic projective plane can be expressed as

$$\bar{K}(X, Y) = 1 + 3|Pr_{\langle Y, J_1Y, J_2Y, J_3Y\rangle}X|^2/\epsilon(X), \tag{2}$$

for arbitrary orthonormal tangent vectors X and Y, where $\epsilon(X)$ denotes the sign of square length of the vector X (see [4]).

5 The orbit space $\mathcal{K}/G$ and its metric

Our construction is a modification of quaternionic-Kähler reduction modified to the particular action of $\mathbb{R}$ on the para-quaternionic projective plane. We adopt this approach. First, because the curvature tensor of self-dual manifolds has many properties in common both with the quaternionic and also the para-quaternionic manifolds – see [8, 9] for further details. Second, because the Clifford algebra $C(1,1)$ provides a natural framework for describing metrics of neutral signatures.

The general para-quaternionic-Kähler reduction was introduced and studied by Vukmirović ([21, 22]).

Let p and q be positive coprime distinct integers. We define an isometric action ϕ of the group $G := \{e^{jt} \mid t \in \mathbb{R}\} \cong (\mathbb{R}, +)$ on $\tilde{\mathbb{H}}P^2$ by setting

$$\phi_t(u_0, u_1, u_2) := (e^{jqt}u_0, e^{jpt}u_1, e^{jpt}u_2),$$

where $e^{jt} := \cosh t + j \sinh t$, $t \in \mathbb{R}$. This action preserves the para-quaternionic structure on $\tilde{\mathbb{H}}P^2$. The induced Killing vector field at a point $[u]$ of $\tilde{\mathbb{H}}P^2$ is $V_u := \pi_*(j(qu_0, pu_1, pu_2)) = \Lambda u$, where the skew-symmetric operator $\Lambda : T_{[u]}\tilde{\mathbb{H}}P^2 \to T_{[u]}\tilde{\mathbb{H}}P^2$ is determined by the map $(x_0, x_1, x_2) \mapsto j(qx_0, px_1, px_2)$ where (x_0, x_1, x_2) is the horizontal lift of $X \in T_{[u]}\tilde{\mathbb{H}}P^2$ to $T_uS^{6,5}$. Let $\mathcal{K}_0$ be the algebraic submanifold of $\tilde{\mathbb{H}}P^2$ which is defined by the equation

$$\mathcal{K}_0 := \{[u_0, u_1, u_2] \in \tilde{\mathbb{H}}P^2 \mid q\bar{u}_0ju_0 + p\bar{u}_1ju_1 + p\bar{u}_2ju_2 = 0\}.$$

Then K_0 has real codimension 3 in $\tilde{\mathbb{H}}P^2$ so $dim_{\mathbb{R}}\mathcal{K}_0 = 5$. It is interesting that the action ϕ_t of the group G preserves the algebraic submanifold $\mathcal{K}_0$. It acts freely on $\mathcal{K}_0$. Thus, the vector V_u is tangent to $\mathcal{K}_0$ at the point $[u] \in \mathcal{K}_0$. One can check that the vectors

$$J_1V_u, \;\; J_2V_u, \;\; J_3V_u \in T_{[u]}\tilde{\mathbb{H}}P^2$$

are normal to the submanifold $\mathcal{K}_0$ at the point $[u]$. At points $[u] \in \mathcal{K}_0$ such that $|V_u|^2 \neq 0$, the vectors $V_u, J_1V_u, J_2V_u, J_3V_u$ are linearly independent. Hence, in order to proceed with calculations, we shall restrict to the subset $\mathcal{K}$ of $\mathcal{K}_0$ consisting of such points, i.e. to the set

$$\mathcal{K} := \{[u] = [u_0, u_1, u_2] \in \mathcal{K}_0 \mid n^2 := |V_u|^2 = q^2|u_0|^2 + p^2|u_1|^2 + p^2|u_2|^2 \neq 0\}.$$

We have the metric $\langle\cdot,\cdot\rangle$ induced from $\tilde{\mathbb{H}}P^2$ on $\mathcal{K}$. We shall denote the associated connection, curvature, and sectional curvature by $\overset{\circ}{\nabla}, \overset{\circ}{R}, \overset{\circ}{K}$, respectively.

Now, $\mathcal{K}$ is a differentiable manifold of real dimension 5. The isometric action ϕ_t of the group G preserves $\mathcal{K}$ and it is free on $\mathcal{K}$. The quotient manifold $\mathcal{K}/G$ is a pseudo-Riemannian manifold of real signature $(2,2)$.

