Deformation Quantization: Observable Algebras, States and Representation Theory

Lecture notes for the summer school in Kopaonik, 2002

Stefan Waldmann*

Fakultät für Mathematik und Physik

Albert-Ludwigs-Universität Freiburg

Physikalisches Institut

Hermann Herder Straße 3

D 79104 Freiburg

Germany

March 2003

Abstract

In these lecture notes I give an introduction to deformation quantization. The quantization problem is discussed in some detail thereby motivating the notion of star products. Starting from a deformed observable algebra, i.e. the star product algebra, physical applications require to study representations of this algebra. I review the recent development of a representation theory including techniques like Rieffel induction and Morita equivalence. Applications beyond quantization theory are found in noncommutative field theories.

*E-mail: Stefan.Waldmann@physik.uni-freiburg.de

1 Deformation Quantization

1.1 Motivation: Why quantization?

Let me start with a brief motivation why quantization is still an important issue in mathematical physics. Of course we know that the quantum theory of nature provides the more fundamental description compared to its classical counterpart. Thus quantization is an artificial problem since nature is already 'quantum'. However, it seems to be very difficult to find *a priori* quantum descriptions in general. Usually, for us the classical picture is much easier to obtain. Thus it is reasonable to take this as a starting point in order to find the more correct quantum description.

Of course we now need some guidelines how to pass from classical physics to quantum physics. A very vague collection is given by the *correspondence principle*:

- There exists a *classical limit*. The classical description is not so bad for a wide range of energies, distances, momenta etc.
- To any classical observable there corresponds a quantum observable. There can not be more classical observables as the fundamental description is quantum and classical observables only arise as classical limits of quantum observables. On the other hand, if there are more quantum observables not having a reasonable classical limit then 'quantization' is hopeless for such a system.
- Classical *Poisson brackets* should correspond to quantum commutators of the corresponding observables.

Now we have to make these ideas more precise and provide a more concrete and detailed formulation. In particular one should give proper definitions of the words 'observable', 'classical limit', etc. This turns out to be a much more profound problem than our naive approach may suggest. It will be useful to separate the problem into two kind of sub-problems:

- Generic features and questions which are common to all 'quantizations'.
- Specific features and questions depending on the particular example.

In *deformation quantization* the main emphasize lies on the generic questions but one can also perform explicit computations for examples in order to obtain more specific answers. To get an impression of the complexity of the problem we shall first take a look at the 'generic' classical situation.

1.2 The classical framework

Classical mechanics 'lives' on the classical *phase space* M which is the set of all (pure) states of the system or, equivalently, the set of possible initial conditions for the time development. The *observables*, i.e. the functions on M, have a *Poisson bracket*. This indicates that a good mathematical model for M is a *Poisson manifold*, i.e. a smooth manifold endowed with a smooth Poisson structure for the smooth functions $C^\infty(M)$ on M. Such a Poisson structure is a bilinear map

$$\{\cdot,\cdot\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M) \tag{1}$$

which satisfies the conditions:

- $\overline{\{f,g\}} = \{\overline{f},\overline{g}\}$ (Reality).
- $\{f,g\} = -\{g,f\}$ (Antisymmetry).
- $\{f,gh\} = \{f,g\}h + g\{f,h\}$ (Leibniz rule).
- $\{f,\{g,h\}\} = \{\{f,g\},h\} + \{g,\{f,h\}\}$ (Jacobi identity).

This way the observables $C^\infty(M)$ become a *Poisson algebra*. It follows from the linearity and the Leibniz rule that in local coordinates $x^1, \ldots, x^n$ of M the Poisson bracket is of the form

$$\{f, g\}(x) = \sum_{i,j} \alpha^{ij}(x) \frac{\partial f}{\partial x^i}(x) \frac{\partial g}{\partial x^j}(x) \tag{2}$$

with local real-valued functions $\alpha^{ij} = -\alpha^{ji}$. Then the Jacobi identity is equivalent to the following non-linear partial differential equation

$$\sum_\ell \left(\alpha^{i\ell} \frac{\partial \alpha^{jk}}{\partial x^\ell} + \alpha^{j\ell} \frac{\partial \alpha^{ki}}{\partial x^\ell} + \alpha^{k\ell} \frac{\partial \alpha^{ij}}{\partial x^\ell} \right) = 0. \tag{3}$$

The *Poisson tensor* $\alpha := \frac{1}{2}\alpha^{ij}\partial_i \wedge \partial_j$ gives a globally defined 2-vector field $\alpha \in \Gamma^\infty(\bigwedge^2 TM)$ on M.

Example 1 (Some Poisson manifolds)

1. Take $M = \mathbb{R}^{2n}$ with the *canonical Poisson bracket*

$$\{f, g\} = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q^i} \right). \tag{4}$$

2. Let $\mathfrak{g}$ be a Lie algebra with structure constants c_{ij}^k and let $\mathfrak{g}^*$ be its dual space with linear coordinates $x_1, \ldots, x_n$. Then

$$\{f, g\}(x) = \sum_{i,j,k} x_k c_{ij}^k \frac{\partial f}{\partial x_i}(x) \frac{\partial g}{\partial x_j}(x) \tag{5}$$

   defines a Poisson bracket on $\mathfrak{g}^*$.

A Poisson manifold is called *symplectic* if the matrix $(\alpha^{ij}(x))$ at any point $x \in M$ is non-degenerate. In this case

$$\omega = \frac{1}{2}\omega_{ij} dx^i \wedge dx^j \quad \text{with} \quad (\omega_{ij}) = (\alpha^{ij})^{-1} \tag{6}$$

is a closed non-degenerate two-form on M, called the *symplectic form*. Conversely, any such two-form defines a symplectic Poisson manifold. In the above examples, the first one is symplectic with the *canonical symplectic form* $\omega = dq^i \wedge dp_i$ while the second never is symplectic as e.g. the coefficients of the Poisson tensor vanish at the origin. The theorem of Darboux states that any symplectic manifold looks locally like the above example, i.e. one can always find local coordinates q^i, p_i such that $\omega = dq^i \wedge dp_i$. Note however, that globally this needs not to be the case.

Having such a general mathematical framework we have to investigate which Poisson manifolds actually occur in the daily life of a physicist. Let me just mention the following examples:

- $(\mathbb{R}^{2n}, \omega)$ is the well-known phase space from classical mechanics.
- $\mathfrak{g}^*$ with the above Poisson structure always plays a role for systems with symmetries described by $\mathfrak{g}$.
- The phase space of the rigid body is $SO(3) \times \mathbb{R}^3$. More generally, cotangent bundles T^*Q describe the phase space of a particle moving in the configuration space Q. There is a canonical symplectic structure on T^*Q.
- For physical systems with constraints, in particular if there are gauge degrees of freedom, the *reduced phase space* typically has a very complicated Poisson structure.

For more examples and further reading I suggest [1, 2, 18].

1.3 Canonical quantization and basic examples

Before approaching the general situation let us first recall the well-known case of canonical quantization on $\mathbb{R}^{2n}$ as well as other simple examples.

