An Application of Continuous Analogue of Newton's Method for Modeling of Boson-Fermion Stars in the Generalized Scalar-Tensor Theories of Gravity

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Abstract

We investigate numerically models of the static spherically symmetric boson-fermion stars in the scalar-tensor theory of gravity with massive dilaton field. The proper mathematical model of such stars is interpreted as a nonlinear two-parametric eigenvalue problem with an unknown internal boundary. We formulate the boundary value problem in two ways and employ the Continuous Analogue of Newton's Method (CANM) for each of them. We obtain the behaviour of the basic geometric quantities and functions describing a dilaton field and matter fields, which build the star.

Keywords: two-parametric nonlinear eigenvalue problem, Continuous Analogue of Newton's Method, mixed fermion-boson stars, scalar-tensor theory of gravity, massive dilaton field

1 Introduction

In the Einstein frame the field equations in presence of fermion and boson matter are:

$$
\begin{aligned}
G_\mu^\nu = \kappa_* \left(\overset{B}{T}{}_\mu^\nu + \overset{F}{T}{}_\mu^\nu\right) + 2\partial_\mu\varphi\partial^\nu\varphi - \partial^\sigma\varphi\partial_\sigma\varphi\delta_\mu^\nu + \tfrac{1}{2}U(\varphi)\delta_\mu^\nu \\
\nabla_\mu\nabla^\mu\varphi + \tfrac{1}{4}U'(\varphi) = -\frac{\kappa_*}{2}\alpha(\varphi)\left(\overset{B}{T} + \overset{F}{T}\right) \\
\nabla_\mu\nabla^\mu\Psi + 2\alpha(\varphi)\partial^\sigma\varphi\partial_\sigma\Psi = -2A^2(\varphi)\frac{\partial\tilde{W}(\Psi^+\Psi)}{\partial\Psi^+} \\
\nabla_\mu\nabla^\mu\Psi^+ + 2\alpha(\varphi)\partial^\sigma\varphi\partial_\sigma\Psi^+ = -2A^2(\varphi)\frac{\partial\tilde{W}(\Psi^+\Psi)}{\partial\Psi}
\end{aligned}
\tag{1}
$$

where ∇_μ is the Levi-Civita connection with respect to the metric $g_{\mu\nu}$, $(\mu = 0, ..., 3; \nu = 0, ..., 3)$. The constant κ_* is given by $\kappa_* = 8\pi G_*$, where G_* is the bare Newtonian gravitational constant. The physical gravitational "constant" is $G_* A^2(\varphi)$ where $A(\varphi)$ is a function of the dilaton field φ depending on the concrete scalar-tensor theory of gravity.

The function $\alpha(\varphi) = \dfrac{d}{d\varphi}[\ln A(\varphi)]$ determines the strength of the coupling between the dilaton filed φ and the matter. The functions $\overset{B}{T}$ and $\overset{F}{T}$ are correspondingly the trace of the energy-momentum tensor of the fermionic matter $\overset{F}{T}{}_\mu^\nu$ and bosonic matter $\overset{B}{T}{}_\mu^\nu$. Here Ψ is a complex scalar field describing the bosonic matter while Ψ^+ is its complex conjugated function and $\tilde{W}(\Psi^+\Psi)$ is the potential for the boson field (for detail see [1]).

We will consider a static and spherically symmetric mixed boson-fermion star in asymptotic flat space-time.

2 First Formulation of the Problem

Taking into account the assumption that have been made the system of the field equations (1) is reduced to the following system of dimensionless ordinary differential equations (ODEs):

