Background fields contribution to space-time geometry *

B. Sazdović†
Institute of Physics, 11001 Belgrade, P.O.Box 57, Serbia

Abstract

We study the space-time geometry felt by probe bosonic string moving in antisymmetric and dilaton background fields. This space-time geometry we shall call the stringy geometry. In particular, the presence of the antisymmetric field leads to the space-time torsion, and the presence of the dilaton field leads to the space-time non-metricity. The field equations produce the equality of mean extrinsic curvature and dual mean extrinsic curvature, which we call C-duality. The presence of the dilaton field, breaks conformal invariance, and determine the factor between intrinsic and induced metrics. We also derive the integration measure for the space-time with stringy non-metricity.

1 Bosonic string in background fields

It is well known that the classical test particle, in external gravitational field, follows geodesic line. In the present article we will try to understand: how test string (as an extended object) feels the external metric, antisymmetric and dilaton fields? The space-time with the features recognized by the string, we shall call the stringy space-time. Its properties have been studied in more details in ref. [1].

Let us start with the particle string comparison in the metric background $G_{\mu\nu}$ (for details see [2]). The particle action is proportional to the length of the world-line (Fig.1)

$$S = -m \int_{s_i}^{s_f} ds = \int_{\tau_i}^{\tau_f} d\tau \mathcal{L} \,. \tag{1.1}$$

*Work supported in part by the Serbian Ministry of Science, Technology and Development under contract No. 1486.

†e-mail address: sazdovic@phy.bg.ac.yu

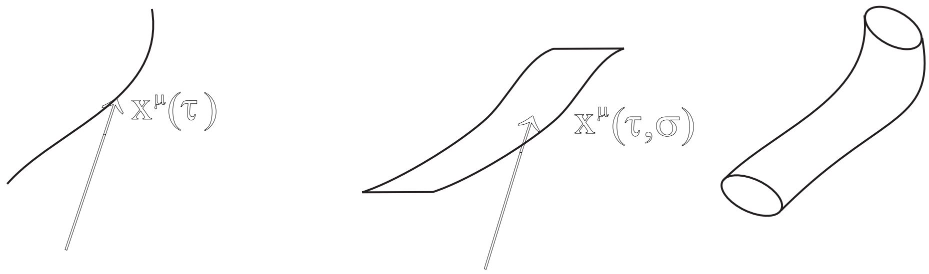

Figure 1: *World-line and world-sheet.*

Here $ds^2 = G_{\mu\nu} dx^\mu dx^\nu$ is space-time interval measured by metric tensor $G_{\mu\nu}$. By $x^\mu(\tau)$, $(\mu, \nu = 0, 1, ..., D-1)$, we denote the position of the particle and by τ a parameter along the particle trajectory.

The string is one dimensional elementary object. It exists in two topologies: open and closed. During the evolution in D dimensional Minkowski space-time the string spans the two dimensional surface (world-sheet) (Fig.1). In analogy with the particle case, the string action is proportional to the area of the world-sheet Σ

$$S = \kappa \int_\Sigma d^2 A = \int_\Sigma d^2\xi \mathcal{L} \,. \tag{1.2}$$

The world-sheet is parameterized by the coordinated $\xi^\alpha = (\xi^0 = \tau, \xi^1 = \sigma)$, so that the string position is described by D dimensional space-time coordinates $x^\mu(\xi)\, (\mu = 0, 1, ..., D-1)$.

The world-sheet interval can be written in terms of space-time metric tensor $G_{\mu\nu}(x)$

$$ds^2 = G_{\mu\nu}(x) dx^\mu dx^\nu = G_{\mu\nu}(x) \partial_\alpha x^\mu \partial_\beta x^\nu d\xi^\alpha d\xi^\beta \,, \tag{1.3}$$

where $\partial_\alpha \equiv \frac{\partial}{\partial \xi^\alpha}$. The world-sheet **induced metric tensor**

$$G_{\alpha\beta}(x) = G_{\mu\nu}(x) \partial_\alpha x^\mu \partial_\beta x^\nu \,, \tag{1.4}$$

is defined by the requirement that it gives the same interval as the target space metric. Then the string action obtains the Nambu-Goto form

$$S_{NG} = -\kappa \int_\Sigma d^2\xi \sqrt{-G_2} \,. \qquad (G_2 \equiv det G_{\alpha\beta}) \tag{1.5}$$

This action can be simplified by introducing an intrinsic world-sheet metric $g^{\alpha\beta}$, as a auxiliary variable. The result is the Polyakov form of the action

$$S_P = \frac{\kappa}{2} \int_\Sigma d^2\xi \sqrt{-g} g^{\alpha\beta} \partial_\alpha x^\mu \partial_\beta x^\nu G_{\mu\nu}(x) \,. \tag{1.6}$$

The equation of motion for the variable $g^{\alpha\beta}$ leads to the vanishing of the energy-momentum tensor, $T_{\alpha\beta} = G_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} g^{\gamma\delta} G_{\gamma\delta} = 0$. It is easy to check that

$$\lambda g_{\alpha\beta} = G_{\alpha\beta}\,, \qquad \frac{g_{\alpha\beta}}{\sqrt{-g}} = \frac{G_{\alpha\beta}}{\sqrt{-G_2}}\,, \tag{1.7}$$

is a solution of this field equation, so that the intrinsic world-sheet metric, up to the conformal factor, is on-shell equivalent to the induced world-sheet metric.

It is possible to include the other modes in the string action. With the requirement of the reparametrization invariance, we can add the terms with no more then two world-sheet derivatives. We obtain the action for the bosonic string propagating in the non-trivial background

$$S = \kappa \int_\Sigma d^2\xi \sqrt{-g} \left\{ \left[\frac{1}{2} g^{\alpha\beta} G_{\mu\nu}(x) + \frac{\varepsilon^{\alpha\beta}}{\sqrt{-g}} B_{\mu\nu}(x) \right] \partial_\alpha x^\mu \partial_\beta x^\nu + \Phi(x) R^{(2)} \right\}, \tag{1.8}$$

which has been investigated in the literature [3]. Here $R^{(2)}$ is two dimensional scalar curvature. The background gravitational field $G_{\mu\nu}$, antisymmetric tensor field $B_{\mu\nu} = -B_{\nu\mu}$ and dilaton field Φ, depend on the coordinates x^μ.

