On the Octonionic M-algebra and superconformal M-algebra.

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Abstract

It is shown that the M-algebra related with the M theory comes in two variants. Besides the standard M algebra based on the real structure, an alternative octonionic formulation can be consistently introduced. This second variant has striking features. It involves only 52 real bosonic generators instead of 528 of the standard M algebra and moreover presents a novel and surprising feature, its octonionic $M5$ (super-5-brane) sector is no longer independent, but coincides with the octonionic $M1$ and $M2$ sectors. This is in consequence of the non-associativity of the octonions. An octonionic version of the superconformal M-algebra also exists. It is given by $OSp(1, 8|\mathbf{O})$ and admits 239 bosonic and 64 fermionic generators. It is speculated that the octonionic M-algebra can be related to the exceptional Lie and Jordan algebras that apparently play a special role in the Theory Of Everything.

1 Introduction

The generalized supersymmetries going beyond the standard HŁS scheme [1] admit the presence of bosonic abelian tensorial central charges associated with the dynamics of extended objects (branes). It is widely known since the work of [2] that supersymmetries are related to division algebras.

Indeed, even for the generalized supersymmetries, classification schemes based on the associative division algebras (**R**, **C**, **H**) are now available, see [3]. For what concerns the remaining division algebra, the octonions, much less is known due to the complications arising from their non-associativity. Octonionic structures were, nevertheless, investigated in [4, 5] in application to superstring theory.

Octonions are not just a curiosity. They are the maximal division algebra. This fact alone already justifies that they should receive the same kind of attention paid to, let's say, the maximal supergravity. However, their importance is more than that, they are at the very heart of many exceptional structures in mathematics and can be held responsible for their existence. Among these exceptional structures we can cite the 5 exceptional Lie algebras and the exceptional Jordan algebras. Indeed, the G_2 Lie algebra is the automorphism group of the octonions, while F_4 is the automorphism group of the 3×3 octonionic-valued hermitian matrices realizing the exceptional $J_3(\mathbf{O})$ Jordan algebra. F_4 and the remaining exceptional Lie algebras E_6, E_7, E_8 are recovered from the so-called "magic square Tits' construction" which associates a Lie algebra to any given pair of division algebras, if at least one of these algebras coincides with the octonionic algebra [6].

It has been pointed out several times, [7, 8] that the exceptional Lie algebras fit well into the grand-unification scenario. Moreover, the E_8 Lie algebra enters, through the $E_8 \times E_8$ tensor product, the anomaly-free heterotic string, while the G_2 holonomy of seven-dimensional manifolds is required, on phenomenological basis, to produce 4-dimensional $N = 1$ supersymmetric field theories by compactification of the eleven dimensions. This partial list of scattered pieces of evidence has brought to suggest, see e.g. [8], that for some deep reasons, Nature seems to prefer exceptional structures. In this context it deserves to be mentioned the special role of the exceptional Jordan algebra $J_3(\mathbf{O})$, not only associated to the unique consistent quantum mechanical system (in the Jordan framework, see [9]) based on a non-associative algebra, but also leading to a unique matrix Chern-Simon theory of Jordan type, see [10].

In this talk I will discuss the investigations presented in [11, 12] concerning the possibility of realizing general supersymmetries in terms of the non-associative division algebra of the octonions. In particular in [11] it was shown that the M algebra which supposedly underlines the M-theory comes in two (and only two, due to the absence of the complex and of the quaternionic structures) variants. Besides the standard realization of the M-algebra which involves real spinors and makes therefore use of the real structure, an alternative formulation, requiring the introduction of the octonionic structure, is also possible and can be exploited. This is made possible due to the existence of an octonionic description for the Clifford algebra defining the 11-dimensional Minkowskian spacetime and its related spinors. The features of this second variant, the octonionic M-superalgebra, are puzzling. While it is not at all surprising that it contains fewer bosonic generators, 52, w.r.t. the 528 of the standard M-algebra (this is after all expected, since the imposition of an extra structure always puts a constraint on a theory), what really came as an unexpected surprise is the fact that new conditions, not present in the standard M-theory, are now found. These conditions imply that the different brane-sectors are no longer independent. The octonionic 5-brane alone contains the whole set of degrees of freedom and is therefore equivalent to the octonionic $M1$ and $M2$ sectors. We can write this equivalence, symbolically, as $M5 \equiv M1 + M2$. This result is indeed very intriguing. It implies that quite non-trivial structures are found when investigating the octonionic construction of the M-theory. It is quite tempting to think that the exceptional structures that we mentioned before should be better understood from this octonionic variant of the M-algebra, rather than the standard real M-algebra.