1.3.1 Canonical quantization on $\mathbb{R}^{2n}$

Let us consider $n = 1$ for convenience. Then we assign to the classical observables q and p the quantum operators $Q = q$ and $P = -\mathrm{i}\hbar\frac{\partial}{\partial q}$ acting on wave functions depending on q. However, in order to get all interesting observables quantized we have to specify what should happen to other classical observables than q and p. In particular, we have to specify an *ordering prescription* for the polynomials in q and p since Q and P do no longer commute. Here one can chose between several possibilities, let me just mention two:

- *Standard ordering*: $\varrho_{\mathrm{S}}(q^n p^m) = Q^n P^m$.
- *Weyl ordering*: $\varrho_{\mathrm{Weyl}}(q^n p^m)$ is the corresponding totally symmetrized polynomial in Q and P, e.g.

$$\varrho_{\mathrm{Weyl}}(qp^2) = \frac{1}{3}(QP^2 + QPQ + PQ^2). \tag{7}$$

One has the explicit formulas for ϱ_{S} and ϱ_{Weyl}, which can be verified easily:

$$\varrho_{\mathrm{S}}(f) = \sum_{r=0}^{\infty} \frac{1}{r!} \left(\frac{\hbar}{\mathrm{i}}\right)^r \left.\frac{\partial^r f}{\partial p^r}\right|_{p=0} \frac{\partial^r}{\partial q^r} \tag{8}$$

$$\varrho_{\mathrm{Weyl}}(f) = \varrho_{\mathrm{S}}(Nf) \quad \text{where} \quad N = \mathrm{e}^{\frac{\mathrm{i}\hbar}{2}\Delta} \quad \text{and} \quad \Delta = \frac{\partial^2}{\partial q \partial p}. \tag{9}$$

From these formulas one observes that ϱ_{S} as well as ϱ_{Weyl} can be extended to all smooth functions of q and p which depend polynomially on the momentum p. These functions shall be denoted by $\mathrm{Pol}(T^*\mathbb{R})$. Then the maps ϱ_{S}, ϱ_{Weyl} provide *linear isomorphisms*

$$\varrho_{\mathrm{S}}, \varrho_{\mathrm{Weyl}} : \mathrm{Pol}(T^*\mathbb{R}) \to \mathrm{Diffops}(\mathbb{R}), \tag{10}$$

where $\mathrm{Diffops}(\mathbb{R})$ denotes the space of differential operators with smooth coefficients acting on functions depending on q.

The *main idea* is now to pull-back the noncommutative product of the differential operators via these quantization maps to obtain a *new, noncommutative product* for the classical observables $\mathrm{Pol}(T^*\mathbb{R})$. Thus one defines the *standard ordered* and the *Weyl ordered star product*, also called the *Moyal star product*

$$f \star_{\mathrm{S}} g = \varrho_{\mathrm{S}}^{-1}(\varrho_{\mathrm{S}}(f)\varrho_{\mathrm{S}}(g)), \tag{11}$$

$$f \star_{\mathrm{Weyl}} g = \varrho_{\mathrm{Weyl}}^{-1}(\varrho_{\mathrm{Weyl}}(f)\varrho_{\mathrm{Weyl}}(g)). \tag{12}$$

With some little computation one obtains the following explicit formulas

$$f \star_{\mathrm{S}} g = \sum_{r=0}^{\infty} \frac{1}{r!}\left(\frac{\hbar}{\mathrm{i}}\right)^r \frac{\partial^r f}{\partial p^r}\frac{\partial^r g}{\partial q^r} \tag{13}$$

$$\begin{aligned} f \star_{\mathrm{Weyl}} g &= N^{-1}(Nf \star_{\mathrm{S}} Ng) \\ &= \sum_{r=0}^{\infty} \frac{1}{r!}\left(\frac{\hbar}{2\mathrm{i}}\right)^r \sum_{s=0}^{r} \binom{r}{s}(-1)^{r-s}\frac{\partial^r f}{\partial q^s \partial p^{r-s}}\frac{\partial^r g}{\partial q^{r-s}\partial p^s}, \end{aligned} \tag{14}$$

where the operator N is as in (9).

Let me just mention a few properties of the new products $\star_{\mathrm{S}}$ and $\star_{\mathrm{Weyl}}$ which are obvious from the above explicit formulas (13) and (14).

1. $\star$ is associative since $\mathrm{Diffops}(\mathbb{R})$ is an associative algebra.
2. $f \star g = fg +$ higher orders in $\hbar$.
3. $f \star g - g \star f = \mathrm{i}\hbar\{f, g\} +$ higher orders in $\hbar$.
4. $f \star 1 = f = 1 \star f$.
5. $f \star g = \sum_{r=0}^{\infty} \hbar^r C_r(f, g)$ with bidifferential operators C_r.

Furthermore one has $\varrho_{\mathrm{S}}(f)^\dagger = \varrho_{\mathrm{S}}(N^2\overline{f}) \neq \varrho_{\mathrm{S}}(\overline{f})$ whence the standard ordered quantization is physically *not* appropriate as it maps classical observables, i.e. real-valued functions, to not necessarily symmetric operators. The Weyl ordering has this nice property

$$\varrho_{\mathrm{Weyl}}(f)^\dagger = \varrho_{\mathrm{Weyl}}(\overline{f}), \tag{15}$$

whence physically it is the more reasonable choice. On the level of the star products this means that

$$\overline{f \star_{\mathrm{Weyl}} g} = \overline{g} \star_{\mathrm{Weyl}} \overline{f}, \tag{16}$$

while for the standard ordered product no such relation is valid.

Let me finally mention that other orderings like anti-standard, Wick ordering and anti-Wick ordering lead also to star products having analogous properties.

1.3.2 The BCH star product on $\mathfrak{g}^*$

Another rather simple example of a star product can be obtained for the Poisson structure on the dual $\mathfrak{g}^*$ of a Lie algebra $\mathfrak{g}$ as explained in Example 1. The following construction is due to Gutt [29] where she used this BCH star product to obtain a star product on the cotangent bundle T^*G of the corresponding Lie group G.

Denote by $\mathrm{Pol}^\bullet(\mathfrak{g}^*)$ the space of polynomials on $\mathfrak{g}^*$ which is known to be graded isomorphic to the symmetric algebra $\bigvee^\bullet(\mathfrak{g})$ over $\mathfrak{g}$. Then the Poincaré-Birkhoff-Witt theorem states that the later space is isomorphic to the universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$ as a filtered vector space. An explicit isomorphism is given by the total symmetrization map

$$\sigma : v_1 \vee \cdots \vee v_k \mapsto \frac{(\mathrm{i}\hbar)^k}{k!} \sum_{\pi \in S_k} v_{\pi(1)} \circ \cdots \circ v_{\pi(k)} \tag{17}$$

where $\vee$ denotes the symmetric tensor product and $\circ$ the product in $\mathfrak{U}(\mathfrak{g})$. Using this linear isomorphism one defines a new product

$$f \star_{\mathrm{BCH}} g = \sigma^{-1}(\sigma(f) \circ \sigma(g)) \tag{18}$$

for $f, g \in \mathrm{Pol}(\mathfrak{g}^*)$. Again a rather straightforward computation shows that this product deforms the pointwise commutative product and gives in the first order of $\hbar$ of the commutator the Poisson bracket (5). In order to obtain a more explicit formula for $\star_{\mathrm{BCH}}$ we extend it to exponential functions $\mathrm{e}_x(\xi) = \mathrm{e}^{\xi(x)}$, where $x \in \mathfrak{g}$ and $\xi \in \mathfrak{g}^*$. One obtains

$$\mathrm{e}_x \star_{\mathrm{BCH}} \mathrm{e}_y = \mathrm{e}_{\frac{1}{\mathrm{i}\hbar}\mathrm{BCH}(\mathrm{i}\hbar x, \mathrm{i}\hbar y)}, \tag{19}$$

where $\mathrm{BCH}(x,y) = x + y + \frac{1}{2}[x,y] + \cdots$ is the *Baker-Campbell-Hausdorff series*. While the above formula perfectly makes sense for small $x, y \in \mathfrak{g}$ the convergence of the BCH series can not be guaranteed for all elements in $\mathfrak{g}$. Thus in general (19) can only be understood as a *formal power series in* $\hbar$.

This turns out to be a general problem: The naive extension of $\star_{\mathrm{S}}$, $\star_{\mathrm{Weyl}}$ or $\star_{\mathrm{BCH}}$ to all smooth functions on the underlying phase space is *not* possible. In general, the series in $\hbar$ diverges. However, we have seen that there are usually 'nice' subalgebras where the series converge or even terminate after finitely many powers in $\hbar$.