$$
\begin{aligned}
\frac{d\lambda}{dr} &= F_1 \equiv \frac{1-\exp(\lambda)}{r} + r\left\{\exp(\lambda)\left[\overset{F}{T}{}_0^0 + \overset{B}{T}{}_0^0 + \frac{1}{2}\gamma^2 V(\varphi)\right] + \left(\frac{d\varphi}{dr}\right)^2\right\}, \\
\frac{d\nu}{dr} &= F_2 \equiv -\frac{1-\exp(\lambda)}{r} - r\left\{\exp(\lambda)\left[\overset{F}{T}{}_1^1 + \overset{B}{T}{}_1^1 + \frac{1}{2}\gamma^2 V(\varphi)\right] - \left(\frac{d\varphi}{dr}\right)^2\right\}, \\
\frac{d^2\varphi}{dr^2} &= F_3 \equiv -\frac{2}{r}\frac{d\varphi}{dr} + \frac{1}{2}\left(F_1 - F_2\right)\frac{d\varphi}{dr} \\
&\quad + \frac{1}{2}\exp(\lambda)\left[\alpha(\varphi)\,(\overset{F}{T} + \overset{B}{T}) + \frac{1}{2}\gamma^2 V'(\varphi)\right], \qquad (2) \\
\frac{d^2\sigma}{dr^2} &= F_4 \equiv -\frac{2}{r}\frac{d\sigma}{dr} + \left[\frac{1}{2}\left(F_1 - F_2\right) - 2\alpha(\varphi)\frac{d\varphi}{dr}\right]\frac{d\sigma}{dr} \\
&\quad - \sigma\exp(\lambda)\left[\Omega^2\exp(-\nu) + 4\sigma A^2(\varphi)W'(\sigma)\right], \\
\frac{d\mu}{dr} &= F_5 \equiv -\frac{g(\mu)+f(\mu)}{f'(\mu)}\left[\frac{1}{2}F_2 + \alpha(\varphi)\frac{d\varphi}{dr}\right].
\end{aligned}
$$

Here $\lambda(r), \nu(r), \varphi(r), \sigma(r)$ and $\mu(r)$ are unknown functions of r and Ω is an unknown parameter. Having in mind the physical assumptions, we have to solve the equations (2) under the following boundary conditions (BCs):

$$\lambda(0) = \frac{d\varphi}{dr}(0) = \frac{d\sigma}{dr}(0) = 0, \quad \sigma(0) = \sigma_c, \quad \mu(0) = \mu_c, \quad \mu(R_s) = 0 \qquad (3)$$

where σ_c and μ_c are the values of density of, respectively, the bosonic and fermionic matter at the star's center. The first three conditions in (3) guarantee the nonsingularity of the metrics and the functions $\lambda(r)$, $\varphi(r)$, $\sigma(r)$ at the star's center.

As it is required by the asymptotic flatness of space-time, BCs at the infinity are:

$$\nu(\infty) = 0, \quad \varphi(\infty) = 0, \quad \sigma(\infty) = 0 \qquad (4)$$

where $(\cdot)(\infty) = \lim_{r\to\infty}(\cdot)(r)$.

Here functions $\overset{F}{T}{}_0^0$, $\overset{F}{T}{}_1^1$ and $\overset{B}{T}{}_0^0$, $\overset{B}{T}{}_1^1$ depend on the components of the energy-momentum tensor of the fermionic and bosonic matter, the functions $\overset{B}{T}$ and $\overset{F}{T}$ represent respectively the trace of the energy-momentum tensor of fermionic and bosonic matter, functions $f(\mu)$, $g(\mu)$ and $W(\sigma^2)$ are given.

Let us note that for arbitrary functions $f(\mu)$, $g(\mu)$ and $\alpha(\varphi)$ the last equation in (2) has a first integral, which can be presented as:

$$\int_{\mu_1}^{\mu_2} \frac{f'(\mu)}{f(\mu)+g(\mu)}\, d\mu + \frac{1}{2}(\nu_2 - \nu_1) + \ln\frac{A(\varphi_2)}{A(\varphi_1)} = 0 \tag{5}$$

where ν_1, ν_2, φ_1, φ_2, μ_1, μ_2 stand for the functions $\nu(r)$, $\varphi(r)$, $\mu(r)$ at points r_1 and r_2 $(0 \leq r_1 \leq r_2 \leq R_s)$.

Apart from the unknown functions $\nu(r)$, $\sigma(r)$, $\varphi(r)$ and $\mu(r)$ the equations (2) include also two unknown parameters R_s and Ω. For their computation we require two conditions among BCs (3),(4), for example

$$\sigma(0) = \sigma_c, \qquad \mu(R_s) = 0, \quad 0 < R_s < \infty, \tag{6}$$

e.g., the problem (2)-(4) can be considered as a nonlinear eigenvalue problem where R_s and Ω are eigenvalues.

3 Second Formulation of the Problem

With all definitions we have given, the main system of differential equations (2) governing the structure of static and spherically symmetric boson-fermion stars can be divided in two parts: internal fermionic one $(r < R_s)$ and external domain $(r > R_s)$ where is no fermionic matter, *i.e.*, it can be formally supposed that the function $\mu(r) \equiv 0$ there. The fermionic part of the energy-momentum tensor also vanishes identically and, thus, the differential equations with respect to the rest four unknown functions $\lambda(r)$, $\nu(r)$, $\varphi(r)$, and $\sigma(r)$ in the external part are the same as in the internal part and the last equation in (2) can be omitted.

