Regularisation and Renormalisation of Quantum Field Theories on Noncommutative Spaces

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Abstract

We first review regularisation methods based on matrix geometry and show how an ultraviolet cut-off for scalar fields respecting symmetries results. Sections of bundles over the sphere can be quantised too. This procedure even allows to regularise supersymmetry without violating it. This work was extended recently to include quantum group covariant regularisations.

In a second part recent attempts to renormalise four-dimensional deformed quantum field theory models are reviewed. For scalar models the well-known UV/IR-mixing does not allow to use standard techniques. The same applies to the Yang-Mills model in four dimensions. Only additional symmetry, as it occurs in the Wess-Zumino model, may allow to avoid this problem.

There was some hope that circumventing UV/IR-mixing by application of the Seiberg-Witten map one can achieve renormalisability of noncommutative field theories. Although the photon self-energy is renormalisable to all orders, θ-expanded noncommutative QED was shown to be not renormalisable, thus ruling out this approach as well.

1 Introduction

More than ten years ago, John Madore introduced what he called the Fuzzy sphere. If you work within a geometry which does not have points, singularities of quantum field theory models may be cured. We applied these ideas first to simple two-dimensional models [1]

Lateron, together with Klimčik and Presnajder we treated all kind of models from scalar fields to spinor and gauge fields, even supersymmetric models were successfully treated. We also dealt with models defined on $CP(2)$, on a cylinder respectively on an hyperboloid. All this was within what is now called the Lie algebra deformation.

Recently, together with John Madore and Harold Steinacker we went to q-deformed models. We obtained a cut-off procedure based on a sequence of embedded Podles spheres and were able to formulate a differential calculus which led to the formulation of gauge models too. This way we connected the Yang-Mills models and the Chern-Simons model to those obtained from string theory.

Despite many attempts, four-dimensional quantum field theory on commutative space-time is still in bad shape. For two-dimensional models constructive methods as well as new algebraic methods led to enormous insights. Even three-dimensional models are quite well understood. In four dimensions we have to rely on renormalised perturbation theory, and attempts to cure the diseases go from unification ideas to adding additional dimensions, strings etc. It is natural therefore to ask for alternatives.

One idea is to change the geometry. The final goal would be to include full quantised gravity, but on the way we may well study quantum field theoretic models on quantised space-time. This led recently to some surprises and new problems have shown up.

There is an old and simple argument that a smooth space-time manifold contradict quantum physics. If one localises an event within a region of extension l an energy of the order hc/l is transfered. This energy generates a gravitational field. A strong gravity field prevents on the other hand signals to reach an observer. If one inserts the energy density to the rhs of Einstein's equation and puts a length r characterising the curvature of space-time we get

$$1/r^2 = (G/c^4)(hc/l)(1/l^3) \ . \tag{1}$$

Next we put the two length scales to be equal, since it is certainly operational impossible to localise an event beyond this resulting Planck length. To the best of our knowledge the first time this argument was put to precise mathematics was in the work by Doplicher, Fredenhagen and Roberts

[2]. They obtained what is now called the canonical deformation but averaged over two-spheres. We believe that this averaging will not improve the divergences problems, although it seems that nobody did do explicit calculations.

We used in our work part of the ideas of noncommutative geometry. There are a number of books giving an overview [3, 4, 5]. We replace first the algebra of functions over a manifold by a noncommutative deformed algebra. Three kinds of deformations are treated. If the commutator of coordinates is put equal to a constant antisymmetric tensor, we call it the canonical deformation. If it is put to be a linear function of coordinates we call it a Lie algebra deformation, and if it is put to be a quadratic expression we call it a quantum group deformation. In the last case the Yang-Baxter equation has to be fulfilled.

Next step concerns the differential calculus. We replace vector fields by derivations on the algebra, we define differential forms by duality. Although it is somewhat tricky to do this for the q-deformed sphere, in all three cases one succeeds. Hodge duality, Lie-derivatives, the Laplace operator etc., in summary most of the steps of ordinary differential geometry can be simulated.

An essential step concerns fields. Classically they are sections of bundles, but also modules over the algebra. This last notion can be taken over to the noncommutative situation. We study finitely generated projective modules over the deformed algebra and are able to quantise scalar, spinor and gauge fields.

Finally we write down actions and integrate over the algebra, so we use certain trace functionals. Next steps concerns cohomology problems, the formulation of the spectral triples with the help of a Dirac operator and the use of cyclic cocyles. The final version of Alain Connes' ideas [3] concerns the spectral action which allows to unify all the interactions within one principle [6].