For the case when gradient of the dilaton field, $a_\mu \equiv \partial_\mu \Phi$, is not light-cone vector, in ref. [1] we derived the following equations of motion

$$[J^\mu] \equiv \nabla_\mp \partial_\pm x^\mu + {}^\star\Gamma^\mu_{\mp\rho\sigma} \partial_\pm x^\rho \partial_\mp x^\sigma = 0\,, \tag{1.9}$$

$$[h^\pm] \equiv G_{\mu\nu} \partial_\pm x^\mu \partial_\pm x^\nu - 2\nabla_\pm \partial_\pm \Phi = 0\,, \tag{1.10}$$

$$[i^F] \equiv R^{(2)} + \frac{2}{a^2} (D_{\mp\mu} a_\nu) \partial_\pm x^\nu \partial_\mp x^\mu = 0\,. \tag{1.11}$$

In the tangent basis, the light-cone components of the arbitrary vector V_α have the form $V_\pm = \frac{\sqrt{2} e^{-F}}{h^- - h^+}(V_0 + h^\mp V_1)$, (see [6]). The expression

$${}^\star\Gamma^\rho_{\pm\nu\mu} = \Gamma^\rho_{\nu\mu} \pm {P^T}^\rho{}_\sigma B^\sigma_{\nu\mu} + \frac{a^\rho}{a^2} D_\mu a_\nu\,, \tag{1.12}$$

which appears in the $[J^\mu]$ equation, transforms as a connection under space-time general coordinate transformations. Here, $\Gamma^\rho_{\nu\mu}$ is the Christoffel connection, D_μ is the corresponding covariant derivative, $B_{\mu\nu\rho} = D_\mu B_{\nu\rho} + D_\nu B_{\rho\mu} + D_\rho B_{\mu\nu}$ is field strength of the antisymmetric tensor and ${P^T}_{\mu\nu} = G_{\mu\nu} - \frac{a_\mu a_\nu}{a^2}$ is projection operator. The world-sheet covariant derivative we denoted by $\nabla_\mp$.

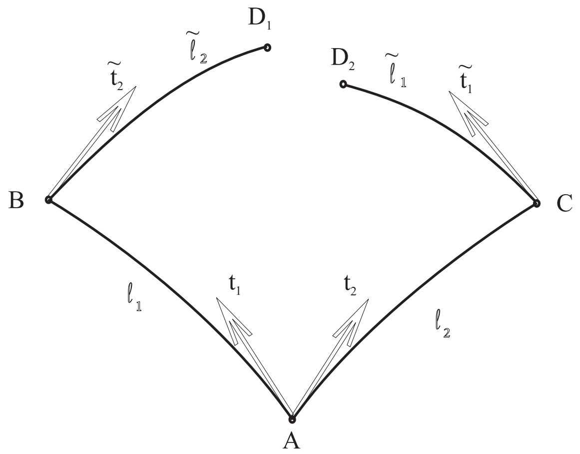

Figure 2: *Geometrical meaning of the torsion.*

2 Geometry of space-time with torsion and non-metricity

Let us introduce some basic elements of the affine space-time (see ref. [5] for more details). In order to compare the vectors from different points we need the rule for vector parallel transport $V^\mu(x) \to {}^\circ V^\mu_{||} = V^\mu + {}^\circ\delta V^\mu$, where

$$ {}^\circ\delta V^\mu = -{}^\circ\Gamma^\mu_{\rho\sigma} V^\rho dx^\sigma \,. \tag{2.1}$$

The variable ${}^\circ\Gamma^\mu_{\rho\sigma}$ is the **affine linear connection**. From the invariance of the vector product under parallel transport, ${}^\circ\delta(V^\mu U_\mu) = 0$, we obtain the rule for the parallel transport of the covector $U_\mu(x)$, ${}^\circ\delta U_\mu = {}^\circ\Gamma^\rho_{\mu\nu} U_\rho dx^\nu$. The covariant derivative is defined as

$$ {}^\circ DV^\mu = V^\mu(x+dx) - {}^\circ V^\mu_{||} = dV^\mu - {}^\circ\delta V^\mu \equiv {}^\circ D_\nu V^\mu dx^\nu \,, \tag{2.2}$$

where ${}^\circ D_\nu V^\mu = \partial_\nu V^\mu + {}^\circ\Gamma^\mu_{\rho\nu} V^\rho$.

The connection is not a tensor, but its antisymmetric part

$$ {}^\circ T^\rho_{\mu\nu} = {}^\circ\Gamma^\rho_{\mu\nu} - {}^\circ\Gamma^\rho_{\nu\mu} \,, \tag{2.3}$$

transforms as a tensor, and define the **torsion**. The geometrical meaning of the torsion is illustrated in the Fig.2. Here, $\tilde{t}^\mu_1(\tilde{t}^\mu_2)$ are the positions of the tangent vectors $t^\mu_1(t^\mu_2)$ after parallel transport along the geodesics $\ell_2(\ell_1)$, respectively. We also have $AB \equiv d\ell_1 = d\tilde{\ell}_1 \equiv CD_2$ and $AC \equiv d\ell_2 = d\tilde{\ell}_2 \equiv$

BD_1. In fact *the torsion measures the non-closure of the curved rectangle ABCD*

$$x^\mu(D_2) - x^\mu(D_1) = {}^\circ T^\mu{}_{\rho\sigma}\, t_1^\rho\, t_2^\sigma\, d\ell_1\, d\ell_2\,. \qquad (2.4)$$

The **metric tensor** $G_{\mu\nu}$ and the connection ${}^\circ\Gamma^\rho_{\mu\nu}$ are independent variables. Usually, the following compatibility between the metric and connection is imposed: after parallel transport the metric is equal to the local one, which is equivalent to the statement that the metric is covariantly constant, ${}^\circ DG_{\mu\nu} = 0$. This requirement, known as a metric postulate, we will not accept here.