On the other hand, on a generic phase space M there is *no* distinguished Poisson subalgebra of $C^\infty(M)$. Thus we will look for deformed products only in the sense of formal power series and study their generic properties. Only *after specifying* a concrete example one can start to look for a suitable subalgebra where the formal series actually converge. The choice of a subalgebra usually requires additional and more specific information than just the knowledge of the Poisson manifold M.

1.4 General definitions and first results

The framework for the general situation will be a Poisson manifold M as classical phase space and the smooth functions $C^\infty(M)$ on M as classical observable algebra. Then the definition of a star product according to Bayen et al. [6] generalizes the properties of $\star_{\mathrm{S}}$, $\star_{\mathrm{Weyl}}$, $\star_{\mathrm{BCH}}$, see also [23, 30, 51] for recent reviews.

Definition 2 *A star product $\star$ for M is an associative $\mathbb{C}[[\lambda]]$-bilinear product for $C^\infty(M)[[\lambda]]$,*

$$f \star g = \sum_{r=0}^{\infty} \lambda^r C_r(f,g), \tag{20}$$

such that

1. $f \star g = fg + \cdots$,
2. $f \star g - g \star f = \mathrm{i}\lambda\{f,g\} + \cdots$,
3. $f \star 1 = f = 1 \star f$,
4. *C_r is a bidifferential operator.*

If in addition $\overline{f \star g} = \overline{g} \star \overline{f}$ then the star product is called Hermitian.

The *formal parameter* λ corresponds to Planck's constant $\hbar$ as soon as one has established convergence of the product for some suitable subalgebra.

If $S = \mathsf{id} + \sum_{r=1}^{\infty} \lambda^r S_r$ is a formal series of differential operators S_r on M and if $\star$ is a star product then

$$f \star' g = S^{-1}(Sf \star Sg) \tag{21}$$

turns out to be again a star product. We have already seen in (14) that $\star_{\mathrm{S}}$ and $\star_{\mathrm{Weyl}}$ are related by the operator N exactly in this way.

Definition 3 *Two star products $\star$ and $\star'$ are called equivalent if there is a formal series of differential operators $S = \mathsf{id} + \sum_{r=1}^{\infty} \lambda^r S_r$ such that (21) holds.*

It is clear that the above relation indeed defines an equivalence relation. In particular, $\star_{\mathrm{S}}$ and $\star_{\mathrm{Weyl}}$ are equivalent. However, the 'equivalence' is first of all only a mathematical one whence we have the following (still not completely answered) question:

Question 4 Which physical properties of 'quantization' do only depend on the equivalence class $[\star]$ of the star product and which properties depend on the specific choice of $\star$?

Before we continue discussing the framework of deformation quantization and its physical interpretation let me just mention a few of the strong results obtained in the last two decades:

The *existence* of star products on symplectic phase spaces was first shown by DeWilde and Lecomte [22] in 1983, Fedosov [24] gave an independent proof in 1986, and Omori, Maeda, Yoshioka [39] gave a third proof in 1991. Gutt [29] considered the example of linear Poisson structures, i.e. the case of $\mathfrak{g}^*$, in 1983. The existence of star products in the general Poisson case turned out to be a much harder problem and it was eventually solved by Kontsevich [36] in 1997. Soon after, Cattaneo and Felder [19] gave a TQFT interpretation of Kontsevich's construction in terms of a Poisson sigma model.

The *classification* of star products up to equivalence was first achieved by Nest and Tsygan [38] in 1995 for the symplectic case and independently by Bertelson, Cahen, and Gutt [7], Deligne [21], as well as by Weinstein and Xu [52]. In the general Poisson case Kontsevich also found the classification [36].

The mathematical framework of deformation quantization is Gerstenhaber's theory of deformations of associative algebras [26, 28]. As an example from this mathematical background let me mention the following quite general construction of an associative deformation. In fact, $\star_{\mathrm{S}}$ and $\star_{\mathrm{Weyl}}$ are of this form:

Example 5 (Commuting Derivations)
Let $\mathcal{A}$ be an associative algebra and let $D_i, E^i : \mathcal{A} \to \mathcal{A}$ be pairwise commuting derivations. Denote the undeformed product of $\mathcal{A}$ by $\mu : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$.

Then

$$a \star b = \mu \circ \mathrm{e}^{\lambda \sum_i D_i \otimes E^i}(a \otimes b) \tag{22}$$

defines an associative product on $\mathcal{A}[[\lambda]]$ deforming the product of $\mathcal{A}$, see [27, Thm. 8].

1.5 Star products beyond quantization

The mathematical structure of deformed algebras appears not only in the theory of quantization but also in many other contexts of recent mathematical physics. In fact, any associative deformation of a commutative associative algebra is 'morally' a star product for some Poisson bracket. Let me now illustrate this for two examples.

1.5.1 The quantum plane

Consider the two vector fields $x\frac{\partial}{\partial x}$ and $y\frac{\partial}{\partial y}$ on $\mathbb{R}^2$. A simple computation shows that they commute whence we can apply the general formula of Example 5 to obtain a star product for the functions $C^\infty(\mathbb{R}^2)$, i.e.

$$f \star g = \mu \circ \mathrm{e}^{\mathrm{i}\lambda x\frac{\partial}{\partial x} \otimes y \frac{\partial}{\partial y}}(f \otimes g) \tag{23}$$

is an associative deformation. The first order commutator specifies the following *quadratic* Poisson bracket

$$\{f, g\} = xy \left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x} \right). \tag{24}$$

A straightforward computation gives then the following commutation relations

$$x \star y = \sum_{r=0} \frac{(\mathrm{i}\lambda)^r}{r!} xy = \mathrm{e}^{\mathrm{i}\lambda} xy \quad \text{and} \quad y \star x = yx$$

whence

$$x \star y = q\, y \star x \quad \text{with} \quad q = \mathrm{e}^{\mathrm{i}\lambda}. \tag{25}$$

Thus one recovers the commutation relations for the quantum plane, see e.g. [35].

1.5.2 Noncommutative field theories

Let me now mention very briefly the noncommutative field theories. Here one considers the following very simple model. Take a symplectic constant two-form $B = \frac{1}{2}B_{ij}dx^i \wedge dx^j$ on Minkowski space-time M together with the Weyl product $\star$ with respect to B as discussed before. Then $(M, \star)$ can be considered as a 'quantized space time' and one may ask the question whether there are some interesting models for a field theory on such a space time. The idea is to replace an ordinary Lagrangean by a noncommutative counterpart, e.g.

$$\mathcal{L} = \partial_\mu\phi\partial^\mu\phi - m^2\phi^2 + \alpha\phi^4 + \cdots \tag{26}$$

will be replaced by

$$\tilde{\mathcal{L}} = \partial_\mu\phi \star \partial^\mu\phi - m^2\phi \star \phi + \alpha\phi \star \phi \star \phi \star \phi + \cdots . \tag{27}$$

Such models are studied by various groups, see e.g. [3, 32, 43, 44] and [46] for a recent review with additional references.

Describing such a model within the framework of deformation quantization has several advantages: First one can treat *arbitrary* Poisson structures and not only constant B. Furthermore, one can formulate gauge theories in this context and even extend the deformation program to situations where the classical fields take their values in arbitrary non-trivial vector bundles. The later requires the notion of a deformation quantization of vector bundles [13, 48], to which I shall come back in Section 3.3.

2 States and Representations

2.1 The notion of positivity: ordered rings

Up to now the star product gives us a model for the quantum observable algebra build out of the classical observables. However, for a complete description one also needs a notion for *states*. Here a conceptual problem arises: Usually, states are described by vectors (or better: rays) in a complex Hilbert space. But how should the algebra $C^\infty(M)[[\lambda]]$ act on a Hilbert space. The formal power series in λ certainly do not fit very well to the analytic structure of a Hilbert space. So is this already the point where one has to impose convergence conditions for a replacement $\lambda \rightsquigarrow \hbar$? As we shall see, this is not yet the case and there is a completely intrinsic description of states for formal star products, see e.g. [11, 50] and references therein.