The next passage consists in defining the closed algebraic structure which realizes the octonionic superconformal M-algebra. It turns out that the $OSp(1,64)$ superconformal algebra of the real M-theory is replaced in the octonionic case by the $OSp(1,8|\mathbf{O})$ superalgebra of supermatrices with octonionic-valued entries and total number of $7 + 232 = 239$ bosonic generators.

2 Octonionic Clifford algebras and spinors.

In the $D = 11$ Minkowskian spacetime, where the M-theory should be found, the spinors are real and have 32 components. Since the most general symmetric 32×32 matrix admits 528 components, one can easily prove that the most general supersymmetry algebra in $D = 11$ can be presented as

$$\{Q_a, Q_b\} = (C\Gamma_\mu)_{ab} P^\mu + (C\Gamma_{[\mu\nu]})_{ab} Z^{[\mu\nu]} + (C\Gamma_{[\mu_1\ldots\mu_5]})_{ab} Z^{[\mu_1\ldots\mu_5]} \qquad (1)$$

(where C is the charge conjugation matrix), while $Z^{[\mu\nu]}$ and $Z^{[\mu_1\ldots\mu_5]}$ are totally antisymmetric tensorial central charges, of rank 2 and 5 respectively, which correspond to extended objects [13, 14], the p-branes. Please notice that the total number of 528 is obtained in the r.h.s as the sum of the three distinct sectors, i.e.

$$528 \quad = \quad 11 + 66 + 462. \qquad (2)$$

The algebra (1) is called the M-algebra. It provides the generalization of the ordinary supersymmetry algebra, recovered by setting $Z^{[\mu\nu]} \equiv Z^{[\mu_1\ldots\mu_5]} \equiv 0$.

In the next section we will prove the existence of an octonionic version of Eq. (1). For this purpose we need at first to introduce the octonionic realizations of Clifford algebras and spinors. They exist only in a restricted class of spacetime signatures which includes the Minkowskian $(10,1)$ spacetime.

The most convenient way to construct realizations of Clifford algebras is to iteratively derive them with the help of the following algorithm, allowing the recursive construction of $D+2$ spacetime dimensional Clifford algebras by assuming known a D dimensional representation. Indeed, it is a simple exercise to verify that if γ_i's denotes the d-dimensional Gamma matrices of a $D=p+q$ spacetime with (p,q) signature (namely, providing a representation for the $C(p,q)$ Clifford algebra) then $2d$-dimensional $D+2$ Gamma matrices (denoted as Γ_j) of a $D+2$ spacetime are produced according to either

$$\begin{aligned} \Gamma_j &\equiv \begin{pmatrix} 0 & \gamma_i \\ \gamma_i & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & \mathbf{1}_d \\ -\mathbf{1}_d & 0 \end{pmatrix}, \quad \begin{pmatrix} \mathbf{1}_d & 0 \\ 0 & -\mathbf{1}_d \end{pmatrix} \\ (p,q) &\mapsto (p+1, q+1). \end{aligned} \tag{3}$$

or

$$\begin{aligned} \Gamma_j &\equiv \begin{pmatrix} 0 & \gamma_i \\ -\gamma_i & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & \mathbf{1}_d \\ \mathbf{1}_d & 0 \end{pmatrix}, \quad \begin{pmatrix} \mathbf{1}_d & 0 \\ 0 & -\mathbf{1}_d \end{pmatrix} \\ (p,q) &\mapsto (q+2, p). \end{aligned} \tag{4}$$