Let us compare the square of the vector V^μ at the point x, $V^2(x) = G_{\mu\nu}(x)V^\mu(x)V^\nu(x)$, with its parallel transport to the point $x+dx$, ${}^\circ V_{||}^2(x+dx) = G_{\mu\nu}(x+dx){}^\circ V_{||}^\mu\, {}^\circ V_{||}^\nu$. Because the scalar product is invariant under parallel transport, we have

$${}^\circ\delta V^2 = {}^\circ V_{||}^2(x+dx) - V^2(x) = {}^\circ DG_{\mu\nu}V^\mu V^\nu \equiv -dx^\rho\, {}^\circ Q_{\rho\mu\nu}V^\mu V^\nu\,, \quad (2.5)$$

where we introduced the **nonmetricity** as a covariant derivative of the metric tensor

$${}^\circ Q_{\mu\rho\sigma} = -{}^\circ D_\mu G_{\rho\sigma}\,. \qquad (2.6)$$

The nonmetricity measures the deformation of lengths and angles during the parallel transport. In other words it *measures the breaking of the metric postulate.*

Following ref. [5], we can decomposed the connection ${}^\circ\Gamma^\mu_{\nu\rho}$ in terms of the Christoffel connection, contortion and nonmetricity

$${}^\circ\Gamma_{\mu,\rho\sigma} = \Gamma_{\mu,\rho\sigma} + {}^\circ K_{\mu\rho\sigma} + \frac{1}{2}{}^\circ Q_{\{\mu\rho\sigma\}}\,. \qquad (2.7)$$

With the help of the Schouten braces, $\{\mu\rho\sigma\} = \sigma\mu\rho + \rho\sigma\mu - \mu\rho\sigma$, the Christoffel connection can be expressed as $\Gamma_{\mu,\rho\sigma} = \frac{1}{2}\partial_{\{\mu}G_{\rho\sigma\}}$ and the contortion ${}^\circ K_{\mu\rho\sigma}$ is defined in terms of the torsion

$${}^\circ K_{\mu\rho\sigma} = \frac{1}{2}{}^\circ T_{\{\sigma\mu\rho\}} = \frac{1}{2}({}^\circ T_{\rho\sigma\mu} + {}^\circ T_{\mu\rho\sigma} - {}^\circ T_{\sigma\mu\rho})\,. \qquad (2.8)$$

The first and third terms in (2.7) are symmetric in ρ,σ indices. In the second term we can separate symmetric and antisymmetric parts, ${}^\circ K_{\mu\rho\sigma} = {}^\circ K_{\mu(\rho\sigma)} + \frac{1}{2}{}^\circ T_{\mu\rho\sigma}$ which produces

$${}^\circ\Gamma^\mu_{\rho\sigma} = {}^\circ\Gamma^\mu_{(\rho\sigma)} + \frac{1}{2}{}^\circ T^\mu_{\rho\sigma}\,. \qquad (2.9)$$

Let us apply this general considerations to the string case. Instead of general mark $\circ$, the mark $\star$ will indicates the presence of dilaton field Φ, and lower indices $\pm$ will indicate the presence of the antisymmetric field $B_{\mu\nu}$ in the corresponding two forms of connections (1.12).

The **stringy torsion** and the **stringy nonmetricity** take the forms

$$
{}^\star T_{\pm}{}^{\rho}_{\mu\nu} = \pm 2 P^{T\rho}{}_{\sigma} B^{\sigma}_{\mu\nu} \, , \qquad {}^\star Q_{\pm\mu\rho\sigma} = \frac{1}{a^2} D_{\pm\mu} (a_\rho a_\sigma) \, . \tag{2.10}
$$

So, in the presence of the dilaton field Φ, the metric $G_{\mu\nu}$ is not compatible with the stringy connection ${}^\star\Gamma^{\mu}_{\pm\nu\rho}$, and during stringy parallel transport, the lengths and angles deformations depend on the vector field a_μ.

The stringy Weyl vector, ${}^\star Q_\mu = \frac{1}{D} G^{\rho\sigma\star} Q_{\pm\mu\rho\sigma} = \frac{-4}{D}\partial_\mu \varphi$, is a gradient of the new scalar field

$$
\varphi = -\frac{1}{4} \ln a^2 = -\frac{1}{4} \ln (G^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi) \, . \tag{2.11}
$$

3 Induced and extrinsic geometry

The geometry of world-sheet embedded in curved space-time described in [4], has been generalized for the space-times with nontrivial torsion and nonmetricity in ref. [1]. We will shortly repeat the main results.

We will use the local space-time basis, relating with the coordinate one by the vielbein $E^\mu_A = \{\partial_\alpha x^\mu, n^\mu_i\}$. Here, $\partial_\alpha x^\mu = \{\dot{x}^\mu, x'^\mu\}$ is local world-sheet basis and n^μ_i $(i = 2, 3, ..., D-1)$ are local unit vectors, normal to the world-sheet.

The world-sheet and the orthogonal projections of the arbitrary space-time covector V_μ, we will denote as $v_\alpha = \partial_\alpha x^\mu V_\mu$ and $v_i = n^\mu_i V_\mu$. The tangent and normal vectors to the world-sheet Σ, have the forms $V^\mu_\Sigma = \partial_\alpha x^\mu v^\alpha$ and $V^\mu_\perp = n^\mu_i v^i$.

After parallel transport of the covector V^Σ_μ, along world-sheet line from the point ξ^α to the point $\xi^\alpha + d\xi^\alpha$, (Fig. 3), we obtain the covector ${}^\circ V^\Sigma_{\|\mu} = V^\Sigma_\mu + {}^\circ\delta V^\Sigma_\mu$, where

$$
{}^\circ\delta V^\Sigma_\mu = {}^\circ\Gamma^\nu_{\mu\rho} V^\Sigma_\nu dx^\rho = {}^\circ\Gamma^\nu_{\mu\rho} V^\Sigma_\nu \partial_\gamma x^\rho d\xi^\gamma \, . \tag{3.1}
$$

Its world-sheet projection at the point $\xi + d\xi$, has the form ${}^\circ v^\Sigma_{\|\alpha} = \partial_\alpha x^\mu (\xi + d\xi) {}^\circ V^\Sigma_{\|\mu}$. The **induced connection** ${}^\circ\Gamma^\beta_{\alpha\gamma}$ defines the rule for the parallel transport of the world-sheet covector v_α, along the same world-sheet line, to the projection ${}^\circ v^\Sigma_{\|\alpha}$. So, we have ${}^\circ v^\Sigma_{\|\alpha} = v_\alpha + {}^\circ\delta v_\alpha$, where

$$
{}^\circ\delta v_\alpha = {}^\circ\Gamma^\beta_{\alpha\gamma} v_\beta d\xi^\gamma \, . \tag{3.2}
$$