2 Regularisation of Quantum Field Theory

2.1 Scalar Fields

The ideas of deforming or quantising the commutative algebra of functions over a manifold can best be explained for the simple example of the two-sphere and leads to the Fuzzy Sphere. The euclidean action of a scalar field is given by

$$S[\Phi] = \frac{1}{4\pi} \int_{S^2} d\Omega \; [(J_i\Phi)^2 + \mu^2\Phi^2 + \mathrm{POL}(\Phi)] \tag{2}$$

and is invariant under isometries or rotations of the sphere. The generators J_i are angular momentum operators, they close under $su(2)$. The field Φ can be expanded according to the infinite set of irreducible representations of $su(2)$.

Next we truncate this tower: Consider vector spaces transforming according to the first N representations. They can be identified with mappings from the representation space $N/2$ to itself. These mappings are $(N+1) \times (N+1)$-dimensional matrices, the noncommutative product of these is taken as the product within the algebra. In addition we have to give a precise description of the embedding of these algebras for different N, which gives a precise meaning also to the limit. For details see [7, 8, 9]. The Lie algebra of the generators of this algebra is easy to describe. They form the $su(2)$ Lie algebra with suitable rescaling, such that the Casimir operator still fulfils the defining relation for the two-sphere as an operator:

$$[\hat{X}_i, \hat{X}_j] = \mathrm{i}\lambda\varepsilon_{ijk}\hat{X}_k \ , \qquad \sum_{i=1}^{3} \hat{X}_i^2 = R^2 \ , \tag{3}$$

$$R\lambda^{-1} = \sqrt{\frac{N}{2}\left(\frac{N}{2}+1\right)} \ . \tag{4}$$

Since we work on a matrix algebra it is easy to introduce a differential calculus. All derivations are inner, they are given by the adjoint action with the generators themselves. For the two-sphere we take the generators introduced above and can develop a derivation-based differential calculus. Our aim next is to make sense out of a functional integral of the type

$$\langle F[\Phi] \rangle = \frac{\int D\Phi \ \mathrm{e}^{-S[\Phi]} F[\Phi]}{\int D\Phi \ \mathrm{e}^{-S[\Phi]}} \ . \tag{5}$$

The action (2) will be replaced by

$$S[\phi] = (1/N)\mathrm{Tr}\big([\hat{X}_i, \Phi][\hat{X}_i, \Phi] + \mathrm{POL}(\Phi)\big) \ , \tag{6}$$

where everything depends on N. As for the measure we take just the product measure for the finite number of degrees of freedom. This makes the functional integral well-defined.

In the limit where N tends to infinity, the old divergences show up. Feynman rules can be developed, a tadpole graph for the Φ^4 model diverges logarithmically in the limit.

In the same spirit it is possible to quantise Kähler manifolds like $CP(n)$, models on the cylinder and on hyperboloids have been treated too.

In addition it is possible to quantise sections of line bundles over S^2, which are characterised by the Chern number. This way we construct projective modules which lead in a certain limit to the sections of the line bundles over S^2. We start from the Hopf fibration of S^3 over S^2 with fibre $U(1)$. Let χ_1 and χ_2 be components of a spinor. Ristrict the sum of the squares of these two complex numbers to be R. This defines the sphere S^3. Next we study expansions in terms of these two complex coordinates and their complex conjugates. Define $X_i = \chi^\dagger \sigma_i \chi$. The squares of X_i sum up to R^2 and define therefore S^2. If in an expansion an equal number of χ's and $\chi^\dagger$'s occur, it becomes a well-defined function on S^2. Otherwise it is a section of a bundle.

Next we quantise this scheme: Replace the complex quantities by creation and annihilation operators of two bosons:

$$[A_i, A_j^\dagger] = \delta_{ij} , \qquad i,j = 1,2 . \tag{7}$$

This means we use the Jordan-Schwinger representation of $su(2)$. Next we define N-dimensional subspaces F_N of the Fock space given by a fixed number of $N-1$ creation operators. They form an irreducible representation of $su(2)$. We study now maps from one such subspace to another one. If they are of equal dimension, square matrices will map one to the other and the sequence of them forms again the Fuzzy Sphere. If they are of different dimensions and their difference equals twice the topological charge, we quantise in a certain sense sections of the bundles. This way we obtain a sequence of embedded modules formed from non-square matrices of special size.

The Dirac operator and spinors were obtained from a supersymmetric treatment. We extended $su(2)$ to the supergroup $osp(2/1)$ and obtained a quantisation of superfields through an embedded sequence of graded matrix algebras.

2.2 The Fuzzy q-Sphere

The above described Fuzzy Sphere is invariant under the action of $SO(3)$, or equivalently under the action of $U(so(3))$. Together with John Madore and Harold Steinacker we defined a sequence of finite algebras, which have analogous properties, but which are covariant under the quantised universal enveloping algebra $U_q(su(2))$ [10, 11]. This has been done for real q as well as for q being a root of unity. In the later case certain restrictions have to be obeyed. Covariance of an algebra A under $U_q(su(2))$ means that there exists an action

$$U_q(su(2)) \times A \to A \ , \qquad (u,a) \mapsto u \triangleright a \ , \tag{8}$$

such that $u \triangleright (ab) = (u_{(1)} \triangleright a)(u_{(2)} \triangleright b)$ for $a, b \in A$. Here $\Delta(u) = u_{(1)} \otimes u_{(2)}$ is the Sweedler notation for the coproduct.