The guideline for the following will be the theory of C^*-algebras, see e.g. [12, 34]. Here a state is identified with its expectation value functional $\omega : \mathcal{A} \to \mathbb{C}$ which is a linear functional on the observables such that $\omega(A^*A) \geq 0$, i.e. ω is a *positive linear functional.*

Example 6 Let $\mathcal{A}$ be the algebra $\mathcal{B}(\mathcal{H})$ of bounded operators on a Hilbert space $\mathcal{H}$ and let $0 \neq \phi \in \mathcal{H}$. Then

$$\omega(A) = \frac{\langle \phi, A\phi \rangle}{\langle \phi, \phi \rangle} \tag{28}$$

defines a state for $\mathcal{A}$. More generally, one can also consider the *mixed states* $\omega(A) = \mathsf{tr}(\varrho A)$ where ϱ is a density matrix.

Thus the idea will be to use positive functionals as states for star products as well. Thus we are looking for functionals

$$C^\infty(M)[[\lambda]] \to \mathbb{C}[[\lambda]], \tag{29}$$

which are now required to be $\mathbb{C}[[\lambda]]$-linear and positive in the sense of formal power series:

Definition 7 *A real formal power series $a = \sum_{r=r_0}^\infty \lambda^r a_r \in \mathbb{R}[[\lambda]]$ is positive if $a_{r_0} > 0$.*

This definition is now used to make sense out of the requirement

$$\omega(\overline{f} \star f) \geq 0 \tag{30}$$

for $f \in C^\infty(M)[[\lambda]]$. In fact, it turns $\mathbb{R}[[\lambda]]$ into an *ordered ring.*

Definition 8 *An associative, commutative, unital ring* R *is called ordered with positive elements* $\mathsf{P} \subseteq \mathsf{R}$ *if* $\mathsf{P} \cdot \mathsf{P} \subseteq \mathsf{P}$, $\mathsf{P} + \mathsf{P} \subseteq \mathsf{P}$ *and* R *is the disjoint union* $\mathsf{R} = -\mathsf{P} \cup \{0\} \cup \mathsf{P}$.

Examples for ordered rings in this sense are $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{R}[[\lambda]]$. Moreover, if R is ordered then $\mathsf{R}[[\lambda]]$ is ordered as well by an analogous definition as for $\mathbb{R}$.

2.2 *-Algebras over ordered rings

Since the crucial concept of states in deformation quantization is now based on the notion of ordered rings I shall continue with some arbitrary ordered ring and consider *-algebras over ordered rings in general. This way, I shall develop a representation theory for such *-algebras parallel to the well-known theory of C^*-algebras, generalizing the latter. The star products will then provide a particular example.

Thus let R be an ordered ring and denote by $\mathsf{C} = \mathsf{R}(\mathrm{i})$ its ring extension by a square root i of -1, i.e. $\mathrm{i}^2 = -1$. This way one obtains a replacement for the real and complex numbers $\mathbb{R}$ and $\mathbb{C}$. For deformation quantization $\mathsf{R} = \mathbb{R}[[\lambda]]$ and $\mathsf{C} = \mathbb{C}[[\lambda]]$ will be the relevant choice.

Definition 9 *A pre Hilbert space* $\mathcal{H}$ *over* C *is a* C*-module with inner product* $\langle \cdot, \cdot \rangle : \mathcal{H} \times \mathcal{H} \to \mathsf{C}$ *such that*

1. $\langle \phi, z\psi + w\chi \rangle = z\langle \phi, \psi \rangle + w\langle \phi, \chi \rangle$ *for* $\phi, \psi, \chi \in \mathcal{H}$ *and* $z, w \in \mathsf{C}$.
2. $\langle \phi, \psi \rangle = \overline{\langle \psi, \phi \rangle}$ *for* $\phi, \psi \in \mathcal{H}$.

3. $\langle \phi, \phi \rangle > 0$ *for* $\phi \neq 0$.

An operator $A : \mathcal{H} \to \mathcal{H}$ is called *adjointable* if there exists an operator A^* such that

$$\langle \phi, A\psi \rangle = \langle A^* \phi, \psi \rangle \tag{31}$$

for all $\phi, \psi \in \mathcal{H}$. This allows to define

$$\mathcal{B}(\mathcal{H}) = \{ A \in \mathsf{End}(\mathcal{H}) \mid A \text{ is adjointable } \}. \tag{32}$$

Note that for a complex Hilbert space this definition exactly yields the bounded operators thanks to the Hellinger-Toeplitz theorem, see e.g. [42, p. 117].

Lemma 10 *Let* $A, B \in \mathcal{B}(\mathcal{H})$ *and* $z, w \in \mathsf{C}$.

1. A^* *is unique and* $A^* \in \mathcal{B}(\mathcal{H})$, $A^{**} = A$.
2. $zA + wB \in \mathcal{B}(\mathcal{H})$ *and* $(zA + wB)^* = \overline{z}A^* + \overline{w}B^*$.
3. $AB \in \mathcal{B}(\mathcal{H})$ *and* $(AB)^* = B^* A^*$.

The algebra $\mathcal{B}(\mathcal{H})$ gives now the proto-type for a *-algebra over C:

Definition 11 *An associative algebra* $\mathcal{A}$ *over* C *is called* **-algebra if it is equipped with an involutive anti-linear anti-automorphism* $^* : \mathcal{A} \to \mathcal{A}$, *called the* **-involution. A* **-homomorphism* $\Phi : \mathcal{A} \to \mathcal{B}$ *between* **-algebras is an algebra morphism with* $\Phi(A^*) = \Phi(A)^*$.

In the following I shall mainly focus on *unital* *-algebras and *-homomorphisms are required to be unit preserving.

Example 12 (*-Algebras)

1. Any C^*-algebra is a *-algebra over $\mathbb{C}$.

2. $(C^\infty(M)[[\lambda]], \star)$ with a Hermitian star product is a *-algebra over $\mathbb{C}[[\lambda]]$.

3. $\mathcal{B}(\mathcal{H})$ is a *-algebra over C for any pre Hilbert space $\mathcal{H}$ over C.

4. If $\mathcal{A}$ is a *-algebra then $M_n(\mathcal{A})$ becomes a *-algebra as well.

Definition 13 *A *-representation π of $\mathcal{A}$ on $\mathcal{H}$ is a *-homomorphism $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$.*

Given two *-representations $(\mathcal{H}, \pi)$ and $(\mathcal{K}, \varrho)$ then an *intertwiner* is an adjointable isometric map $T : \mathcal{H} \to \mathcal{K}$ such that

$$T\pi(A) = \varrho(A)T \tag{33}$$

for all $A \in \mathcal{A}$. Two *-representations are called *equivalent* if there exists a unitary intertwiner between them.

Definition 14 *The representation theory of $\mathcal{A}$ is the category* *-Rep$(\mathcal{A})$ *of all *-representations of $\mathcal{A}$ with intertwiners as morphisms.*

2.3 Positive functionals

Since the observable algebras in deformation quantization are particular examples of *-algebras over C we shall now come back to the question of states in the general framework.

Definition 15 *Let $\mathcal{A}$ be a *-algebra over C. A linear functional $\omega : \mathcal{A} \to \mathsf{C}$ is called positive if*

$$\omega(A^*A) \geq 0 \tag{34}$$

for all $A \in \mathcal{A}$. It is called a state of $\mathcal{A}$ if in addition $\omega(\mathbb{1}) = 1$. Then the value $\omega(A)$ is called the expectation value of $A \in \mathcal{A}$ in the state ω.