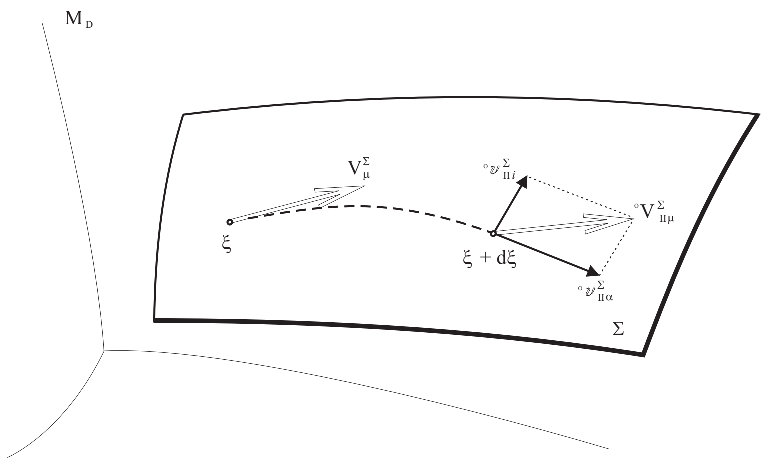

Figure 3: *The parallel transport of the world-sheet tangent covector.*

It produce the expression for the induced connection

$$^\circ\Gamma^{\gamma}_{\alpha\beta} = G^{\gamma\delta}\partial_\delta x^\mu G_{\mu\nu}{}^\circ D_\beta \partial_\alpha x^\nu \,, \tag{3.3}$$

where $^\circ D_\alpha V^\mu = \partial_\alpha x^{\nu\circ} D_\nu V^\mu$ is space-time covariant derivative along world-sheet direction.

The orthogonal projection of the covector $^\circ V^\Sigma_{\|\mu}$

$$n^\mu_i(\xi + d\xi)^\circ V^\Sigma_{\|\mu} \equiv {}^\circ v^\Sigma_{\|i} = -{}^\circ\bar{b}_{i\alpha\beta} v^\alpha d\xi^\beta \,, \tag{3.4}$$

is proportional to the **second fundamental form** (SFF)

$$^\circ\bar{b}_{i\alpha\beta} = n^\mu_i\, {}^\circ D_\beta (G_{\mu\nu}\partial_\alpha x^\nu) = -\partial_\alpha x^\nu G_{\mu\nu}{}^\circ D_\beta n^\mu_i \,, \tag{3.5}$$

which define the extrinsic geometry.

The parallel transport of the covector $V^\perp_\mu$, orthogonal to the world-sheet (Fig.4), produces $^\circ V^\perp_{\|\mu} = V^\perp_\mu + {}^\circ\delta V^\perp_\mu$, where $^\circ\delta V^\perp_\mu = {}^\circ\Gamma^\nu_{\mu\rho} V^\perp_\nu dx^\rho = {}^\circ\Gamma^\nu_{\mu\rho} V^\perp_\nu \partial_\alpha x^\rho d\xi^\alpha$. Its world-sheet projection

$$\partial_\alpha x^\mu(\xi + d\xi)^\circ V^\perp_{\|\mu} \equiv {}^\circ v^\perp_{\|\alpha} = v^i\, {}^\circ b_{i\alpha\beta} d\xi^\beta \,, \tag{3.6}$$

defines new SFF

$$^\circ b_{i\alpha\beta} = n^\mu_i G_{\mu\nu}{}^\circ D_\beta \partial_\alpha x^\nu = -\partial_\alpha x^{\nu\circ} D_\beta (G_{\mu\nu} n^\mu_i) \,, \tag{3.7}$$

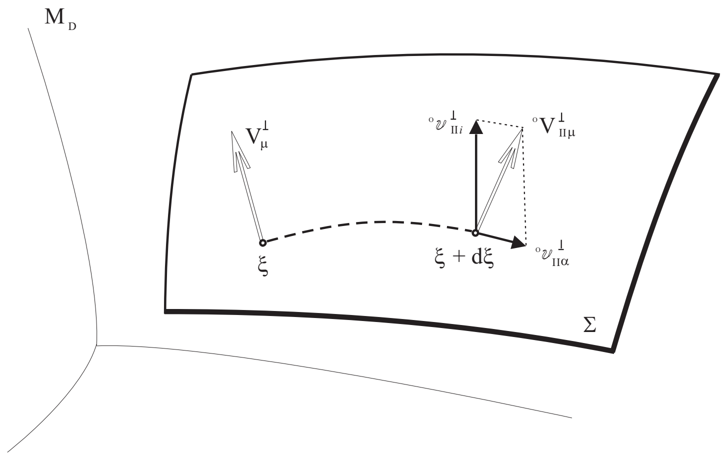

Figure 4: *The parallel transport of the world-sheet orthogonal covector.*

which differs from the previous one (3.5) by the term containing the covariant derivative of the metric. So, in the spaces with nonmetricity, there are two forms of SFF, connected by the relation

$$^\circ b_{i\alpha\beta} = {}^\circ\bar{b}_{i\alpha\beta} + {}^\circ Q_{\rho\mu\nu}\partial_\beta x^\rho n_i^\mu \partial_\alpha x^\nu \,. \tag{3.8}$$

Similarly, there are also two forms of the world-sheet induced connections and world-sheet induced covariant derivatives. We will use the variables without bar, because the bar variables can be expressed in terms of the first ones and nonmetricity.