Some remarks are in order. The two-dimensional real-valued Pauli matrices τ_A, τ_1, τ_2 which realize the Clifford algebra $C(2,1)$ are obtained by applying either (3) or (4) to the number 1, i.e. the one-dimensional realization of $C(1,0)$. We have indeed

$$\tau_A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{5}$$

The above algorithms can be applied to "lift" the Clifford algebra $C(0,7)$, furnishing higher-dimensional Clifford algebras. $C(10,1)$ is constructed by successively applying (no matter in which order) (3) and (4) to $C(0,7)$. For what concerns $C(0,7)$, it must be previously known. Two inequivalent realizations of $C(0,7)$ can be constructed. The first one is associative and admits a matrix realization. Without loss of generality (the associative irreducible representation of $C(0,7)$ is unique) we can choose expressing it through

$$C(0,7) \quad \equiv \quad \begin{array}{l} \tau_A \otimes \tau_1 \otimes \mathbf{1}_2, \\ \tau_A \otimes \tau_2 \otimes \mathbf{1}_2, \\ \mathbf{1}_2 \otimes \tau_A \otimes \tau_1, \\ \mathbf{1}_2 \otimes \tau_A \otimes \tau_2, \\ \tau_1 \otimes \mathbf{1}_2 \otimes \tau_A, \\ \tau_2 \otimes \mathbf{1}_2 \otimes \tau_A, \\ \tau_A \otimes \tau_A \otimes \tau_A. \end{array} \tag{6}$$

On the other hand another, inequivalent, realization is at disposal. It is based on the identification of the $C(0,7)$ Clifford algebra generators with the seven imaginary octonions τ_i satisfying the algebraic relation

$$\tau_i \cdot \tau_j \quad = \quad -\delta_{ij} + C_{ijk}\tau_k, \tag{7}$$

for $i,j,k = 1,\cdots,7$ and C_{ijk} the totally antisymmetric octonionic structure constants given by

$$C_{123} = C_{147} = C_{165} = C_{246} = C_{257} = C_{354} = C_{367} = 1 \tag{8}$$

and vanishing otherwise. This octonionic realization of the seven-dimensional Euclidean Clifford algebra will be denoted as $C_{\mathbf{O}}(0,7)$. Similarly, the octonionic realization $C_{\mathbf{O}}(10,1)$, obtained through the lifting procedure, is realized in terms of 4×4 matrices with octonionic entries.

One should be aware of the properties of the non-associative realizations of Clifford algebras. In the octonionic case the commutators $\Sigma_{\mu\nu} = [\Gamma_\mu, \Gamma_\nu]$ are no longer the generators of the Lorentz group. They correspond instead to the generators of the coset $SO(p,q)/G_2$, being G_2 the 14-dimensional exceptional Lie algebra of automorphisms of the octonions. As an example, in the Euclidean 7-dimensional case, these commutators give rise to $7 = 21 - 14$ generators, isomorphic to the imaginary octonions. Indeed

$$[\tau_i, \tau_j] \quad = \quad 2C_{ijk}\tau_k. \tag{9}$$

The algebra (9) is not a Lie algebra, but a Malcev algebra (due to the alternativity property satisfied by the octonions, a weaker condition w.r.t. associativity, see [15]). It can be regarded [16, 17] as the "quasi" Lorentz algebra of homogeneous transformations acting on the seven sphere S^7. This is so because S^7 is a parallelizable manifold with a quasi (due to the lack of associativity) group structure which can be identified with the unit octonions

$$X^\dagger \cdot X = 1. \tag{10}$$

Here $X^\dagger$ denotes the principal conjugation for the octonions, namely

$$\begin{aligned} X &= x_0 + x_i\tau_i, \\ X^\dagger &= x_0 - x_i\tau_i. \end{aligned} \tag{11}$$

On the seven sphere, infinitesimal homogeneous transformations which play the role of the Lorentz algebra can be introduced through

$$\delta X = a \cdot X, \tag{12}$$

with a an infinitesimal constant octonion. The requirement of preserving the unitary norm (10) implies the vanishing of the a_0 component, so that $a \equiv a_i\tau_i$.