Let $g_{p,q}$ be the metric induced on $\mathcal{K}/G$ by the Riemannian submersion ξ from $\mathcal{K}$ to $\mathcal{K}/G$. It may be represented by the following diagram.

$$\begin{array}{ccccccc}
\tilde{\mathbb{H}}_1 = S^{2,1} & \rightarrow & G = \mathbb{R} & \rightarrow & G = \mathbb{R} & \rightarrow & G = \mathbb{R} \\
\downarrow & & \downarrow \phi_{p,q} & & \downarrow \phi_{p,q} & & \downarrow \phi_{p,q} \\
S^{6,5} & \xrightarrow{\pi} & \tilde{\mathbb{H}}P^2 & \supset & \mathcal{K}_0 & \supset & \mathcal{K} = \mathcal{K}_0 \setminus \mathcal{S} \\
\cap & & & & & & \downarrow \xi \\
\tilde{\mathbb{H}}^3 & & & & & & M_{p,q} = \mathcal{K}/G
\end{array}$$

It is not difficult to check that it can be expressed in terms of global coordinates $u \in S^{6,5}$ in the following way

$$(\xi \circ \pi)^*(g_{p,q}) = d\bar{u} \otimes_{\mathbb{R}} du + (d\bar{u} \cdot u) \otimes_{\mathbb{R}} (\bar{u} \cdot du) + \frac{(d\bar{u} \cdot V_u) \otimes_{\mathbb{R}} (\overline{V_u} \cdot du)}{|V_u|^2}.$$

We note that the first two terms determine the metric of the para-quaternionic projective plane. Let $i = 1,2,3$. The endomorphisms J_i of $T_{[u]}\tilde{\mathbb{H}}P^n$ induce corresponding endomorphisms $\tilde{J}_i$ of tangent space of the orbit space $\mathcal{K}/G$ Denote the induced curvature and the sectional curvature of $\mathcal{K}/G$ by R and K, respectively.

6 Curvature and a local dynamics of $\mathcal{O}_{p,q}$

We shall study the curvature tensor and the sectional curvature of the orbit space $\mathcal{K}/G = \mathcal{O}_{p,q}$ in this section. Furthermore, a study of a local geometry and dynamics of these orbit spaces is presented.

The starting point for our computations is O'Neill approach in the the the study of submersions, i.e. the basic relation is

$$\begin{aligned} \overset{\circ}{K}(X,Y) &= \bar{K}(X,Y) - \frac{1}{Q}(|B(X,Y)|^2 - \langle B(X,X), B(Y,Y)\rangle), \\ Q &:= Q(X,Y) = |X|^2|Y|^2 - \langle X,Y\rangle^2, \end{aligned}$$

between second fundamental form of the submersion and the sectional curvature of the submersion (see [16]).

The sectional curvature of $\mathcal{K}/G$ determines its Jacobi operator $\mathcal{K}$ as $\mathcal{K}_{\tilde{X}}(\tilde{Y}) := R(\tilde{X},\tilde{Y})\tilde{X}$ and

$$\begin{aligned} K_{\tilde{X}}(\tilde{Y}) &= h(\bar{R}(X,Y)X) + \tfrac{1}{n}\left[\textstyle\sum_{i=1}^{3} \epsilon_i \left(-\langle X, J_i\Lambda X\rangle h(J_i\Lambda Y)\right.\right. \\ &\quad \left.\left. + +\langle X, J_i\Lambda Y\rangle h(J_i\Lambda X)\right) - 3\langle Y, \Lambda X\rangle h(\Lambda X)\right], \end{aligned}$$

where X, Y are the unique horizontal lifts of the vectors $\tilde{X}, \tilde{Y}$ and $h(X)$ denotes the horizontal part of the vector $X \in T_{[u]}\tilde{\mathbb{H}}P^2$, i.e. the orthogonal projection of the vector X onto $\mathbb{R}\langle V_u, J_1V_u, J_2V_u, J_3V_u\rangle^{\perp}$.

Now, we shall use the curvature of $\mathcal{K}/G$ to describe some of the interesting geometric properties of this space.

First, we recall some definitions. Let R be the curvature tensor of a pseudo-Riemannian manifold (M,g). The Jacobi operator K_X is defined by $K_X : Y \to R(X,Y)X$. Let $S^+(M,g)$ be the bundle of all unit spacelike vectors. The manifold M is said to be pointwise spacelike Osserman if the eigenvalues of the Jacobi operator are constant on $S^+_m(M,g)$ for every point $m \in M$; the eigenvalues being in principle allowed to vary from point to point. The manifold M is said to be (globally) spacelike Osserman if the eigenvalues constant on $S(M,g)$ and does not vary from point to point. The notions pointwise spacelike Jordan Osserman and spacelike Jordan Osserman are defined similarly using the Jordan normal form rather than the eigenvalue structures.