The induced connection and the SFF are projections of the space-time covariant derivative of the world-sheet basis vectors $\partial_\alpha x^\mu$, to the world-sheet and its normal, respectively. Consequently, we have the generalization of the Gauss-Weingarten equation

$$^\circ D_\beta \partial_\alpha x^\mu = {}^\circ\Gamma^\gamma_{\alpha\beta}\partial_\gamma x^\mu + {}^\circ b^i_{\alpha\beta}\, n_i^\mu \,. \tag{3.9}$$

In analogy with the general rule for connection decomposition, we can decompose the induced connection

$$^\circ\Gamma_{\gamma,\alpha\beta} = \Gamma_{\gamma,\alpha\beta} + {}^\circ K_{\gamma\alpha\beta} + \frac{1}{2}{}^\circ Q_{\{\gamma\alpha\beta\}} \,, \tag{3.10}$$

in terms of induced Christoffel connection $\Gamma^\gamma_{\alpha\beta} = G^{\gamma\delta}\partial_\delta x^\mu\, G_{\mu\nu}{}^\circ D_\beta \partial_\alpha x^\nu$, **induced torsion** and **induced nonmetricity**

$$^\circ T_{\alpha\beta\gamma} = {}^\circ T_{\mu\rho\sigma}\partial_\alpha x^\mu \partial_\beta x^\rho \partial_\gamma x^\sigma = {}^\circ\Gamma_{\alpha,\beta\gamma} - {}^\circ\Gamma_{\alpha,\gamma\beta} \,, \tag{3.11}$$

$$
^\circ Q_{\alpha\beta\gamma} = {}^\circ Q_{\mu\rho\sigma}\partial_\alpha x^\mu \partial_\beta x^\rho \partial_\gamma x^\sigma = -{}^\circ\nabla_\alpha G_{\beta\gamma}\,, \tag{3.12}
$$

where ${}^\circ K_{\gamma\alpha\beta} = \frac{1}{2}{}^\circ T_{\{\gamma\alpha\beta\}}$ is the induced contortion. Similarly, we can decompose the SFF

$$
{}^\circ b_{i\alpha\beta} = b_{i\alpha\beta} + {}^\circ K_{i\alpha\beta} + \tfrac{1}{2}{}^\circ Q_{\{i\alpha\beta\}}\,, \tag{3.13}
$$

where

$$
b_{i\alpha\beta} = n_i^\mu G_{\mu\nu} D_\beta \partial_\alpha x^\nu\,, \qquad {}^\circ K_{i\alpha\beta} = \tfrac{1}{2}{}^\circ T_{\{i\alpha\beta\}}\,, \tag{3.14}
$$

$$
{}^\circ T_{i\alpha\beta} = {}^\circ T_{\mu\rho\sigma} n_i^\mu \partial_\alpha x^\rho \partial_\beta x^\sigma\,, \qquad {}^\circ Q_{i\alpha\beta} = {}^\circ Q_{\mu\rho\sigma} n_i^\mu \partial_\alpha x^\rho \partial_\beta x^\sigma\,. \tag{3.15}
$$

When the torsion is present, the SFF is not symmetric in α, β indices. Similarly as in the case of the connection, we can write

$$
{}^\circ b_{i\alpha\beta} = {}^\circ b_{i(\alpha\beta)} + \tfrac{1}{2}{}^\circ T_{i\alpha\beta}\,, \qquad {}^\circ b_{i(\alpha\beta)} = b_{i\alpha\beta} + {}^\circ K_{i(\alpha\beta)} + \tfrac{1}{2}{}^\circ Q_{\{i\alpha\beta\}}\,. \tag{3.16}
$$

4 C-duality

In the torsion free case, the SFF is symmetric in the world-sheet indices and its properties are well known in the literature. Here, we will investigate the features of the SFF, when nontrivial torsion and nonmetricity are present.

The curvature of the normal section ℓ_i, as a space-time curve, is the orthogonal projection of the covariant derivative along the curve ${}^\circ k_i = {}^\circ D_s t^\mu G_{\mu\nu} n_i^\nu$, or explicitly

$$
{}^\circ k_i = {}^\circ b_{i\beta\alpha}\, t^\alpha t^\beta = {}^\circ b_{i(\beta\alpha)}\, t^\alpha t^\beta = \frac{{}^\circ b_{i(\beta\alpha)} d\xi^\alpha d\xi^\beta}{G_{\alpha\beta} d\xi^\alpha d\xi^\beta}\,. \tag{4.1}
$$

It depends on the direction of the world-sheet tangent vector t^α and only on the symmetric part of the SFF.

The eigenvalues ${}^\circ\kappa^i$, of the quadratic forms ${}^\circ b^i_{\alpha\beta}$ with respect to the metric $G_{\alpha\beta}$, we define as a principal curvatures in Minkowski space-time. The condition for nontrivial solutions, $\det({}^\circ b^i_{\alpha\beta} - {}^\circ\kappa^i G_{\alpha\beta}) = 0$, produces the expression ${}^\circ\kappa^i_{0,1} = {}^\circ H^i \pm \sqrt{({}^\circ H^i)^2 - {}^\circ K^i}$. Here

$$
{}^\circ H^i = \tfrac{1}{2} G^{\alpha\beta}\,{}^\circ b^i_{\alpha\beta} = \tfrac{1}{2}({}^\circ\kappa^i_0 + {}^\circ\kappa^i_1)\,, \tag{4.2}
$$

is the trace of the SFF and ${}^\circ K^i = \frac{\det {}^\circ b^i_{\alpha\beta}}{\det G_{\alpha\beta}} = {}^\circ\kappa^i_0\,{}^\circ\kappa^i_1$ (no summation over i) is Gauss curvature.

In the frame when both symmetric matrices, $G_{\alpha\beta}$ and ${}^\circ b_{i(\alpha\beta)}$, are diagonal, the curvature of the normal section ℓ_i becomes ${}^\circ k^i_{tl}(\alpha) = {}^\circ\kappa^i_0 \cosh^2(\alpha) -$

${}^\circ\kappa_1^i \sinh^2(\alpha)$, where $\alpha \in (-\infty, \infty)$ parameterize all time-like vectors. The i-th mean extrinsic curvature (MEC) is mean value of the i-th normal section curvature, with fixed normal n_i^μ. It is well defined in Euclidean spaces. In Minkowski spaces, the curvature (4.1) is divergent in the light-like directions. For the time-like region we define regularized mean curvature

$$ {}^\circ H_{tl}^i(\Lambda) = \frac{1}{2\Lambda}\int_{-\Lambda}^{\Lambda} {}^\circ k_{tl}^i(\alpha) d\alpha \,. \tag{4.3} $$

For the space-like normal section we have ${}^\circ k_{sl}^i(\alpha) = {}^\circ\kappa_0^i \sinh^2(\alpha) - {}^\circ\kappa_1^i \cosh^2(\alpha)$ and similar expression as (4.3).