3 The octonionic M-superalgebra

The octonionic M-superalgebra is introduced by assuming an octonionic structure for the spinors which, in the $D = 11$ Minkowskian spacetime, are octonionic-valued 4-component vectors. The algebra replacing (1) is given by

$$\{Q_a, Q_b\} = \{{Q^*}_a, {Q^*}_b\} = 0, \qquad \{Q_a, {Q^*}_b\} = Z_{ab}, \tag{13}$$

where $*$ denotes the principal conjugation in the octonionic division algebra and, as a result, the bosonic abelian algebra on the r.h.s. is constrained to be hermitian

$$Z_{ab} = {Z_{ba}}^*, \tag{14}$$

leaving only 52 independent components.

The Z_{ab} matrix can be represented either as the $11 + 41$ bosonic generators entering

$$Z_{ab} = P^\mu (C\Gamma_\mu)_{ab} + Z_{\mathbf{O}}^{\mu\nu} (C\Gamma_{\mu\nu})_{ab}, \tag{15}$$

or as the 52 bosonic generators entering

$$Z_{ab} = Z_{\mathbf{O}}^{[\mu_1\ldots\mu_5]} (C\Gamma_{\mu_1\ldots\mu_5})_{ab}\,. \tag{16}$$

Due to the non-associativity of the octonions, unlike the real case, the sectors individuated by (15) and (16) are not independent. Furthermore, as we have already seen for $k = 2$, in the antisymmetric products of k octonionic-valued matrices, a certain number of them are redundant (for $k = 2$, due to the G_2 automorphisms, 14 such products have to be erased). In the general case [18] a table can be produced expressing the number of independent components in D odd-dimensional spacetime octonionic realizations of Clifford algebras, by taking into account that out of the D Gamma matrices, 7 of them are octonionic-valued, while the remaining $D-7$ are purely real. We get the following table, with the columns labeled by k, the number of antisymmetrized Gamma matrices and the rows by D (up to $D = 13$)

$D \setminus k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13
7	1	7	7	1	1	7	7	1						
9	1	9	22	22	10	10	22	22	9	1				
11	1	11	41	75	76	52	52	76	75	41	11	1		
13	1	13	64	168	267	279	232	232	279	267	168	64	13	1

(17)

For what concerns the octonionic equivalence of the different sectors, it can be symbolically expressed, in different odd space-time dimensions, according to the table

$D=7$	$M0 \equiv M3$
$D=9$	$M0+M1 \equiv M4$
$D=11$	$M1+M2 \equiv M5$
$D=13$	$M2+M3 \equiv M6$
$D=15$	$M3+M4 \equiv M0+M7$

(18)

In $D = 11$ dimensions the relation between $M1 + M2$ and $M5$ can be made explicit as follows. The 11 vectorial indices μ are split into the 4 real indices, labeled by $a, b, c, \ldots$ and the 7 octonionic indices labeled by $i, j, k, \ldots$. The 52 independent components are recovered from $52 = 4 + 2 \times 7 + 6 + 28$, according to

4	$M1_a$	$M5_{[aijkl]} \equiv M5_a$
7	$M1_i$, $M2_{[ij]} \equiv M2_i$	$M5_{[abcdi]} \equiv M5_i$,$M5_{[ijklm]} \equiv \widetilde{M5}_i$
6	$M2_{[ab]}$	$M5_{[abijk]} \equiv M5_{[ab]}$
$4 \times 7 = 28$	$M2_{[ai]}$	$M5_{[abcij]} \equiv M5_{[ai]}$

(19)