We may follow now the undeformed scheme and define q-deformed creation and annihilation operators:

$$A^{\dagger i} A_j = \delta^i_j + q \hat{R}^{ik}_{jl} A_k A^{\dagger l} \ . \tag{9}$$

Next one considers again N-dimensional subspaces of Fock space which form irreducible representations of $U_q(su(2))$. It is possible to define the q-deformed spheres in terms of $Z_i = A^{\dagger\alpha} \epsilon_{\alpha\beta} \sigma_i^{\beta\gamma} A_\gamma$ by

$$\epsilon^{ij}_k Z_i Z_j = \text{constant} \cdot Z_k \tag{10}$$

and to prove that the full matrix algebra is generated. After a study of the reality structure we introduced an invariant integral and studied a differential calculus. There exists a unique three-dimensional module of one-forms. As opposed to the classical case, an additional radial one-form shows up. This leads to the addition of a scalar field.

Next it is possible to write down actions for scalar fields as well as for gauge fields. Three possible actions can be formed for the later:

$$S_1 = \int A * A + 2A\Theta \ , \tag{11}$$

$$S_2 = 2 \int A^3 + 3(A\, dA + A(^*A)) + 6A\Theta + 2\Theta^3 \ , \tag{12}$$

$$S_3 = \int (dA + A^2)(^*(dA + A^2)) \ , \tag{13}$$

were $*$ denotes the Hodge star operation and Θ a special one-form which plays the rôle of the Maurer-Cartan form. The commutator of algebra elements with this special form gives one-forms:

$$df = [\Theta, f] \ . \tag{14}$$

As for the step from one-forms to two-forms one has to take the commutator but subtract the Hodge star of the one form in order to get two-forms,

$$d\alpha = [\Theta, \alpha]_+ - {}^*\alpha \ , \tag{15}$$

where the anticommutator enters. The step to three-forms is given again by the commutator with this special form.

It is interesting to note that special linear combinations of S_i's correspond to the Yang-Mills action and to the Chern-Simons action. A special combination of both resulted from string theory after taking a particular limit [12].

Very recently we studied second quantisation of a field theory on the q-deformed fuzzy sphere for q real. For this case it was necessary to perform a path integral over modes, which generate a quasi-associative algebra. This way we kept the symmetry and obtained a smooth limit for q going to 1.

3 Renormalisation of deformed quantum field theory

3.1 UV/IR-mixing

Inspired to a large extent by the work of Seiberg and Witten [13], numerous investigations of field theories on deformed space(-time) have appeared. The deformed space under consideration—called noncommutative $\mathbb{R}^4$—can be introduced in several ways. First, in analogy to quantum mechanics one assumes that the four-dimensional coordinate operators obey a commutator relation like

$$[x^\mu, x^\nu] = \mathrm{i}\theta^{\mu\nu} = -\mathrm{i}\theta^{\nu\mu} \,. \tag{16}$$

Thanks to the Baker-Campbell-Hausdorff formula, plane waves $u_p = \mathrm{e}^{\mathrm{i}p_\mu x^\mu}$ obey the algebra

$$u_p u_q = \mathrm{e}^{-\frac{1}{2}\mathrm{i}\theta^{\mu\nu}p_\mu q_\nu} u_{p+q} \,, \tag{17}$$

were p and q denote commutative d-dimensional momenta. Derivations on this algebra are defined through the multiplication with these momenta, an integral on this algebra is defined by mapping to the $p = 0$ part. In the form (17) we make contact to the noncommutative torus, where p and q run over an integer lattice. The later occured through compactification of M-theory on noncommutative 2-tori [14] and corresponds to turning on a background three-form.

Superpositions of plane waves with smooth functions $f(x)$ gives the Weyl operators $W[f]$. A standard question is which product of functions allows to encode this algebra. The answer is given by the Moyal-Weyl product

$$W[f]W[g] = W[f \star g] \,, \qquad (f \star g)(x) = \mathrm{e}^{\frac{1}{2}\mathrm{i}\theta^{\mu\nu}\partial^y_\mu \partial^z_\nu} f(y)g(z)\Big|_{x=y=z} \,. \tag{18}$$

Unless the functions f, g are analytic, the derivatives are meant in a distributional sense. For Schwartz class functions a useful formula is

$$(f \star g)(x) = \int \frac{d^4k}{(2\pi)^4} \int d^4y \; f(x+\tfrac{1}{2}\theta{\cdot}k)\, g(x+y)\, \mathrm{e}^{\mathrm{i}k\cdot y} \,, \tag{19}$$

where $(\theta\cdot k)^\mu = \theta^{\mu\nu}k_\nu$ and $k\cdot y = k_\mu y^\mu$.