The **mean extrinsic curvature** we define as a mean value of the time-like mean curvature with sign $+$, and space-like mean curvature with sign $-$, with the same cut-off Λ going to the infinity

$$ \lim_{\Lambda\to\infty} \tfrac{1}{2}[{}^\circ H_{tl}^i(\Lambda) - {}^\circ H_{sl}^i(\Lambda)] = \tfrac{1}{2}({}^\circ\kappa_0^i + {}^\circ\kappa_1^i) = {}^\circ H^i \,. \tag{4.4} $$

Because the Λ dependence disappears, we obtain just the expression (4.2).

For ${}^\circ\kappa_0^i = {}^\circ\kappa_1^i \equiv {}^\circ\kappa^i$, the line curvature does not depend on the tangent vector direction, ${}^\circ k_{tl}^i(\alpha) = {}^\circ\kappa^i = {}^\circ H_{tl}^i$, and ${}^\circ k_{sl}^i(\alpha) = -{}^\circ\kappa^i = {}^\circ H_{sl}^i$. Consequently, the MEC has the same interpretation $\frac{1}{2}({}^\circ H_{tl}^i - {}^\circ H_{sl}^i) = {}^\circ\kappa^i = {}^\circ H^i$.

The surface defined by the equation ${}^\circ H^i = 0$ is the **minimal surface**.

The above considerations are torsion independent, because the antisymmetric part of the SFF disappears from (4.1) and (4.2). Following [1], we are going to investigate the case when the torsion is present.

Let us first generalize the eigenvalue problem. We introduce the **dual eigenvalue** problem, such that linear transformation of the vector v^α with operator ${}^\circ b_{\alpha\beta}^i$ is proportional to the two dimensional dual vector ${}^*v_\alpha = \sqrt{-G_2}\,\varepsilon_{\alpha\beta}\, v^\beta$

$$ {}^\circ b_{\alpha\beta}^i\, v^\beta = {}^*\kappa^i\, {}^*v_\alpha \,. \tag{4.5} $$

The dual eigenvalues, ${}^*\kappa_{0,1}^i = {}^*H^i \pm \sqrt{({}^*H^i)^2 + {}^\circ K^i}$, are solutions of the condition $\det({}^\circ b^i - {}^*k^i\, \varepsilon\sqrt{-G_2})_{\alpha\beta} = 0$. In analogy with the previous case, we will call them dual principal curvatures, and the variable

$$ {}^*H^i = \tfrac{1}{2}({}^*\kappa_0^i + {}^*\kappa_1^i) = \tfrac{1}{2} G^{\alpha\beta}\, {}^*b_{\alpha\beta}^i = \tfrac{1}{2}\frac{\varepsilon^{\alpha\beta}}{\sqrt{-G_2}}\, {}^\circ b_{\beta\alpha}^i = \tfrac{1}{4}\frac{\varepsilon^{\alpha\beta}}{\sqrt{-G_2}}\, {}^\circ T_{\beta\alpha}^i \,, \tag{4.6} $$

the **dual mean extrinsic curvature** (DMEC). Here ${}^\circ K^i = \frac{\det {}^\circ b_{\alpha\beta}^i}{\det G_{\alpha\beta}} = {}^*\kappa_0^i\, {}^*\kappa_1^i$ (no summation over i) is the same Gauss curvature as before.

Let us now turn to the geometrical meaning of the DMEC. In the case when t_1^μ and t_2^μ are world-sheet tangent vectors, (Fif.2), we can rewrite (2.4) in the form

$$ {}^\circ T^\mu \equiv \frac{x^\mu(D_2) - x^\mu(D_1)}{2dP_{12}} = \frac{\varepsilon^{\beta\alpha}}{4\sqrt{-G_2}}\, {}^\circ T^\mu_{\rho\sigma}\partial_\alpha x^\rho \partial_\beta x^\sigma \,. \tag{4.7} $$

Here, $dP_{12} = \sqrt{-G_2}\det(\frac{\partial \xi^\alpha}{\partial s_r})d\ell_1 d\ell_2$ is area of the parallelogram, spanned by the vectors $d\ell_1^\mu = t_1^\mu d\ell_1$ and $d\ell_2^\mu = t_2^\mu d\ell_2$. The variable ${}^\circ T^\mu$ does not depend on the directions t_1^μ and t_2^μ, and on the lengths $d\ell_1$ and $d\ell_2$. So, we will call it the **mean torsion**. Its normal projection, the **extrinsic mean torsion**

$$ {}^\circ T_i = {}^\circ T^\mu G_{\mu\nu} n_i^\mu = \frac{\varepsilon^{\beta\alpha}}{4\sqrt{-G_2}}\, {}^\circ T_{i\alpha\beta} = {}^* H_i \,, \tag{4.8} $$

is exactly the same variable as DMEC, defined above in (4.6).

We can reformulate the dual eigenvalue problem (4.5), as an ordinary eigenvalue problem $({}^* b^i_{\alpha\beta} - {}^*\kappa^i G_{\alpha\beta})v^\alpha = 0$, with the **dual SFF**

$$ {}^* b^i_{\alpha\beta} = \frac{G_{\alpha\gamma}\varepsilon^{\gamma\delta}}{\sqrt{-G_2}}\, {}^\circ b^i_{\delta\beta} \,. \tag{4.9} $$

We define the **C-duality** (**C**urvature duality), which maps SFF to dual SFF, ${}^\circ b^i_{\alpha\beta} \to {}^* b^i_{\alpha\beta}$, and interchanges the roles of the symmetric and the antisymmetric parts of the SFF. Consequently, under C-duality MEC maps to DMEC, allowing the exchange of the mean curvature and mean torsion.

The self-dual and self-antidual configurations correspond to the conditions on the SFF

$$ {}^\circ H^i = \pm {}^* H^i \,, \qquad \Longleftrightarrow \qquad (G^{\alpha\beta} \mp \frac{\varepsilon^{\alpha\beta}}{\sqrt{-G_2}})\, {}^\circ b^i_{\beta\alpha} = 0 \,. \tag{4.10} $$

The last equations define **C-dual (antidual) surfaces**. In the torsion free case, they turn to the standard minimal surface, ${}^\circ H^i = 0$.

5 Background fields contribution to the space-time geometry

In this section we are going to analyze the field equations (1.9)-(1.11), and make the space-time classification, depending on the background fields. We will see, that the probe string feels more space-time features (torsion and nonmetricity) then the probe particle.