4 The octonionic superconformal M-algebra

The conformal algebra of the octonionic M-theory can be introduced [12] adapting to the eleven dimensions the procedure discussed in [5] for the 10 dimensional case. It requires the identification of the conformal algebra of the octonionic $D = 11$ M-algebra with the generalized Lorentz algebra in the $(11, 2)$-dimensional space-time. In such a space-time the octonionic Clifford's Gamma-matrices are 8-dimensional. The basis of the hermitian generators is given by the 64 antisymmetric two-tensors $C\Gamma_{[\mu_1\mu_2]}\mathcal{Z}^{\mu_1\mu_2}$ and the 168 antisymmetric three tensors $C\Gamma_{[\mu_1\mu_2\mu_3]}\mathcal{Z}^{\mu_1\mu_2\mu_3}$ (or, equivalently, by the 232 antisymmetric six-tensors $C\Gamma_{[\mu_1\cdots\mu_6]}\mathcal{Z}^{\mu_1\cdots\mu_6}$). This is already an indication that the total number of generators in the conformal algebra is 232. We will show that this is the case.

According to [5] the conformal algebra can be introduced as the algebra of transformations leaving invariant the inner product of Dirac's spinors. In $(11, 2)$ this is given by $\psi^\dagger C\eta$, where the matrix C, the analogous of the Γ^0, given by the product of the two space-like Clifford's Gamma matrices, is real-valued and totally antisymmetric. Therefore, the conformal transformations are realized by the octonionic-valued 8-dimensional matrices $\mathcal{M}$ leaving C invariant, i.e. satisfying

$$\mathcal{M}^\dagger C + C\mathcal{M} = 0. \tag{20}$$

This allows identifying the (quasi)-group of conformal transformations with the (quasi-)group of symplectic transformations. Indeed, under a simple change of variables, C can be recast in the form

$$\Omega = \begin{pmatrix} 0 & \mathbf{1}_4 \\ -\mathbf{1}_4 & 0 \end{pmatrix}. \tag{21}$$

The most general octonionic-valued matrix leaving invariant Ω can be expressed through

$$\mathbf{M} = \begin{pmatrix} D & B \\ C & -D^\dagger \end{pmatrix}, \tag{22}$$

where the 4×4 octonionic matrices B, C are hermitian

$$B = B^\dagger, \qquad C = C^\dagger. \tag{23}$$

It is easily seen that the total number of independent components in (22) is precisely 232, as we expected from the previous considerations.

It is worth noticing that the set of matrices $\mathbf{M}$ of (22) type forms a closed algebraic structure under the usual matrix commutation. Indeed $[\mathbf{M},\mathbf{M}] \subset \mathbf{M}$ endows the structure of $Sp(8|\mathbf{O})$ to $\mathbf{M}$. For what concerns the supersymmetric extension of the superconformal algebra, we have to accommodate the 64 real components (or 8 octonionic) spinors of $(11,2)$ into a supermatrix enlarging $Sp(8|\mathbf{O})$. This can be achieved as follows. The two 4-column octonionic spinors α and β can be accommodated into a supermatrix of the form

$$\left(\begin{array}{c|cc} 0 & -\beta^\dagger & \alpha^\dagger \\ \hline \alpha & 0 & 0 \\ \beta & 0 & 0 \end{array}\right). \tag{24}$$

Under anticommutation, the lower bosonic diagonal block reduces to $Sp(8|\mathbf{O})$. On the other hand, extra seven generators, associated to the 1-dimensional antihermitian matrix A

$$A^\dagger = -A, \tag{25}$$

i.e. representing the seven imaginary octonions, are obtained in the upper bosonic diagonal block. Therefore, the generic bosonic element is of the form

$$\left(\begin{array}{c|cc} A & 0 & 0 \\ \hline 0 & D & B \\ 0 & C & -D^\dagger \end{array}\right), \tag{26}$$

with A, B and C satisfying (25) and (23).

The closed superalgebraic structure, with (24) as generic fermionic element and (26) as generic bosonic element, will be denoted as $OSp(1,8|\mathbf{O})$. It is the superconformal algebra of the M-theory and admits a total number of 239 bosonic generators.