We seek for a solution $[\lambda(r), \nu(r), \varphi(r), \sigma(r), \mu(r), R_s, \Omega]$ subjected to the nonlinear ODEs (2), satisfying the BCs (3) in fermionic part $(0 < r < R_s)$ and for a solution $[\lambda(r), \nu(r), \varphi(r), \sigma(r), R_s, \Omega]$ satisfying the BCs (4) in bosonic part $(r > R_s)$. The so-posed two BVPs are two-parametric eigenvalue problems with respect to the quantities R_s and Ω. After resolving them we match the two solutions at the internal boundary R_s - the fermionic boundary of the star. At that we assume the function $\mu(r)$ is continuous in the interval $[0, R_s]$, while the functions $\lambda(r)$, $\nu(r)$ are continuous and the functions $\varphi(r)$, $\sigma(r)$ are smooth in the whole interval $[0, \infty)$, including the unknown internal boundary $r = R_s$.

4 Method of Solution

4.1 First formulation of the problem

For solving the above posed nonlinear eigenvalue problem CANM is used. The presence of the *a priori* unknown quantity R_s, however, is an obstacle for direct treatment by CAMN - the problem has an unknown internal boundary R_s. In order to overcome it we introduce a new scaled coordinate $x = r/R_s$ [6]. In this way the physical domain $r \in [0, \infty]$ renders to the domain $x \in [0, \infty)$ and the point $r = R_s$ maps into point $x = 1$.

For the sake of convenience we eliminate the function $\lambda(r)$ from the first equation in (2), transform equivalently the second one into a second order ODE (see [1]) and introduce the vector $\mathbf{y}(x) = \{\nu(x), \varphi(x), \sigma(x)\}$. Then the three equations (2) (without the first one and the fourth one) can be written as follows:

$$\mathbf{f}(\mathbf{y}, \mu, R_s, \Omega) \equiv -x\mathbf{y}'' - \mathbf{y}' + \mathbf{F} = 0 \tag{7}$$

where $\mathbf{F} = \mathbf{F}(x, \mathbf{y}, \mathbf{y}', \mu, R_s, \Omega)$ is 3D vector consisting of the right hand sides of the equations (2) and by $(.)'$ is denoted the differentiation with respect to the new independent variable x.

Following CANM [3], [4], [5] we introduce a "time"-like parameter $t \in [0, \infty)$ and assume the unknown quantities depend on t as well: $\mathbf{y} = \mathbf{y}(x, t)$, $R_s = R_s(t)$, $\Omega = \Omega(t)$. If we suppose that the function $\mu = \mu(x)$ is known, then after manipulations obtain the following linearized system of ODEs:

$$\begin{aligned}
-x\mathbf{u}'' - \mathbf{u}' + \frac{\partial \mathbf{F}}{\partial \mathbf{y}}\mathbf{u} + \frac{\partial \mathbf{F}}{\partial \mathbf{y}'}\mathbf{u}' &= x\mathbf{y}'' + \mathbf{y}' - \mathbf{F} \\
-x\mathbf{v}'' - \mathbf{v}' + \frac{\partial \mathbf{F}}{\partial \mathbf{y}}\mathbf{v} + \frac{\partial \mathbf{F}}{\partial \mathbf{y}'}\mathbf{v}' &= -\left(\frac{2}{R_s}\mathbf{F} + \frac{\partial \mathbf{F}}{\partial R_s}\right) \\
-x\mathbf{w}'' - \mathbf{w}' + \frac{\partial \mathbf{F}}{\partial \mathbf{y}}\mathbf{w} + \frac{\partial \mathbf{F}}{\partial \mathbf{y}'}\mathbf{w}' &= -\frac{\partial \mathbf{F}}{\partial \Omega}.
\end{aligned} \tag{8}$$

Here $\dfrac{\partial \mathbf{F}}{\partial (.)}$ are the respective Frechét derivatives at the point $(\mathbf{y}, R_s, \Omega)$, $\mathbf{u}(x), \mathbf{v}(x)$ and $\mathbf{w}(x)$ are unknown 3D vector-functions of x. On its turn

$$\mathbf{z} = \mathbf{u} + \rho\mathbf{v} + \omega\mathbf{w}, \quad \dot{\mathbf{y}} = \mathbf{z}\,, \quad \dot{R}_s = \rho\,, \quad \dot{\Omega} = \omega. \tag{9}$$