Riemann and Stringy Riemann-Cartan space-time. The absence of the dilaton field Φ, leads to the conformal invariance of the action. Consequently, the field F and the corresponding equation $[i^F]$ are absent.

The $[h^\pm]$ equation can be written in the form $G_{\pm\pm} \equiv e^\alpha_\pm e^\beta_\pm G_{\alpha\beta} = 0$. With the help of the relation $g_{\pm\pm} = 0$, it produces (1.7). So, in the case of the conformal invariance, the metric densities are equivalent and relation for the intrinsic connection $\omega^\gamma_{\alpha\beta}$, in terms of the induced one $\Gamma^\gamma_{\alpha\beta}$, $\omega^\gamma_{\alpha\beta} = \Gamma^\gamma_{\alpha\beta} - \delta^\gamma_\alpha \lambda_\beta - \delta^\gamma_\beta \lambda_\alpha + g_{\alpha\beta}\, g^{\gamma\delta}\lambda_\delta$ $(\lambda_\alpha \equiv \frac{1}{2}\partial_\alpha \ln \lambda)$, satisfies the field equations. Therefore, both intrinsic world-sheet metric tensor and intrinsic connection, are equal to the induced ones from the space-time up to the conformal factor.

There are two forms of connection $\Gamma^\rho_{\pm\nu\mu} = \Gamma^\rho_{\nu\mu} \pm B^\rho_{\nu\mu}$ and corresponding two forms of covariant derivative $D_{\pm\mu}$. The torsion $T_\pm{}^\rho_{\mu\nu} = \pm 2B^\rho_{\mu\nu}$, is proportional to the new term in the connection, which is the field strength of the antisymmetric tensor.

The orthogonal projection of the $[J^\mu]$ equation obtains the form

$$(G^{\alpha\beta} \pm \frac{\varepsilon^{\alpha\beta}}{\sqrt{-G_2}}) b_{\pm i\beta\alpha} = 0\,, \qquad \Longleftrightarrow \qquad H_i \pm {}^*H_{\pm i} = 0\,. \tag{5.1}$$

So, MEC is equal to DMEC, which means that world-sheet is *C-dual (antidual) surface*. Consequently, the string with background fields $G_{\mu\nu}$ and $B_{\mu\nu}$ feels the target space as *Riemann-Cartan space-time.*

For $B_{\mu\nu} = 0$, the last equation turns to the condition of *minimal surface*

$$H^i \equiv \tfrac{1}{2} G^{\alpha\beta} b^i_{\beta\alpha} = 0\,, \tag{5.2}$$

where all MECs, vanish. The string with the background field $G_{\mu\nu}$ does not see torsion and nonmetricity. It feels the target space as *Riemann space-time* of general relativity.

Stringy space-time. When we include the dilaton field Φ, the equations of motion are the complete ones, (1.9)-(1.11).

The theory loses conformal invariance and the coefficient λ, from eq. (1.7), is determined. The relations between intrinsic and induced metric tensors and connections are

$$g_{\alpha\beta} = a_2^2 G_{\alpha\beta}\,, \qquad \omega^\gamma_{\alpha\beta} = {}^\star\Gamma^\gamma_{\alpha\beta} - \delta^\gamma_\alpha\lambda_\beta - \delta^\gamma_\beta\lambda_\alpha + G_{\alpha\beta}G^{\gamma\delta}\lambda_\delta - \tfrac{1}{2}{}^\star Q^{\{\gamma}{}_{\alpha\beta\}}\,, \tag{5.3}$$

where $\lambda_\alpha \equiv -\partial_\alpha \ln\sqrt{a_2^2}$ and $a_2^2 = G^{\alpha\beta}a_\alpha a_\beta$ is a length of the world-sheet projection of the vector field a_μ.

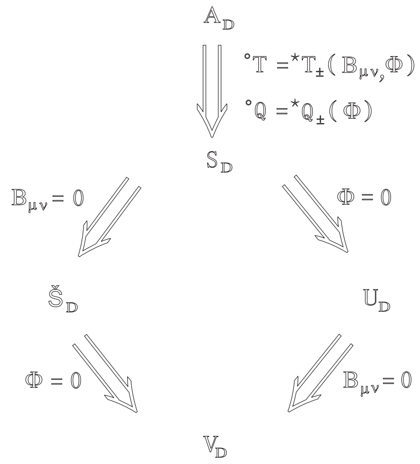

Figure 5: *Classification of space-time.* A_D *is affine space-time,* S_D *is stringy space-time,* $\check{S}_D$ *is stringy torsion free space-time,* U_D *is stringy Riemann-Cartan space-time and* V_D *is Riemann space-time.*

From the equation $[i^F]$ we obtain the expression for the intrinsic and induced world-sheet curvatures

$$R^{(2)} = -\frac{G^{\alpha\beta}}{a_2^4} \stackrel{\Gamma}{\nabla}_\alpha a_\beta \,, \qquad \stackrel{\Gamma}{R}{}^{(2)} = \frac{1}{a_2^4}\Delta\Phi - \Delta \ln a_2^2 \,, \tag{5.4}$$

where $\stackrel{\Gamma}{\nabla}_\alpha$ is world-sheet covariant derivative with induced Christoffel connection and Δ is Laplace operator for the induced metric.

The normal projection of the $[J^\mu]$ equation takes the form

$$\left(G^{\alpha\beta} \pm \frac{\varepsilon^{\alpha\beta}}{\sqrt{-G_2}} \right) {}^\star b_{\pm i\beta\alpha} = 0 \,, \qquad \Longleftrightarrow \qquad {}^\star H_i \pm {}^* H_{\pm i} = 0 \,. \tag{5.5}$$

So, the world-sheet is *stringy C-dual (antidual) surface.* The string propagating in the presence of all three background fields $G_{\mu\nu}$, $B_{\mu\nu}$ and Φ, feels both space-time torsion and nonmetricity. The corresponding target space we call *stringy space-time.* For $\Phi = 0$, eq. (5.5) turns to eq. (5.1). For $B_{\mu\nu} = 0$ it turns to the *stringy minimal surface* condition

$${}^\star H_i = \tfrac{1}{2} G^{\alpha\beta}\, {}^\star b_{i\beta\alpha} = 0 \,, \tag{5.6}$$

because all stringy MECs vanish. The string with background fields $G_{\mu\nu}$ and Φ can see the space-time nonmetricity. The corresponding target space we call *stringy torsion free space-time.*