Since divergences come from singularities due to point like interactions, there was hope that field theories on these deformed algebras may be better behaved. This is not the case. Loop integrals in noncommutative field theories differ from their commutative counterpart by phase factors $\mathrm{e}^{\pm\frac{1}{2}\mathrm{i}\theta^{\mu\nu}p_\mu q_\nu}$ [15]. In planar diagrams the resulting phases of inner momenta cancel, i.e. the same integral with the same divergences occurs as in undeformed models.

Next question concerns renormalisability: By direct evaluation of Feynman graphs it was shown that Yang-Mills theory is one-loop renormalisable, i.e. the divergences can be absorbed such that Ward-Slavnov identities are fulfilled [16, 17]. For higher loop contributions the new phenomenon of UV/IR-mixing occurs, which was first discovered for scalar fields [18] and shortly later for Yang-Mills theories as well [19]. Although non-planar one-loop diagrams are convergent for generic external momenta, they diverge for exceptional external momenta, where the regulating phase becomes inefficient. Inserted as subgraphs into a bigger graph, the external momenta of the subgraphs are internal momenta for the bigger graph, and the exceptional momenta are realised by the loop integration. The total graph thus becomes divergent. Due to the integrated phase factors involving inner momenta of non-planar graphs, the resulting counterterms are of different structure than the initial action was: the model is not renormalisable.

The problem was analysed to all loop orders by Chepelev and Roiban [20, 21]. They represented the integrals of noncommutative field theories by ribbon graphs drawn on a genus-g Riemann surface with boundary and proved a power-counting theorem for such a situation. The conclusion is that non-planar graphs are convergent except for two dangerous cases. So called rings stacked on the came cycle lead (in four dimensions) to singularities of arbitrarily high order. There are proposals to reorder the perturbation series in order to sum up these divergences. Commutants occur if exceptional momenta are forced by momentum conservation for subgraphs. They lead to non-local counterterms.

One possible way out concerns additional symmetries. We used a superfield formulation and established a proof that the Wess-Zumino model does not have the above mentioned disease [22]. Since only logarithmically divergent diagrams occur, and we showed earlier that even iterated integrals of this type are integrable, the standard renormalisability proof may be adapted.

3.2 Seiberg-Witten map

The phase factors $e^{\pm\frac{1}{2}i\theta^{\mu\nu}p_\mu q_\nu}$ were kept intact in the previous treatment. There is also the possibility of expanding the phases into a Taylor series in θ. Limiting the expansion to some finite order in θ, we actually get a local (commutative) field theory involving higher-order partial derivatives of the fields. For example, limiting the Taylor expansion of the noncommutative Maxwell action

$$\Gamma = -\frac{1}{4g^2}\int \hat{F}_{\mu\nu} \star \hat{F}^{\mu\nu} ,$$
$$\hat{F}_{\mu\nu} = \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu - i\hat{A}_\mu \star \hat{A}_\nu + i\hat{A}_\nu \star \hat{A}_\mu , \tag{20}$$

to first order in θ, we get

$$\Gamma = \int d^4x \Big(-\frac{1}{4g^2}\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu} - \frac{1}{2g^2}\theta^{\alpha\beta}\tilde{F}^{\mu\nu}\partial_\alpha \hat{A}_\mu \partial_\beta \hat{A}_\nu + \mathcal{O}(\theta^2)\Big) ,$$
$$\tilde{F}_{\mu\nu} = \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu . \tag{21}$$

At higher order in θ the result of the Taylor expansion becomes extremely complicated.

Guided by an equivalence argument for regularisation schemes (point splitting versus Pauli-Villars), Seiberg and Witten discovered for gauge theories a change of variables so that the θ-expanded action becomes at each order in θ invariant under commutative gauge transformations [13]. Denote by $\hat{\lambda}$, $\hat{A}$ and $\hat{F}$ gauge parameter, gauge connection and field strength for the deformed model, and by λ, A and F the appropriate quantities for the undeformed case. For the Seiberg-Witten map one requires that

$$\delta_{\hat{\lambda[\lambda,A]}}\hat{A}[A] = \hat{A}[\delta_\lambda A] , \tag{22}$$

which means that the gauge transformed noncommutative field should be equal to the noncommutative field of the gauge transformed commutative one. The requirement (22) can be reformulated as a differential equation $\frac{d\hat{A}_\mu}{d\theta^{\alpha\beta}} = f_{\mu\alpha\beta}[\hat{A}]$ for some functional $f_{\mu\alpha\beta}$ (see below) and has at first order in θ the solution

$$\hat{A}_\mu = -\frac{1}{2}\theta^{\alpha\beta}A_\alpha(\partial_\beta A_\mu + F_{\beta\mu}) + \mathcal{O}(\theta) , \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu . \tag{23}$$