The above three equations are coupled with the following six BCs:

$$\begin{aligned}
\mathbf{u}'(0) = -\mathbf{y}'(0), \quad \mathbf{v}'(0) = \mathbf{w}'(0) = 0 \\
\mathbf{u}(\infty) = -\mathbf{y}(\infty), \quad \mathbf{v}(\infty) = \mathbf{w}(\infty) = 0,
\end{aligned} \tag{10}$$

which are obtained from the original BCs (3)-(4) applying CAMN over them. Let us emphasize that above equations have an equivalent structure of the left hand sides, which essentially eases their numerical treatment.

In order to calculate the derivatives $\dot{R}_s = \rho$ and $\dot{\Omega} = \omega$ we use the additional BCs (6). Applying CAMN both to them and Eq.(5) for $x_1 = 0$ (the center of the star) and $x_2 = 1$ (the radius of the star), we obtain simply following linear system of algebraic equations:

$$\begin{aligned} a_1\rho + b_1\omega = c_1 \\ a_2\rho + b_2\omega = c_2 \end{aligned} \tag{11}$$

with respect to the unknown derivatives ρ and ω (comprehensive formulae one can see in [1]).

4.2 Second formulation of the problem

For our further considerations, it is convenient to present the system (2) in the following equivalent forms as systems of first order ODEs:

$$-\mathbf{y}_i' + R_s\mathbf{F}_i(R_s x, \mathbf{y}_i, \Omega) = 0 \quad \text{and} \quad -\mathbf{y}_e' + R_s\mathbf{F}_e(R_s x, \mathbf{y}_e, \Omega) = 0 \tag{12}$$

with respect to the unknown vector functions

$$\begin{aligned} \mathbf{y}_i(x) \equiv& (\lambda(x), \nu(x), \varphi(x), \xi(x), \sigma(x), \eta(x), \mu(x))^T, \\ \mathbf{y}_e(x) \equiv& (\lambda(x), \nu(x), \varphi(x), \xi(x), \sigma(x), \eta(x))^T \end{aligned}$$

and right hand sides $\mathbf{F}_i \equiv (F_1, F_2, \xi, F_3, \eta, F_4, F_5)^T$, $\mathbf{F}_e \equiv (F_1, F_2, \xi, F_3, \eta, F_4)^T$, where $(.)'$ stands for differentiation towards the new variable x, $(.)_i$ and $(.)_e$ denote the inner (inside the star) and outer (outside the star) functions.

For given values of the parameters R_s and Ω, the independent solving of the inner system in (12) requires seven BCs. At the same time we have at disposal only six conditions of the kind (3). In order to complete the problem, we set additionally one more parametric condition (the value of someone from among the functions $\lambda(x), \nu(x), \varphi(x), \xi(x), \sigma(x)$ or $\eta(x)$) at the point $x = 1$). Let us set for example:

$$\varphi_i(1) = \varphi_s, \quad \varphi_s - \text{parameter}. \tag{13}$$

In the external domain $x \geq 1$ the vector of solutions $\mathbf{y}_e(x)$ is 6D. Thereupon, six BCs are indispensable for solving the equation (12). At the same time only the three BCs (4) are known. Let us consider that the solution $\mathbf{y}_i(x)$ in the internal domain $x \in [0, 1]$ is known. Then, we postulate the rest three deficient conditions to be the continuity conditions at the point $x = 1$. The first of them is similar to the condition (13) and the other two we assign to two arbitrary functions from among $\lambda(x)$, $\nu(x)$, $\xi(x)$, $\sigma(x)$, and $\eta(x)$, for example $\lambda_e(1) = \lambda_i(1)$, $\varphi_e(1) = \varphi_s$, $\sigma_e(1) = \sigma_i(1)$. Let the solutions $\mathbf{y}_i = \mathbf{y}_i(x, \Omega, R_s, \varphi_s)$ and $\mathbf{y}_e = \mathbf{y}_e(x, \Omega, R_s, \varphi_s)$ be supposed known. Generally speaking, for given arbitrary values of the parameters R_s, Ω, and φ_s the continuity conditions with respect to the functions $\nu(x), \xi(x)$, and $\eta(x)$ at the point $x = 1$ are not satisfied. We choose the parameters R_s, Ω, and φ_s in such manner that the continuity conditions for the functions $\nu(x)$, $\xi(x)$, and $\eta(x)$ are held, *i.e.*,

$$\begin{aligned} \nu_e(1, R_s, \Omega, \varphi_s) - \nu_i(1, R_s, \Omega, \varphi_s) &= 0, \\ \xi_e(1, R_s, \Omega, \varphi_s) - \xi_i(1, R_s, \Omega, \varphi_s) &= 0, \\ \eta_e(1, R_s, \Omega, \varphi_s) - \eta_i(1, R_s, \Omega, \varphi_s) &= 0. \end{aligned} \tag{14}$$