6 The space-time measure

Let us discuss possible forms of the actions in the stringy space-time. Generally it has the form

$$ {}^\star S = \int d^D x \, {}^\star\Omega \, {}^\star\mathcal{L} \,, \tag{6.1} $$

where ${}^\star\Omega$ is a measure factor, and ${}^\star\mathcal{L}$ is a Lagrangian. We demand that the invariant measure, should satisfy three requirements: R_1: it is invariant under space-time general coordinate transformations; R_2: it is preserved under parallel transport; R_3: it enable integration by parts. The second requirement is equivalent to the condition ${}^\star D_{\pm\mu} {}^\star\Omega = 0$. The third one, is consequence of the Leibniz rule, and the relation

$$ \int d^D x \, {}^\star\Omega \, {}^\star D_{\pm\mu} V^\mu = \int d^D x \partial_\mu ({}^\star\Omega V^\mu) \,, \tag{6.2} $$

so that we are able to use Stoke's theorem.

For spaces with nonmetricity, the standard measure $\Omega = \sqrt{-G}$, is not preserved under the stringy parallel transport, and R_2 and R_3 are not satisfy. We are looking for the stingy measure in the form ${}^\star\Omega = \Lambda(x)\sqrt{-G}$. With the help of R_2 and the fact that stringy Weyl vector ${}^\star Q_\mu$ is a gradient of the scalar field φ, we obtain

$$ {}^\star\Omega = \sqrt{-G} e^{-2\varphi} \,. \tag{6.3} $$

Note that now we have ${}^\star\Gamma^\rho_{\pm\mu\rho} = \partial_\mu \ln {}^\star\Omega$, and consequently (6.2) is satisfied. So, we can use the integration by parts for stringy derivatives ${}^\star D_{\pm\mu}$, if we use the stringy measure ${}^\star\Omega$. Therefore, all requirements are satisfy.

Let us shortly compare the action (6.1) with the action of the ref. [3]. In those papers the classical space-time Lagrangian has been obtained from the world-sheet quantum conformal invariance. It is scheme dependent and can be reproduced with suitable combinations of our invariants: the stringy scalar curvature, stringy torsion and stringy nonmetricity.

We are particularly interested to compare the integration measures. The measure factor of the papers [3], $\sqrt{-G}e^{-2\Phi}$, has the same form as our one and confirm the existence of space-time nonmetricity.

7 Conclusions

In the present article, we are interested in the space-time geometry felt by the bosonic string propagating in the nontrivial background. We defined torsion and nonmetricity and obtained expression for the stringy torsion and stringy nonmetricity, (2.10), originating from the antisymmetric field $B_{\mu\nu}$ and dilaton fields Φ, respectively.

We investigated geometry of the surface embedded into space-time with torsion and nonmetricity. We cleared the meaning of MEC in Minkowski space-time ${}^\circ H^i$ (see (4.2) and (4.4)), and introduced the concept of DMEC ${}^\star H^i$, (4.6), as orthogonal projection of the mean torsion (4.8). We defined C-duality, which maps MEC to DMEC. The torsion changes the equation of embedded surfaces. Instead of the usual minimal surface ${}^\circ H_i = 0$, we introduced C-dual (antidual) surface defined by the self-duality (self-antiduality) conditions, ${}^\circ H_i = \pm {}^* H_i$.

The $[J^\mu]$ equation has D components. Two of them determine the contracted intrinsic connection in terms of the corresponding induced expression. The other $D-2$ are of the form ${}^\star H_i = \pm {}^* H_{\mp i}$, and define world-sheet as a stringy C-dual (antidual) surface. In two particular cases, —the vanishing torsion and the vanishing nonmetricity— the field equations turn to the equations of stringy minimal world-sheet ${}^\star H_i = 0$ and C-dual (antidual) world-sheet $H_i = \pm {}^* H_{\mp i}$, respectively. In the case of Riemann space-time, when both torsion and nonmetricity vanish, they turn to the equations of minimal world-sheet $H_i = 0$. The space-time geometry dependence on the background fields, has been presented in Fig.5.

In the absence of the dilaton field Φ the intrinsic metric and connection are equal to the induced ones, up to conformal transformation. The appearance of the dilaton field Φ broke the compatibility between the space-time metric tensor and stringy connection. It also broke conformal invariance, introducing new component of the intrinsic metric tensor F, and consequently, the new equation of motion $[i^F]$. The relations between the intrinsic and induced variables are fixed (5.3). This new field equation allows us to calculate the induced world-sheet curvature (5.4) as a function of the dilaton field.

We constructed the integration measure for the theories with nonmetricity. In fact, the stringy Weyl vector is a gradient of the scalar field φ, necessary to make the integration measure invariant under parallel transport.

References

[1] B. Sazdović, hep-th/0304085; hep-th/0304086.

[2] M. B. Green, J. H. Schwarz and E. Witten, *Superstring Theory*, Cambridge University Press, 1987.

[3] E. S. Fradkin and A. A. Tseytlin, *Phys.Lett.* **B 158** (1985) 316; *Nucl.Phys.* **B 261** (1985) 1; C. G. Callan, D. Friedan, E. J. Martinec and M. J. Perry, *Nucl.Phys.* **B 262** (1985) 593; T. Banks, D. Nemeschansky and A. Sen, *Nucl.Phys.* **B 277** (1986) 67; A. A. Tseytlin, *Int. J. Mod. Phys.* **A 4** (1989 1257; A. A. Tseytlin, *Int. J. Mod. Phys.* **A 4** (1989 1257.

[4] B. M. Barbashov and V. V. Nesterenko, *Introduction to relativistic string theory*, World Scientific, Singapore, 1990; D. Sorokin, *Phys. Rept.* **329** (2000) 1.

[5] M. Blagojević, *Gravitation and gauge symmetries*, IoP Publishing, Bristol, 2002; F. W. Hehl, J. D. McCrea, E. W. Mielke and Y. Ne'eman, *Phys. Rept.* **258** (1995) 1.

[6] M. Blagojević, D. S. Popović and B. Sazdović, *Mod. Phys. Lett.* **A 13** (1998) 911; *Phys. Rev.* **D 59** (1999) 044021; B. Sazdović, *Phys. Rev.* **D 59** (1999) 084008.