Thus we have to investigate the positive functionals for star products. The following examples show that this is indeed a non-trivial notion:

Example 16 (Positive functionals [11])

1. The δ-functional on $\mathbb{R}^{2n}$ is *not* positive with respect to the Weyl star product $\star_{\mathrm{Weyl}}$ since e.g.

$$\delta(\overline{H} \star_{\mathrm{Weyl}} H) = -\frac{\lambda^2}{4} < 0, \tag{35}$$

   where H is the Hamiltonian of the harmonic oscillator.

   Thus points in phase space are (in general) no longer states in quantum mechnics, which is physically to be expected from the uncertainty relations.

2. Consider $f \in C_0^\infty(\mathbb{R}^{2n})[[\lambda]]$ then the Schrödinger functional

$$\omega(f) = \int_{\mathbb{R}^n} f(q, p = 0) \; d^n q \tag{36}$$

   turns out to be a positive functional with respect to the Weyl star product $\star_{\mathrm{Weyl}}$. In this case the positivity can be shown by successive partial integrations.

Having the above examples in mind the following question naturally arises: How many positive functionals does the algebra $(C^\infty(M)[[\lambda]], \star)$ have? Classically one knows that the positive functionals of $C^\infty(M)$ are precisely the compactly supported positive Borel measures. In particular, there are 'many' positive functionals as e.g. all the δ-functionals are positive.

Definition 17 *A Hermitian deformation $\star$ of a *-algebra $\mathcal{A}$ over C is called a positive deformation if for any positive linear functional $\omega_0 : \mathcal{A} \to \mathsf{C}$ there* *exist 'quantum corrections'* ω_r *such that*

$$\omega = \sum_{r=0}^{\infty} \lambda^r \omega_r : \mathcal{A}[[\lambda]] \to \mathsf{C}[[\lambda]] \tag{37}$$

is positive with respect to $\star$*.*

For star products one has the following characterization:

Theorem 18 (Positive deformations [14]) *Any Hermitian star product on a symplectic manifold is a positive deformation.*

The example (35) shows that the 'quantum corrections' are indeed necessary in some cases. Physically speaking, the above theorem states that every classical state arises as classical limit of a quantum state. This is to be expected in order to have a consistent classical limit.

2.4 The GNS construction

The GNS construction will now provide a way how to pass from the abstractly given observable algebra back to usual operators on a pre Hilbert space, i.e. it allows to construct a *-representation of the observable algebra. We start with a *-algebra $\mathcal{A}$ over C as before and a positive functional $\omega : \mathcal{A} \to \mathsf{C}$. As in the well-known case of C^*-algebras a positive functional is real in the sense that

$$\omega(A^*B) = \overline{\omega(B^*A)} \tag{38}$$

and it satisfies the *Cauchy-Schwarz inequality*

$$\omega(A^*B)\overline{\omega(A^*B)} \le \omega(A^*A)\omega(B^*B) \tag{39}$$

for all $A, B \in \mathcal{A}$. Now consider the following subset, the so-called Gel'fand ideal of ω,

$$\mathfrak{J}_\omega = \{A \in \mathcal{A} \mid \omega(A^*A) = 0\}, \tag{40}$$

which indeed turns out to be a left ideal of $\mathcal{A}$ thanks to (39). Being a left ideal makes the quotient

$$\mathcal{H}_\omega = \mathcal{A}/\mathcal{J}_\omega \tag{41}$$

a $\mathcal{A}$-left module. The module action is usually denoted by $\pi_\omega(A)\psi_B = \psi_{AB}$ where $\psi_B \in \mathcal{H}_\omega$ is the equivalence class of $B \in \mathcal{A}$. Furthermore, $\mathcal{H}_\omega$ is a pre Hilbert space over C with the inner product

$$\langle \psi_A, \psi_B \rangle = \omega(A^* B). \tag{42}$$

Indeed, $\langle \cdot, \cdot \rangle$ is well-defined and positive definite, again according to the Cauchy-Schwarz inequality. The final step in the GNS construction consists in observing that the module action π_ω is actually a *-representation, i.e.

$$\langle \pi_\omega(A)\psi_B, \psi_C \rangle = \langle \psi_B, \pi_\omega(A^*)\psi_C \rangle, \tag{43}$$

which can be verified easily. Thus one obtains a *-representation $(\mathcal{H}_\omega, \pi_\omega)$ of $\mathcal{A}$, called the *GNS representation* induced by ω. In some sense the GNS construction is a generalization of the usual Fock space construction starting from a vacuum vector. In fact, if ω is even a state, i.e. $\omega(\mathbb{1}) = 1$, then it can be written as expecation value

$$\omega(A) = \langle \psi_{\mathbb{1}}, \pi_\omega(A)\psi_{\mathbb{1}} \rangle, \tag{44}$$

whence the vector $\psi_{\mathbb{1}}$ plays the role of the vacuum vector.

Example 19 (GNS representations)

1. Let $\phi \in \mathcal{H}$ with $\langle \phi, \phi \rangle = 1$ and consider the positive functional $\omega(A) = \langle \phi, A\phi \rangle$ on the *-algebra $\mathcal{B}(\mathcal{H})$. Then $\mathcal{J}_\omega = \{A \mid A\phi = 0\}$ whence

$$\mathcal{H}_\omega = \mathcal{B}(\mathcal{H})/\mathcal{J}_\omega \ni \psi_A \mapsto A\phi \in \mathcal{H} \tag{45}$$

   is a unitary isomorphism and in fact an intertwiner. Thus one recovers the usual action of $\mathcal{B}(\mathcal{H})$ on $\mathcal{H}$ by the GNS construction.

2. Consider again the Schrödinger functional ω as in (36) which is positive for the Weyl star product. Then the GNS representation induced by this ω is canonically unitarily equivalent to the usual (formal) Schrödinger representation in Weyl ordering ϱ_{Weyl} on the pre Hilbert space $C_0^\infty(\mathbb{R}^n)[[\lambda]]$ of formal wave functions, see e.g. [11].

The last example is important as it shows that deformation quantization has the capability to obtain intrinsically the usual formulation of quantum mechanics starting from the star product algebras.

Having the nice construction of representations out of positive functionals one has to investigate whether there are any of them. In particular, one can construct (somehow pathological) examples of *-algebras without *any* positive linear functional, see e.g. [15]. This motivates the following definition:

Definition 20 *A *-algebra $\mathcal{A}$ over C has sufficiently many positive linear functionals if for any $0 \neq H = H^* \in \mathcal{A}$ there exists a positive functional ω with $\omega(H) \neq 0$.*

Physically speaking, this means that we can 'measure' whether an observable is zero or not by examining all possible expectation values.

By taking the direct sum of all GNS representations one arrives at the following theorem:

Theorem 21 (Faithful *-representations [15]) *Let $\mathcal{A}$ be a unital *-algebra. Then $\mathcal{A}$ has sufficiently many positive linear functionals if and only if it has a faithful *-representation. In this case $\mathcal{A}$ can be viewed as a *-subalgebra of some $\mathfrak{B}(\mathfrak{H})$.*

Lemma 22 *Let $(\mathcal{A}[[\lambda]], \star)$ be a positive Hermitian deformation. Then $\mathcal{A}$ has sufficiently many positive linear functionals iff $(\mathcal{A}[[\lambda]], \star)$ has sufficiently many positive linear functionals.*

In particular, since $C^\infty(M)$ has sufficiently many positive linear functionals (take e.g. the δ-functionals) it follows that Hermitian star products also have sufficiently many positive linear functionals. Hence the star product algebra $(C^\infty(M)[[\lambda]], \star)$ has a faithful *-representation. This can also be obtained more directly and explicitly.

Let me finally mention some more applications of the GNS construction in deformation quantization:

1. There are analogues of $\star_{\mathrm{S}}$ and $\star_{\mathrm{Weyl}}$ for any cotangent bundle T^*Q such that one has a Schrödinger representation on $C_0^\infty(Q)[[\lambda]]$ being the GNS representation for a Schrödinger functional consisting of integration over Q with respect to some positive density [9].