Inserting (23) into (21) one gets

$$\Gamma = \int d^4x \Big(-\frac{1}{4g^2}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2g^2}\theta^{\alpha\beta}\Big(F_{\alpha\mu}F_{\beta\nu}F^{\mu\nu} - \frac{1}{4}F_{\alpha\beta}F_{\mu\nu}F^{\mu\nu}\Big) + \mathcal{O}(\theta^2)\Big) . \tag{24}$$

At higher order in θ the requirement (22) leaves a great degree of freedom open. Various methods are available to get the mapping from A to $\hat{A}$, or better to get the expansion of the later in terms of A. This map is meant as a formal power expansion series in $\theta^{\mu\nu}$. The Munich group [23, 24, 25] used these ideas and argued that this is the way to obtain a finite number of degrees of freedom in non-Abelian noncommutative Yang-Mills theories.

The Seiberg-Witten map of the noncommutative Yang-Mills action leads to a gauge field theory with an infinite number of vertices and Feynman graphs with unbounded degree of divergences, which seems to rule out a perturbative renormalisation. The advantage is, however, that there cannot be any UV/IR-mixing. An explicit quantum field theoretical investigation [26] of the action (24) led to the surprising result that the divergent terms of the one-loop photon self-energy are gauge-invariant, independent of the choice of a linear or non-linear gauge and independent of the gauge parameter. These terms could not be absorbed in a standard wave function renormalisation. We have thus interpreted our result as the need to extend the classical action by terms of order θ^2.

Shortly later we found a possibility to absorb the divergences of the photon self-energy to any loop order and any order in θ through a more general type of wave function renormalisation—a field redefinition [27]. This possibility is related to the freedom in the Seiberg-Witten map. For noncommutative photons $\hat{A}_\mu$,

$$\Big(\frac{d\hat{A}_\mu}{d\theta^{\rho\sigma}}\Big)' = \frac{d\hat{A}_\mu}{d\theta^{\rho\sigma}} + \Omega_{\rho\sigma\mu} \;, \qquad \Omega_{\rho\sigma\mu} \text{ covariant} \;, \tag{25}$$

is an admissible Seiberg-Witten differential equation if $\frac{d\hat{A}_\mu}{d\theta^{\rho\sigma}}$ is. If $\Gamma^{(n)}$ is the order-n term of the θ-expansion of the noncommutative Maxwell action (20), and if $\Omega_{\rho\sigma\mu}$ contains explicitly $n{-}1$ factors of θ, we have

$$\big(\Gamma^{(n)}\big)' - \Gamma^{(n)} = \frac{1}{g^2}\int d^4x\; \theta^{\rho\sigma}\Omega^{(n-1)}_{\rho\sigma\mu}\,\partial_\nu F^{\nu\mu} \;. \tag{26}$$

On the other hand, divergences in 1PI Green's functions of order n in θ are for an invariant renormalisation scheme of the symbolic form (omitting Lorentz indices)

$$\sum_{N=2}^{2+n} \frac{1}{N!}\int dx_1 \dots dx_N \;\delta(x_1{-}x_2)\cdots\delta(x_{N-1}{-}x_N) \\ \times \sum_{(i)} \kappa_{(i)}[\varepsilon]\Big(\theta^n\,\partial^{4+2n-2N}\big(F(x_1)\dots F(x_N)\big)\Big)^{(i)} \;. \tag{27}$$

The label (i) runs over different tensorial structures. For $N = 2$ (i.e. the photon selfenergy) we notice:

1. All ∂'s can be regarded as differentiation with respect to x_1.
2. There are as many Lorentz indices on ∂ as on θ.
3. $\theta^{\mu\nu}\partial_\mu\partial_\nu = 0$.
4. There is always the expression $\partial^\mu F_{\mu\nu}$ or $\partial^\alpha\partial_\alpha F_{\mu\nu} = \partial_\mu\partial^\alpha F_{\alpha\nu} - \partial_\nu\partial^\alpha F_{\alpha\mu}$ in (27).
5. As a conclusion, any such term occurs in (26), and choosing $\Omega = \Omega[\hbar, \varepsilon]$ appropriately we can absorb any divergence of the photon self-energy.

This tells us that field redefinitions are capable to absorb a considerable amount of divergences in θ-expanded noncommutative field theories. There was even some hope that θ-expanded field theories may be renormalisable. The crucial question for renormalisability is whether the freedom due to field redefinitions is big enough to absorb all divergences. In the above proof the steps 1. and 2. can not be extended to $N > 2$. A direct calculation of the 3-point function at lowest order θ^3 leads to ~ 200000 different contributions which are impossible to analyse. We have therefore tried to find symmetries of the θ-expanded action which limit the structure of possible divergences. The existence of such symmetries is already suggested by (24) where the relative coefficient between the two possible gauge invariants of dimension 4 linear in θ, which are $\theta^{\alpha\beta}F_{\alpha\mu}F_{\beta\nu}F^{\mu\nu}$ and $\theta^{\alpha\beta}F_{\alpha\beta}F_{\mu\nu}F^{\mu\nu}$, is $-\frac{1}{4}$. This particular combination seems to be related to the energy-momentum tensor so that one could expect that the θ-expansion of noncommutative Lorentz transformations provides the symmetries looked for.