Let us give some physics motivation to explain the importance of the *Jacobi operator*, known also as a *tidal force*. We will consider a free-falling family of particles γ_t which starts from a given point p. These particles determine the geodesic lines and there is a variational vector field V defined along some geodesic γ_0. The vector field V provides an information about the local dynamics in the neighborhood of the point p and satisfies the well known equation

$$V'' = -R(V, \gamma')\gamma' = K_{\gamma'}(V)$$

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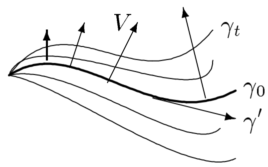

Figure: 4. Tidal force

It is important to notice that the manifold $\mathcal{K}/G$ is pointwise spacelike Jordan Osserman of neutral signature $(--++)$. A verification of this fact is by a direct computation (see [6], Lemma 9.1).

Theorem 6.1 *The manifold $\mathcal{K}/G$ is Einstein, self-dual and not locally homogenous of the signature $(--++)$.*

Proof. In [2], the self-dual, Einstein manifolds of neutral signature are characterized as the pointwise-Osserman manifolds. Thus the previous fact, (see [6], Lemma 9.1) implies that $\mathcal{K}/G$ is Einstein and self-dual. A pointwise-Osserman, locally homogenous manifold is Osserman, i.e. the eigenvalues its Jacobi operator is also independent of the point. But for our examples, we can check that the eigenvalues of the Jacobi operator, λ_1, λ_2 and λ_3 are are not constant on $\mathcal{K}/G$. After some computations, where a few unexpected cancellations happened, we obtain

$$\lambda_1 = \lambda_2 = -\tfrac{2p^4q^2}{n^6} - 4, \quad \lambda_3 = \tfrac{4p^4q^2}{n^6} - 4. \tag{3}$$

Clearly, n^2 is not a constant along $\mathcal{K}/G$ ($\mathcal{K}/G$ is not globally Osserman). Thus, the manifold $\mathcal{K}/G$ is not locally homogeneous, and hence not locally symmetric.

Remark: It is interesting that there exist a three-dimensional submanifold of $\mathcal{K}/G$ along which the eigenvalues of the Jacobi operator are constant.

7 Study of the example of Galicki and Lawson

We return to the starting point for our work, the compact Riemannian orbifolds

$$\mathcal{O}_{q,p} = \mathcal{L}_H/S^1,$$

constructed and studied by Galicki and Lawson [9]. They proved that $\mathcal{O}_{q,p}(n-1)$ is not locally symmetric for $0 < q/p < 1$. For $n = 2$ in [6] another proof of this result based on the study of the curvature operators along the lines performed in Section §5 is given. This approach provides an opportunity to see that $\mathcal{O}_{p,q}(1)$ is not even locally homogeneous for any p, q. The proof is based on the study of curvature of $\mathcal{O}_{p,q}(1)$ and the eigenvalues of its Jacobi operator $\mathcal{K}_X$. These eigenvalues are independent on the choice of the unit direction X but depends on a point u (poinwise Osserman manifolds). It can be also interpreted as follows: the local dynamics around some point u doesn't depend on the choice of direction, i.e. from a dynamical view point space looks the same in all directions from a given point.

The precise bounds of the sectional curvature can be established using the expressions for the sectional curvature. For example, we have a theorem concerning the orbifolds $\mathcal{O}_{p,q}(1)$ with a positive sectional curvature.

Theorem 7.1 ([6]) *If $p^2 < \sqrt{2}q^2 < 2\sqrt{2}p^2$, then the orbifold $\mathcal{O}_{p,q}(1)$ has the positive sectional curvature (Fig. 5).*

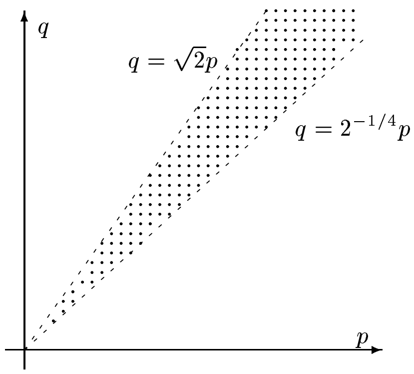

Figure: 5. Orbifolds $\mathcal{O}_{p,q}$ of positive curvature

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