2. The WKB expansion [5] can be obtained in a particular GNS representation where the positive functional is a particular integration over the graph of dS in T^*Q where $S : Q \to \mathbb{R}$ is a solution of the Hamilton-Jacobi equation $H \circ dS = E$, see [8, 9].

3. There is also a characterization of thermodynamical states using the KMS condition. It turns out that for a given inverse temperature β and a Hamiltonian H the KMS states are unique and of the form

$$\omega(f) = \mathsf{tr}\left(\mathrm{Exp}(-\beta H) \star f\right), \tag{46}$$

where Exp is the star exponential and tr the unique trace for a symplectic star product. The GNS representation turns out to be faithful with commutant being (anti-) isomorphic to the algebra itself. This gives a sort of baby-version of the Tomita-Takesaki theorem [4, 10, 47].

3 Representation Theory

As we have seen, formal star products have a rich and physically relevant representation theory. The main advantage of deformation quantization becomes transparent when investigating representations: the algebra of observables is the fundamental object describing the physical system while the representations depend on the current situation of the physical system (pure, thermodynamical etc.) and are thus a derived concept. The algebra determines its states and representations but not vice versa. This point of view allows to consider different representations of the same observable algebra. In particular, one can speak of super selection rules etc. Thus it is of primary importance to develop tools for investigating *-Rep$(\mathcal{A})$.

3.1 Rieffel induction

Rieffel induction was first developed by Rieffel as a tool for understanding the representation theory of C^*-algebras [40, 41]. It turns out that these techniques are much more algebraic and can also be applied in our context. I shall just sketch the construction and refer to [16, 17] for details.

Let two *-algebras $\mathcal{A}$ and $\mathcal{B}$ be given, where I always assume that they are unital. The positive functionals allow us to define positive algebra elements: $A \in \mathcal{A}$ is called *positive* if for all positive $\omega : \mathcal{A} \to \mathsf{C}$ the expectation value $\omega(A) > 0$ is positive. The positive elements are denoted by $\mathcal{A}^+$.

Now consider a $\mathcal{B}$-$\mathcal{A}$ bimodule $\mathcal{E}$. A $\mathcal{A}$-valued completely positive inner product is a map

$$\langle \cdot, \cdot \rangle : \mathcal{E} \times \mathcal{E} \to \mathcal{A} \tag{47}$$

such that $\langle \cdot, \cdot \rangle$ is C-linear in the second argument, satisfies $\langle x, y \rangle = \langle y, x \rangle^*$ and $\langle x, y \cdot A \rangle = \langle x, y \rangle A$ for all $x, y \in \mathcal{E}$ and $A \in \mathcal{A}$, and the matrix

$$(\langle x_i, x_j \rangle) \in M_n(\mathcal{A})^+ \tag{48}$$

is positive for all n and $x_1, \ldots, x_n \in \mathcal{E}$. The inner product is *compatible* with the $\mathcal{B}$-left module structure if in addition $\langle B \cdot x, y \rangle = \langle x, B^* \cdot y \rangle$ for all $x, y \in \mathcal{E}$ and $B \in \mathcal{B}$.

Example 23 The most important example is the free module $\mathcal{E} = \mathcal{A}^n$ over $\mathcal{A}$ and $\mathcal{B} = M_n(\mathcal{A})$ acts from the left in the usual way. The inner product is then defined by

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i^* y_i, \tag{49}$$

and it is an easy check that this indeed gives a compatible, completely positive $\mathcal{A}$-valued inner product.

Now suppose we have a *-representation $(\mathcal{H}, \pi)$ of $\mathcal{A}$. Then we consider the tensor product

$$\widetilde{\mathcal{K}} := \mathcal{E} \otimes_{\mathcal{A}} \mathcal{H}, \tag{50}$$

where we use the representation π to define the $\mathcal{A}$-tensor product. Clearly, $\widetilde{\mathcal{K}}$ is a $\mathcal{B}$-left module as $\mathcal{E}$ was a bimodule. Next, one endows $\widetilde{\mathcal{K}}$ with an inner product by setting

$$\langle x \otimes \phi, y \otimes \psi \rangle := \langle \phi, \pi(\langle x, y \rangle)\psi \rangle, \tag{51}$$

using the $\mathcal{A}$-valued inner product as well as the inner product of the pre Hilbert space $\mathcal{H}$. It turns out that (51) extends sesquilinearily to a well-defined inner product on $\widetilde{\mathcal{K}}$. Moreover, the complete positivity of the $\mathcal{A}$-valued inner product guarantees that (51) is at least positive semi-definite. Thus one can quotient out the (possible) degeneracy space $\widetilde{\mathcal{K}}^\perp$ and obtains a pre Hilbert space $\mathcal{K} := \widetilde{\mathcal{K}}/\widetilde{\mathcal{K}}^\perp$. In a final step one observs that the $\mathcal{B}$-left module structure on $\widetilde{\mathcal{K}}$ is in fact compatible with the inner product and descends to $\mathcal{K}$ to give a *-representation ϱ of $\mathcal{B}$. This is the Rieffel induced representation $(\mathcal{K}, \varrho)$ of $\mathcal{B}$.

Theorem 24 (Rieffel induction [16]) *Let $\mathcal{E}$ be a $\mathcal{B}$-$\mathcal{A}$ bimodule with compatible, completely positive $\mathcal{A}$-valued inner product. Then the construction of the induced representation gives a functor*

$$\mathsf{R}_{\mathcal{E}} : {}^*\text{-Rep}(\mathcal{A}) \ni (\mathcal{H}, \pi) \mapsto (\mathcal{K}, \varrho) \in {}^*\text{-Rep}(\mathcal{B}). \tag{52}$$

The only thing which remains to be shown is the functoriality whence we have to specify how $\mathsf{R}_{\mathcal{E}}$ should act on morphisms. Thus let $U : \mathcal{H}_1 \to \mathcal{H}_2$ be an isometric adjointable intertwiner. Then $V(x \otimes \phi) := x \otimes U\phi$ turns out to define an intertwiner between the $\mathcal{B}$-representations on $\mathcal{K}_1$ and $\mathcal{K}_2$.

3.2 Strong Morita equivalence

Morita equivalence deals with the question under which conditions the categories of representations are actually equivalent in the sense of categories. In our situation we have to investigate the categories *-Rep$(\mathcal{A})$ and *-Rep$(\mathcal{B})$ for two *-algebras $\mathcal{A}$ and $\mathcal{B}$. We are interested whether the Rieffel induction using a bimodule $\mathcal{E}$ gives the equivalence. Since up to now the algebras $\mathcal{A}$ and $\mathcal{B}$ enter quite asymmetric we first have to endow the bimodule with some more structures. Thus we also ask for a $\mathcal{B}$-valued inner product denoted by

$$\mathcal{E} \times \mathcal{E} \ni (x, y) \mapsto \Theta_{x,y} \in \mathcal{B}, \tag{53}$$

which should now be C-linear and $\mathcal{B}$-linear in the *first* argument as $\mathcal{E}$ is a $\mathcal{B}$-*left* module. Beside that it also should be completely positive. Moreover, the compatibility with the $\mathcal{A}$-module structure is $\Theta_{x,y\cdot A} = \Theta_{x\cdot A^*,y}$. Finally, the two inner products $\langle \cdot, \cdot \rangle$ and $\Theta_{\cdot,\cdot}$ are called compatible if

$$\Theta_{x,y} \cdot z = x \cdot \langle y, z \rangle. \tag{54}$$

Given such an $\mathcal{E}$ we can pass to the complex-conjugate bimodule $\overline{\mathcal{E}}$ which is $\mathcal{E}$ as additive group but the C-module structure is now given by $\alpha\overline{x} = \overline{\overline{\alpha}x}$, where $\mathcal{E} \ni x \mapsto \overline{x} \in \overline{\mathcal{E}}$ is the identity map. Indeed, one obtains a $\mathcal{A}$-$\mathcal{B}$ bimodule by the definitions $A \cdot \overline{x} := \overline{x \cdot A^*}$ and $\overline{x} \cdot B := \overline{B^* \cdot x}$. Thus the role of $\mathcal{A}$ and $\mathcal{B}$ is exchanged whence we now have a Rieffel induction functor going the opposite way, namely $\mathsf{R}_{\overline{\mathcal{E}}} : {}^*\text{-Rep}(\mathcal{B}) \to {}^*\text{-Rep}(\mathcal{A})$.