5 Conclusions.

We have seen that, contrary to what is commonly believed, an alternative formulation for the M superalgebra and the M superconformal algebra can be consistently introduced in association with the non-associative maximal division algebra of the octonions. It presents peculiar features, like the non-independence of the different octonionic brane sectors, which is a reflection of the higher-rank antisymmetric octonionic tensorial identities discussed in section **3**. The existence of this second variant of the M algebra is puzzling.

It could be ultimately related with the arising of exceptional structures (exceptional Lie and Jordan algebras) in the "Theory Of Everything" [19].

Since imaginary octonions admits a geometrical description in terms of the seven sphere S^7, it could be speculated that the higher-dimensional octonionic descriptions, e.g. of the eleven dimensions, corresponds to a particular compactification of the eleven-dimensional M theory down to $AdS_4 \times S^7$. This compactification corresponds to a natural solution for the 11 dimensional supergravity, see [20].

The octonionic superconformal algebra $OSp(1,8|\mathbf{O})$ has been explicitly derived. It corresponds to a supersymmetric extension of a bosonic conformal algebra which is mathematically interesting since it corresponds to a closed algebraic structure which goes beyond the standard notion of conformal algebra of a given Jordan algebra, see [12].

References

[1] R. Haag, J. Łopuszański and M. Sohnius,*Nucl.Phys.* **B 88**, (1975), 257.

[2] T. Kugo and P. Townsend, *Nucl. Phys.* **B 221** (1983), 357.

[3] R. D'Auria, S. Ferrara, M.A. Lledo and V.S. Varadarajan, J. Geom. Phys. **40** 101 (2001); R. D'Auria, S. Ferrara and M.A. Lledo, Lett. Math. Phys. **57**, 123 (2001); S. Ferrara and M.A. Lledo, Rev. Math. Phys. **14**, 519 (2002).

[4] D.B. Fairlie and A.C. Manogue, *Phys. Rev.* **D 34**, 1(1986), 1832.

[5] K.W. Chung and A. Sudbery, *Phys. Lett.* **B 198**, (1987), 161.

[6] C.A. Barton and A. Sudbery, math.RA/0203010.

[7] E. Witten, "Quest For Unification", hep-ph/0207124.

[8] P. Ramond, "Algebraic Dreams", hep-th/0112261.

[9] M. Günaydin, C. Piron and H. Ruegg, *Comm. Math. Phys.* **61** (1978), 69.

[10] L. Smolin, hep-th/0104050.

[11] J. Lukierski and F. Toppan, *Phys. Lett.* **B 539**, (2002), 266.

[12] J. Lukierski and F. Toppan, "Octonionic M-theory and $D = 11$ Generalized Conformal and Superconformal Algebras", hep-th/0212201.

[13] E. Bergshoeff, E. Sezgin, P, Townsend, *Ann. Phys.* **18**, 330 (1987).

[14] M.J. Duff. R.R. Khuri, J.X. Lu, *Phys. Rep.* **259**, 213 (1995).

[15] M. Günaydin and S.V. Ketov, *Nucl. Phys.* **B 467** (1996), 215.

[16] J.Lukierski and P. Minnaert, *Phys. Lett.* **B 129** (1983), 392; Z. Hasiewicz and J. Lukierski, *Phys. Lett.* **B 145** (1984), 65.

[17] M. Cederwall and C. Preitschopf, *Comm. Math. Phys.* **167** (1995), 373.

[18] H.L. Carrion, M. Rojas and F. Toppan, "Quaternionic and Octonionic Spinors. A classification", hep-th/0302113.

[19] L. Boya, "Octonions and M-theory", hep-th/0301037.

[20] M.J. Duff and C.N. Pope, "Kaluza-Klein supergravity and the seven-sphere" in Supersymmetry and Supergravity 82, Trieste proceedings; M.J. Duff, B.E.W. Nilsson and C.N. Pope, *Phys. Rep.* **130** (1986), 1.