These conditions should be interpreted as three nonlinear algebraic equations in regard to the unknown quantities R_s, Ω, and φ_s. The usual way for solving the above-mentioned kind of equations (14) is by means of various iteration methods, for example Newton's methods.

In the present work, using CANM we propose a common treatment of both, differential and algebraic problems.

After similar manipulations to those in Section 4.1 we obtain the following linearized systems of ODEs both in inner (fermionic) and outer (bosonic) part of the star

$$\begin{aligned} -\mathbf{s}' + R_s \frac{\partial \mathbf{F}}{\partial \mathbf{y}} \mathbf{s} &= \mathbf{y}'(x) - R_s \mathbf{F}, \\ -\mathbf{u}' + R_s \frac{\partial \mathbf{F}}{\partial \mathbf{y}} \mathbf{u} &= -\left(\mathbf{F} + R_s \frac{\partial \mathbf{F}}{\partial R_s} \right), \\ -\mathbf{v}' + R_s \frac{\partial \mathbf{F}}{\partial \mathbf{y}} \mathbf{v} &= -R_s \frac{\partial \mathbf{F}}{\partial \Omega}, \\ -\mathbf{w}' + R_s \frac{\partial \mathbf{F}}{\partial \mathbf{y}} \mathbf{w} &= 0 \end{aligned} \tag{15}$$

where $\dfrac{\partial \mathbf{F}}{\partial \mathbf{y}}$ stands for a square matrix (7×7 or 6×6), which consists of the Frechét derivatives of operator $\mathbf{F}$ at the point $\{\mathbf{y}(x), R_s, \Omega\}$ and $\mathbf{u}(x)$, $\mathbf{v}(x)$, $\mathbf{w}(x)$ and $\mathbf{s}(x)$ are new unknown vector functions.

Then applying CAMN to BCs (3) and (4) we obtain eight BCs similar to (10) (4 left + 4 right) for each (inner and outer) BVP (for detail see [2]).

In the end, to compute the increments $\{\rho, \omega, \phi\}$ of parameters R_s, Ω, and φ_s we use the three nonlinear conditions (14). Applying CANM to them we attain an algebraic system consisting of three linear scalar equations with respect to the three unknowns ρ, ω, and ϕ.

Taking into account the smoothness of sought solutions, we solve the linear BVPs (8) and (15) employing spline collocation scheme of fourth order of approximation (see for example [7]). At that, we utilize essentially the important feature that each of the above-mentioned two groups vector BVPs (internal and external) has one and the same left-hand side.

5 General Sequence of the Algorithm

The general sequence of the algorithm in the both cases of implementation of CANM is described and discussed in detail in our previous works [1] and [2]. Therefore it will be not presented here again.

6 Some numerical results

For the purpose of illustrating we will present and shortly discuss some results obtained from numerical experiments. Below we consider concrete scalar-tensor model with functions:

$$A(\varphi) = \exp\left(\frac{\varphi}{\sqrt{3}}\right), \quad V(\varphi) = \frac{3}{2}[1 - A^2(\varphi)]^2, \quad W(\sigma) = -\frac{1}{2}\left(\sigma^2 + \frac{1}{2}\Lambda\sigma^4\right),$$

$$f(\mu) = \frac{1}{8}\left[(2\mu - 3)\sqrt{\mu + \mu^2} + 3\ln\left(\sqrt{\mu} + \sqrt{1+\mu}\right)\right],$$

$$g(\mu) = \frac{1}{8}\left[(6\mu + 3)\sqrt{\mu + \mu^2} - 3\ln\left(\sqrt{\mu} + \sqrt{1+\mu}\right)\right].$$

The quantity Λ is given parameter. For completeness, we note that in the concrete case functions $f(\mu)$ and $g(\mu)$ represent the equation of state of noninteracting neutron gas in parametric form, while function $W(\sigma)$ describes the boson field with quartic self-interaction.