However, in general this does not yet give an equivalence of categories, since we do not have made any non-triviality requirements for the inner products. In principle they can all be chosen to be identically zero. One says that the inner products are *full* if

$$\mathsf{C}\text{-span}\{\langle x, y\rangle \mid x, y \in \mathcal{E}\} = \mathcal{A} \tag{55}$$

$$\mathsf{C}\text{-span}\{\Theta_{x,y} \mid x, y \in \mathcal{E}\} = \mathcal{B}. \tag{56}$$

Definition 25 (Strong Morita equivalence [16]) *Two unital* **-algebras* $\mathcal{A}$, $\mathcal{B}$ *are called strongly Morita equivalent if one finds a bimodule* $\mathcal{E}$ *with compatible completely positive inner products which are both full.*

Strong Morita equivalence implies that the algebras behave in many aspects quite similar. In particular the Rieffel induction functors $\mathsf{R}_{\mathcal{E}}$ and $\mathsf{R}_{\overline{\mathcal{E}}}$ give an equivalence of the representation theories ${}^*\text{-Rep}(\mathcal{A})$ and ${}^*\text{-Rep}(\mathcal{B})$. Strong Morita equivalence also implies (ring-theoretical) Morita equivalence. Hence the algebras $\mathcal{A}$ and $\mathcal{B}$ have all the Morita invariants in common, like isomorphic centers, Hochschild cohomology and many more, see e.g. [37]. However, there are more invariants, see [15].

One important consequence for the following is that the bimodule $\mathcal{E}$ viewed as $\mathcal{A}$-right module is always *finitely generated and projective*. The same holds if we view $\mathcal{E}$ as a $\mathcal{B}$-left module.

3.3 An application: deformed vector bundles

If we want to understand strong Morita equivalence of star products we have to investigate the finitely generated projective modules for the star product algebras. Classically, the finitely generated projective modules of $C^\infty(M)$ are precisely the sections $\Gamma^\infty(E)$ of some vector bundle $E \to M$. This is the famous Serre-Swan theorem, see e.g. [45]. In Connes' noncommutative geometry this observation leads to the idea that in a noncommutative context vector bundles have to be replaced by projective modules [20].

Thus we are looking for the deformed analog of a vector bundle: In fact, it is more than an analog as one can easily show that a projective module over an algebra $\mathcal{A}$ can *always* be deformed into a projective module over a deformation $(\mathcal{A}[[\lambda]], \star)$, since one can deform projections. Here one even has an explicit formula, see [25, Eq. (6.1.4)]. If P_0 is the classical projection then

$$\mathbf{P} = \frac{1}{2} + \left(P_0 - \frac{1}{2}\right) \star \frac{1}{\sqrt[\star]{1 + 4(P_0 \star P_0 - P_0)}} \tag{57}$$

defines a projection for the deformed product, i.e. $\mathbf{P} \star \mathbf{P} = \mathbf{P}$, as a simple computation shows. Based on this observation one can prove the following statement:

Theorem 26 (Deformed vector bundles [13]) *Let $\star$ be a star product on M and let $E \to M$ be a vector bundle. Then there exists a right module structure $\bullet$ on $\Gamma^\infty(E)[[\lambda]]$ for $(C^\infty(M)[[\lambda]], \star)$ deforming the usual $C^\infty(M)$-module structure. Moreover, $\bullet$ is unique up to equivalence and induces a deformation $\star'$ for $\Gamma^\infty(\mathsf{End}(E))[[\lambda]]$ which is unique up to isomorphism. Finally, also Hermitian fibre metrics can be deformed in a unique way up to isometry.*

This theorem has at least two important consequences. On one hand it leads to the *classification of star products up to strong Morita equivalence*, see [17, 33]. In particular, if the phase space is a cotangent bundle and a background magnetic field is present, then Morita equivalence of the star product with magnetic field switched off and the star product taking the magnetic field into account means that the *Dirac quantization condition* for the magnetic charges is fulfilled. Moreover, under the Rieffel induction which implements the equivalence, the analog of the usual Schrödinger representation (9) is mapped to a representation on the sections of the line bundle (generalized wave functions) over the configuration space corresponding to the magnetic field which serves as curvature two-form of this line bundle. Hence Morita equivalence has a very natural physical interpretation in quantization theory.

On the other hand, deformed vector bundles provide a playground for noncommutative field theories. As already discussed, in a classical field theory the fields can take their values in a non-trivial vector bundle and a priori it is not clear what the good noncommutative analog should be. But the deformed vector bundle gives exactly what is needed and, even more important, there is no obstruction for it's construction. In fact this construction can be made quite explicit using e.g. an adapted version of Fedosov's construction of a star product [49] or Kontsevich's formality in the more general case of Poisson manifolds but only for line bundles [33]. Also, a local description by deformed transition maps can be obtained [33, 50].

Acknowledgments

I would like to thank the organizers of the summer school in Kopaonik as well as the participants for a wonderful atmosphere and many interesting discussions.

References

[1] Abraham, R., Marsden, J. E.: *Foundations of Mechanics.* Addison Wesley Publishing Company, Reading, Mass., 2. edition, 1985.

[2] ARNOL'D, V. I.: *Mathematical Methods of Classical Mechanics*, vol. 60 in *Graduate Texts in Mathematics*. Springer-Verlag, Berlin, Heidelberg, New York, 2. edition, 1989.

[3] BAHNS, D., DOPLICHER, S., FREDENHAGEN, K., PIACITELLI, G.: *Ultraviolet Finite Quantum Field Theory on Quantum Spacetime*. Preprint **hep-th/0301100** (2003), 22 pages.

[4] BASART, H., FLATO, M., LICHNEROWICZ, A., STERNHEIMER, D.: *Deformation Theory applied to Quantization and Statistical Mechanics*. Lett. Math. Phys. **8** (1984), 483–494.

[5] BATES, S., WEINSTEIN, A.: *Lectures on the Geometry of Quantization*. Berkeley Mathematics Lecture Notes 8, Berkeley, 1995.

[6] BAYEN, F., FLATO, M., FRØNSDAL, C., LICHNEROWICZ, A., STERNHEIMER, D.: *Deformation Theory and Quantization*. Ann. Phys. **111** (1978), 61–151.

[7] BERTELSON, M., CAHEN, M., GUTT, S.: *Equivalence of Star Products*. Class. Quant. Grav. **14** (1997), A93–A107.

[8] BORDEMANN, M., NEUMAIER, N., PFLAUM, M. J., WALDMANN, S.: *On representations of star product algebras over cotangent spaces on Hermitian line bundles*. Preprint (Freiburg FR-THEP-98/24) **math.QA/9811055** (November 1998), 38 pages. To appear in J. Funct. Anal.

[9] BORDEMANN, M., NEUMAIER, N., WALDMANN, S.: *Homogeneous Fedosov star products on cotangent bundles II: GNS representations, the WKB expansion, traces, and applications*. J. Geom. Phys. **29** (1999), 199–234.

[10] BORDEMANN, M., RÖMER, H., WALDMANN, S.: *A Remark on Formal KMS States in Deformation Quantization*. Lett. Math. Phys. **45** (1998), 49–61.

[11] BORDEMANN, M., WALDMANN, S.: *Formal GNS Construction and States in Deformation Quantization*. Commun. Math. Phys. **195** (1998), 549–583.