We have proved in [28] that the θ-expansion reduces the noncommutative Lorentz transformations to a combination of commutative gauge transformations and commutative Lorentz transformations and thus gives no hints at all for new symmetries. As a by-product, however, we have found a new interpretation of the Seiberg-Witten map. We achieved a different kind of θ-expansion which unless Taylor expansion and change of variables which pass through (21) allow us to derive (24) in a single stroke from (20). This derivation is closely related to a deeper discussion of noncommutative Lorentz transformations.

One has to distinguish between two kinds of Lorentz (or more general, conformal) transformations. Lorentz transformations in special relativity relate physical observations made in two inertial reference frames characterised by different velocities and orientations. These transformations can be implemented as coordinate changes, known as *observer Lorentz transformations*. Alternatively one considers transformations which relate physical properties of two particles with different helicities or momenta within one specific inertial frame. These are known as *particle Lorentz transformations*. Usually (without background) these two approaches are equivalent. However, in the presence of a background tensor field this equivalence fails, because the background field will transform as a tensor under observer Lorentz transformation and as a set of scalars under particle Lorentz transformations. Thirdly, having a background tensor field one may consider the transformations of all fields within a specific inertial frame simultaneously, including the background field. These transformations are known as *(inverse) active Lorentz transformations* and are equivalent to observer Lorentz transformations. We demand

1. that the physical action is invariant under observer Lorentz transformations,

2. that the particle Lorentz transformation (which is not a symmetry) of the action is gauge invariant (thus observable).

¿From these requirements we derive in [28] that the particle Lorentz transformation of a field is the sum of its naïve observer Lorentz transformation and an additional part given by the Seiberg-Witten differential equation (which is very conveniently derived in this manner).

Let us give some details of this procedure. Infinitesimal field transformations are implemented by Ward identity operators

$$W=\sum_i \Big\langle \delta\hat{\Phi}_i[\hat{\Phi}_k], \frac{\delta}{\delta\hat{\Phi}_i}\Big\rangle\;, \tag{28}$$

where the index i labels the different sorts of fields, here denoted $\hat{\Phi}_i$. The Ward identity operator (28) acts on (sufficiently regular) functionals $\Gamma[\hat{\Phi}_i]$ in a derivational manner:

$$W\Gamma[\hat{\Phi}_i]=\sum_j \Big\langle \delta\hat{\Phi}_j[\hat{\Phi}_k], \frac{\delta\Gamma[\hat{\Phi}_i]}{\delta\hat{\Phi}_j}\Big\rangle=\lim_{\epsilon\to 0}\frac{1}{\epsilon}\Big(\Gamma[\hat{\Phi}_i+\epsilon\delta\hat{\Phi}_i[\hat{\Phi}_k]]-\Gamma[\hat{\Phi}_i]\Big)\;. \tag{29}$$

We are interested in a set $\mathcal{S}$ of symmetry transformations of the action functional, $W^I\Gamma=0$, $I\in\mathcal{S}$. This set is required to be complete,

$[W^I, W^{I'}] = \sum_n W^{I_n}$, $I_n \in \mathcal{S}$. In particular, we are interested in gauge transformation G and Lorentz transformation L which satisfy

$$[W^L, W^L] \subset W^L\,, \qquad [W^G, W^G] \subset W^G\,, \qquad [W^G, W^L] \subset W^G\,. \tag{30}$$

The Lorentz transformation has for the field $\hat{A}$ of Yang-Mills theory on noncommutative $\mathbb{R}^4$ the symbolic form

$$W^L = \Big\langle \delta^L \hat{A}, \frac{\delta}{\delta \hat{A}} \Big\rangle + \Big\langle \delta^L \theta, \frac{\delta}{\delta \theta} \Big\rangle\,, \tag{31}$$

it is a symmetry of the Yang-Mills action functional, and (30) is satisfied [28]. It is essential that in (31) the sum of the $\hat{A}$ and the θ-transformation appears, the individual transformations do not have any meaning. Neither they are symmetries of the action functional, nor they fulfil (30). But if one really insists on transforming $\hat{A}$ only, then at least this transformation $\tilde{W}^L_{\hat{A}}$, which cannot be a symmetry of the action functional, must satisfy