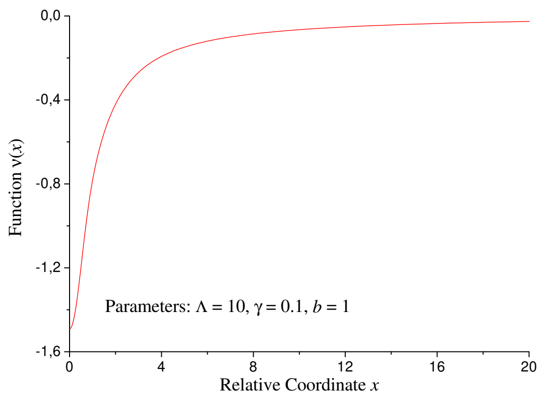

Fig. 1. The function $\nu(x)$.

The calculated eigenfunctions $\nu(x)$, $\varphi(x)$, $\sigma(x)$,and $\mu(x)$ are plotted correspondingly in Figures 1-4 for the values of the parameters $\gamma = 0.1$, $\Lambda = 10$, $b = 1$, $\sigma_c = 0.4$ and $\mu_c = 1.2$. Let us emphasize that the behaviour of the mentioned functions is typical for a wider range of the parameters not only for those values presented in the figures. The function $\nu(x)$ has the largest derivative for $x \in (0, 9)$. After that it approaches slowly zero at infinity like $1/x$, *i.e.*, the asymptotical behaviour of calculated grid function and its derivative agrees very well with the theoretical prediction (see [1]).

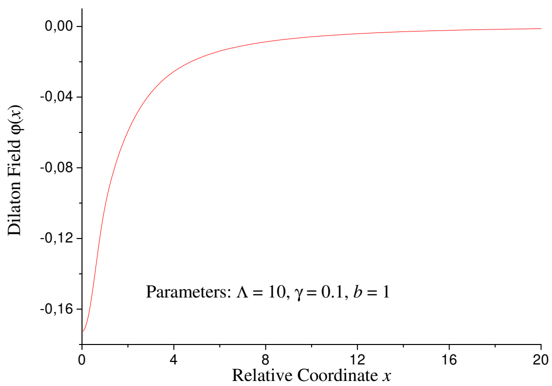

Fig. 2. The function $\varphi(x)$.

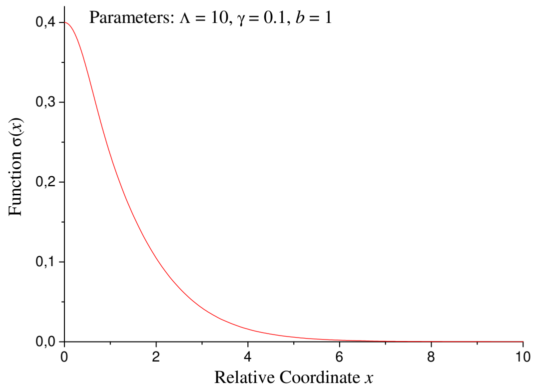

Fig. 3. The function $\sigma(x)$.

Similarly the function $\varphi(x)$ increases rapidly for $x < 4$; besides that it trends asymptotically to zero. Obviously, the quantitative behaviour of $\varphi(x)$ for central value $\sigma_c = 0.4$ is determined by the dominance of the term $\overset{B}{T}$ over the term $\overset{F}{T}$ (see [1]). The function $\sigma(x)$ decreases rapidly from its central value $\sigma_c = 0.4$ (in the case under consideration) to zero. At last the function $\mu(x)$ is nontrivial in the internal domain $x \in [0, 1]$, *i.e.*, inside the star. Here, it varies monotonously and continuously from its central value (in the case under consideration) $\mu_c = 1.2$ until zero at $x = 1$, corresponding to the radius of the star.

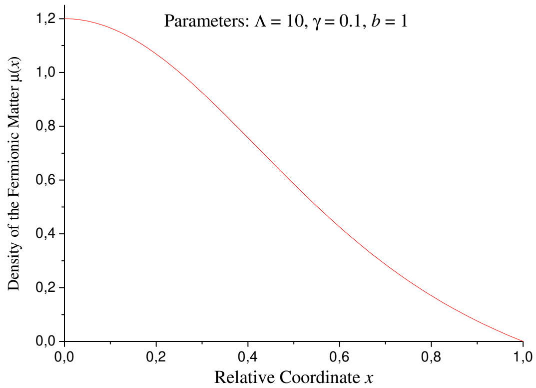

Fig. 4. The function $\mu(x)$.

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