[12] BRATTELI, O., ROBINSON, D. W.: *Operator Algebras and Quantum Statistical Mechanics I: C^*- and W^*-Algebras. Symmetry Groups. Decomposition of States*. Springer-Verlag, New York, Heidelberg, Berlin, 2. edition, 1987.

[13] BURSZTYN, H., WALDMANN, S.: *Deformation Quantization of Hermitian Vector Bundles*. Lett. Math. Phys. **53** (2000), 349–365.

[14] BURSZTYN, H., WALDMANN, S.: *On Positive Deformations of *-Algebras*. In: DITO, G., STERNHEIMER, D. (EDS.): *Conférence Moshé Flato 1999. Quantization,*

Deformations, and Symmetries, *Mathematical Physics Studies* no. **22**, 69–80. Kluwer Academic Publishers, Dordrecht, Boston, London, 2000.

[15] BURSZTYN, H., WALDMANN, S.: **-Ideals and Formal Morita Equivalence of *-Algebras.* Int. J. Math. **12**.5 (2001), 555–577.

[16] BURSZTYN, H., WALDMANN, S.: *Algebraic Rieffel Induction, Formal Morita Equivalence and Applications to Deformation Quantization.* J. Geom. Phys. **37** (2001), 307–364.

[17] BURSZTYN, H., WALDMANN, S.: *The characteristic classes of Morita equivalent star products on symplectic manifolds.* Commun. Math. Phys. **228** (2002), 103–121.

[18] CANNAS DA SILVA, A., WEINSTEIN, A.: *Geometric Models for Noncommutative Algebras. Berkeley Mathematics Lecture Notes.* AMS, 1999.

[19] CATTANEO, A., FELDER, G.: *A Path Integral Approach to the Kontsevich Quantization Formula.* Commun. Math. Phys. **212** (2000), 591–611.

[20] CONNES, A.: *Noncommutative Geometry.* Academic Press, San Diego, New York, London, 1994.

[21] DELIGNE, P.: *Déformations de l'Algèbre des Fonctions d'une Variété Symplectique: Comparaison entre Fedosov et DeWilde, Lecomte.* Sel. Math. New Series **1**.4 (1995), 667–697.

[22] DEWILDE, M., LECOMTE, P. B. A.: *Existence of Star-Products and of Formal Deformations of the Poisson Lie Algebra of Arbitrary Symplectic Manifolds.* Lett. Math. Phys. **7** (1983), 487–496.

[23] DITO, G., STERNHEIMER, D.: *Deformation quantization: genesis, developments and metamorphoses.* In: HALBOUT, G. (EDS.): *Deformation quantization.* [31], 9–54.

[24] FEDOSOV, B. V.: *Quantization and the Index.* Sov. Phys. Dokl. **31**.11 (1986), 877–878.

[25] FEDOSOV, B. V.: *Deformation Quantization and Index Theory.* Akademie Verlag, Berlin, 1996.

[26] GERSTENHABER, M.: *On the Deformation of Rings and Algebras.* Ann. Math. **79** (1964), 59–103.

[27] GERSTENHABER, M.: *On the Deformation of Rings and Algebras III.* Ann. Math. **88** (1968), 1–34.

[28] Gerstenhaber, M., Schack, S. D.: *Algebraic Cohomology and Deformation Theory.* In: Hazewinkel, M., Gerstenhaber, M. (eds.): *Deformation Theory of Algebras and Structures and Applications*, 13–264. Kluwer Academic Press, Dordrecht, 1988.

[29] Gutt, S.: *An Explicit $*$-Product on the Cotangent Bundle of a Lie Group.* Lett. Math. Phys. **7** (1983), 249–258.

[30] Gutt, S.: *Variations on deformation quantization.* In: Dito, G., Sternheimer, D. (eds.): *Conférence Moshé Flato 1999. Quantization, Deformations, and Symmetries*, *Mathematical Physics Studies* no. **21**, 217–254. Kluwer Academic Publishers, Dordrecht, Boston, London, 2000.

[31] Halbout, G. (eds.): *Deformation Quantization*, vol. 1 in *IRMA Lectures in Mathematics and Theoretical Physics.* Walter de Gruyter, Berlin, New York, 2002.

[32] Jurčo, B., Schupp, P., Wess, J.: *Noncommutative gauge theory for Poisson manifolds.* Nucl. Phys. **B584** (2000), 784–794.

[33] Jurčo, B., Schupp, P., Wess, J.: *Noncommutative Line Bundles and Morita Equivalence.* Lett. Math. Phys. **61** (2002), 171–186.

[34] Kadison, R. V., Ringrose, J. R.: *Fundamentals of the Theory of Operator Algebras. Volume I: Elementary Theory*, vol. 15 in *Graduate Studies in Mathematics.* American Mathematical Society, Providence, 1997.

[35] Kassel, C.: *Quantum Groups. Graduate Texts in Mathematics* no. **155**. Springer-Verlag, New York, Berlin, Heidelberg, 1995.

[36] Kontsevich, M.: *Deformation Quantization of Poisson Manifolds, I.* Preprint **q-alg/9709040** (September 1997).

[37] Lam, T. Y.: *Lectures on Modules and Rings*, vol. 189 in *Graduate Texts in Mathematics.* Springer-Verlag, Berlin, Heidelberg, New York, 1999.

[38] Nest, R., Tsygan, B.: *Algebraic Index Theorem.* Commun. Math. Phys. **172** (1995), 223–262.

[39] Omori, H., Maeda, Y., Yoshioka, A.: *Weyl Manifolds and Deformation Quantization.* Adv. Math. **85** (1991), 224–255.

[40] Rieffel, M. A.: *Induced representations of C^*-algebras.* Adv. Math. **13** (1974), 176–257.

[41] Rieffel, M. A.: *Morita equivalence for C^*-algebras and W^*-algebras.* J. Pure. Appl. Math. **5** (1974), 51–96.

[42] Rudin, W.: *Functional Analysis.* McGraw-Hill Book Company, New York, 2. edition, 1991.

[43] Schomerus, V.: *D-branes and deformation quantization.* J. High. Energy. Phys. **06** (1999), 030.

[44] Seiberg, N., Witten, E.: *String Theory and Noncommutative Geometry.* J. High. Energy Phys. **09** (1999), 032.

[45] Swan, R. G.: *Vector bundles and projective modules.* Trans. Amer. Math. Soc. **105** (1962), 264–277.

[46] Szabo, R.: *Quantum Field Theory on Noncommutative Spaces.* Preprint **hep-th/0109162** (September 2001), 111 pages. To appear in Phys. Rep.

[47] Waldmann, S.: *Locality in GNS Representations of Deformation Quantization.* Commun. Math. Phys. **210** (2000), 467–495.

[48] Waldmann, S.: *Deformation of Hermitian Vector Bundles and Non-Commutative Field Theory.* In: Maeda, Y., Watamura, S. (eds.): *Noncommutative Geometry and String Theory*, vol. 144 in *Prog. Theo. Phys. Suppl.*, 167–175. Yukawa Institute for Theoretical Physics, 2001. Proceedings of the International Workshop on Noncommutative Geometry and String Theory.

[49] Waldmann, S.: *Morita equivalence of Fedosov star products and deformed Hermitian vector bundles.* Lett. Math. Phys. **60** (2002), 157–170.

[50] Waldmann, S.: *On the representation theory of deformation quantization.* In: Halbout, G. (eds.): *Deformation quantization.* [31], 107–133.

[51] Weinstein, A.: *Deformation Quantization.* Séminaire Bourbaki 46ème année **789** (1994).

[52] Weinstein, A., Xu, P.: *Hochschild cohomology and characteristic classes for star-products.* In: Khovanskij, A., Varchenko, A., Vassiliev, V. (eds.): *Geometry of differential equations. Dedicated to V. I. Arnold on the occasion of his 60th birthday*, 177–194. American Mathematical Society, Providence, 1998.