$$[\tilde{W}^L_{\hat{A}}, W^G] \subset W^G\,. \tag{32}$$

The condition (32) guarantees that $\tilde{W}^L_{\hat{A}} \Gamma[\hat{A}] \neq 0$, which can be regarded as the particle Lorentz symmetry breaking, is a gauge-invariant quantity [28]. Otherwise $\tilde{W}^L_{\hat{A}}$ is completely unphysical. It is then somehow natural to make the ansatz

$$\begin{aligned} &W^L = \tilde{W}^L_{\hat{A}} + \tilde{W}^L_\theta\,, \\ &\tilde{W}^L_{\hat{A}} = \Big\langle \delta^L \hat{A} - \delta^L \theta \,\frac{d\hat{A}}{d\theta}, \frac{\delta}{\delta \hat{A}} \Big\rangle\,, \quad \tilde{W}^L_\theta = \Big\langle \delta^L \theta, \frac{\delta}{\delta \theta} \Big\rangle + \Big\langle \delta^L \theta \,\frac{d\hat{A}}{d\theta}, \frac{\delta}{\delta \hat{A}} \Big\rangle\,, \end{aligned} \tag{33}$$

where $\frac{d\hat{A}}{d\theta}$ is, for the time being, just a symbol. The condition (32) determines $\frac{d\hat{A}}{d\theta}[\hat{A}]$, which thus becomes a concrete (but not unique) function of $\hat{A}$. The equation $\frac{d\hat{A}}{d\theta} = \frac{d\hat{A}}{d\theta}[\hat{A}]$ looks formally like a differential equation—the Seiberg-Witten differential equation. The general solution has the form

$$\frac{d\hat{A}_\mu}{d\theta^{\rho\sigma}} = -\frac{1}{8}\big\{\hat{A}_\rho, \partial_\sigma \hat{A}_\mu + \hat{F}_{\sigma\mu}\big\}_\star + \frac{1}{8}\big\{\hat{A}_\sigma, \partial_\rho \hat{A}_\mu + \hat{F}_{\rho\mu}\big\}_\star + \hat{\Omega}_{\rho\sigma\mu}\,, \tag{34}$$

where the functional $\hat{\Omega}_{\rho\sigma\mu}[\hat{A}]$, which parametrises the freedom due to field redefinitions, transforms covariantly under noncommutative gauge transformations. The solution (34) is for $\hat{\Omega}_{\rho\sigma\mu} = 0$ known as the Seiberg-Witten differential equation.

Now we can define the following Taylor expansion of the action functional $\Gamma[\hat{A}]$:

$$\Gamma^{(n)}[A] := \sum_{j=0}^{n} \frac{1}{j!}(\theta)^j \Big(\big(\tilde{W}_\theta^1\big)^j \Gamma[\hat{A}] \Big)_{\theta=0} , \quad \delta^1\theta := 1 , \quad A := \big(\hat{A}\big)_{\theta=0} . \tag{35}$$

By construction, the action functional $\Gamma^{(n)}[A]$ describes a commutative Yang-Mills theory (coupled to the external field θ) which is invariant under commutative gauge and Lorentz transformations at any cut-off order n in θ, see [28]. We have thus obtained the generalisation of (24) to any desired order in a single stroke.

Since the search for symmetries of the θ-expanded action was not successful, we continued with the direct computation of one-loop divergent graphs. The simplest model to study other Green's functions than the self-energy is θ-deformed QED. For massless electrons there are 9 independent gauge and Lorentz invariant field polynomials of first order in θ. There are 5 independent field redefinitions so that one should expect 4 types of divergences linear in θ which cannot be removed. The actual one-loop computation of these Green's function in [29] showed that there is only a single independent one-loop divergent Green's functions at first order in θ. In the massive case (where the mass term is inserted directly into the Dirac action) there are 15 invariant terms, 8 field redefinitions and 3 divergences. The computation was very complicated due to the big number of different graphs involving and the (compared with standard QFT computations) high degree of divergence, which results in contributions from the entire Clifford algebra of γ matrices. A typical integral to compute has the form

$$\int d^4k \frac{k_{\mu_1} \cdots k_{\mu_n}}{((k+p_1)^2 - m_1^2) \cdots ((k+p_s)^2 - m_s^2)} . \tag{36}$$

The following table shows the number of graphs to compute for the one-loop Green's functions of first order in θ, together with the order n, s of the above integral:

Green's function	$\Gamma_{\bar{\psi}\psi}$	$\Gamma_{A\bar{\psi}\psi}$	$\Gamma_{\bar{\psi}\psi\bar{\psi}\psi}$	Γ_{AA}	Γ_{AAA}
number of graphs	7	13	14	5	16
(s,n)	$(2,3)$	$(3,4)$	$(4,4)$	$(2,4)$	$(3,5)$

(37)

The contributions to the divergent Green's functions $\Gamma_{AA\bar{\psi}\psi}$, $\Gamma_{AAA\bar{\psi}\psi}$, Γ_{AAAA}, Γ_{AAAAA} and Γ_{AAAAAA} are actually too difficult to compute but fortunately they can determined by Ward identities.

Thus, although these computations made clear that θ-expanded field theories can not be expected to be renormalisable, there is evidence now that additional symmetries for the θ-expanded action exist indeed. It is natural to assume that these symmetries are already present in the original noncommutative action before θ-expansion. One possible source is the spectral origin of the action [6] which might admit a bigger symmetry group than just Lorentz and gauge transformations.

The loop calculations of [26, 29] were performed for the θ-expanded action after application of the Seiberg-Witten map such as (24). As we have shown in [30], very similar computations are possible when starting directly from the Taylor expansion of the action functional such as (21). The only difference is that now the gauge symmetry is non-linearly realised so that the whole machinery of external fields and Slavnov-Taylor identities must be used. In other words, one has to extend the gauge-fixed noncommutative QED action by

$$\Gamma_{ext} = \int d^4x \,\Big(\hat{\bar{\rho}}\big(\hat{s}\hat{\psi}\big) + \big(\hat{s}\hat{\bar{\psi}}\big)\hat{\rho} + \hat{\sigma}^\mu\big(\hat{s}\hat{A}_\mu\big) + \hat{\kappa}\big(\hat{s}\hat{c}\big)\Big)\,, \tag{38}$$

where we define the non-commutative BRST-transformations by

$$\begin{aligned} \hat{s}\hat{A}_\mu &= \hat{D}^{adj}_\mu \hat{c}\,, & \hat{s}\hat{c} &= \mathrm{i}\hat{c}\star\hat{c}\,, \\ \hat{s}\hat{\psi} &= \mathrm{i}\hat{c}\star\hat{\psi}\,, & \hat{s}\hat{\bar{\psi}} &= -\mathrm{i}\hat{\bar{\psi}}\star\hat{c}\,, \\ \hat{s}\hat{\bar{c}} &= \hat{B}\,, & \hat{s}\hat{B} &= 0\,. \end{aligned} \tag{39}$$

We denote by $\hat{A}, \hat{\psi}, \hat{c}$ the noncommutative photon, electron and ghost, respectively, and $\hat{\rho}, \hat{\sigma}, \hat{\kappa}$ are the external fields. Loop calculations do not preserve the θ-expansion of (39). The Slavnov-Taylor identity

$$\mathcal{S}\,(\Gamma) = \int d^4x \,\left(\frac{\delta\Gamma}{\delta\hat{\sigma}^\mu}\frac{\delta\Gamma}{\delta\hat{A}_\mu} + \frac{\delta\Gamma}{\delta\hat{\kappa}}\frac{\delta\Gamma}{\delta\hat{c}} + \frac{\delta\Gamma}{\delta\hat{\rho}}\frac{\delta\Gamma}{\delta\hat{\bar{\psi}}} + \frac{\delta\Gamma}{\delta\hat{\psi}}\frac{\delta\Gamma}{\delta\hat{\bar{\rho}}} + \hat{B}\frac{\delta\Gamma}{\delta\hat{\bar{c}}}\right) = 0\,. \tag{40}$$

is preserved, though. The identity (40) contains several individual identities when differentiating which respect to the fields. To our great surprise we found in [30] that up to first order in θ and first loop order the Taylor expansion of the noncommutative QED leads up to field redefinition to the same result when using the Seiberg-Witten map or not: There is for massless electrons only a single resulting divergence

$$\int d^4x \;\mathrm{i}\theta^{\alpha\beta}\hat{\bar{\psi}}\star\gamma^{\mu\nu}{}_\alpha \hat{D}^{adj}_\beta \hat{F}_{\mu\nu}\star\hat{\psi}\,, \qquad \hat{D}^{adj}_\beta \hat{F}_{\mu\nu} = \partial_\beta \hat{F}_{\mu\nu} - \mathrm{i}[\hat{A}_\beta, \hat{F}_{\mu\nu}]_\star\,, \tag{41}$$

with the numerical coefficient of the divergent term being independent of the application or omission of the Seiberg-Witten map. This seems to indicate that the Seiberg-Witten map is an unphysical change of variables also on quantum level.

This is true to some extent, but there is a subtlety. One can perform the change of variables before or after quantisation. Changing the variables $\Phi' = \Phi'[\Phi]$ after quantisation, i.e. performing a change of the dummy integration variables in the path integral, one obtains exactly the same Green's functions. This was to be expected from the general equivalence theorem [31]. The changes in the Feynman rules from Φ' to Φ are compensated by graphs involving the modified source term $\langle J, \Phi'[\Phi]\rangle$ in the path integral. In principle one would also expect contributions from field redefinition ghosts, but here the propagator equals 1 so that there is no contribution at least for certain regularisation schemes. On the other hand, changing the variables in the action functional before inserting it into the path integral, the outcome is expected to be different. However, at first order in θ only, the difference to the other method is a field redefinition.

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