Topics in String Field Theory

L. Bonora, C. Maccaferri, D. Mamone
International School for Advanced Studies (SISSA/ISAS)
Via Beirut 2–4, 34014 Trieste, Italy, and INFN, Sezione di Trieste

M. Salizzoni
Humboldt Universität zu Berlin, Institut für Physik
Newtonstrasse 15, Berlin, Germany

Abstract. This review of bosonic string field theory is concentrated on two main subjects. In the first part we revisit the construction of the three string vertex and rederive the relevant Neumann coefficients both for the matter and the ghost part following a conformal field theory approach. We use this formulation to solve the VSFT equation of motion for the ghost sector. This part of the paper is based on a new method which allows us to derive known results in a simpler way. In the second part we concentrate on the solution of the VSFT equation of motion for the matter part. We describe the construction of the three strings vertex in the presence of a background B field. We determine a large family of lump solutions, illustrate their interpretation as D–branes and study the low energy limit. We show that in this limit the lump solutions flow toward the so–called GMS solitons.

1 Introduction

Recently, as a consequence of the increasing interest in tachyon condensation, String Field Theory (SFT) has received a renewed attention. There is no doubt that the most complete description of tachyon condensation and related phenomena has been given so far in the framework of Witten's Open String Field Theory, [1]. This is not surprising, since the study of tachyon condensation involves off–shell calculations, and SFT is the natural framework where off–shell analysis can be carried out.

All these developments can be described along the blueprint represented by A.Sen's conjectures, [2]. The latter can be summarized as follows. Bosonic open string theory in D=26 dimensions is quantized on an unstable vacuum, an instability which manifests itself through the existence of the open string tachyon. The effective tachyonic potential has, beside the local maximum where the theory is quantized, a local minimum. Sen's conjectures concern the nature of the theory around this local minimum. First of all, the energy density difference between the maximum and the minimum should exactly compensate for the D25–brane tension characterizing the unstable vacuum: this is a condition for the stability of the theory at the minimum. Therefore the theory around the minimum should not contain any quantum fluctuation pertaining to the original (unstable) theory. The minimum should therefore correspond to an entirely new theory, the bosonic closed string theory. If so, in the new theory one should be able to explicitly find in particular all the classical solutions characteristic of closed string theory, specifically the D25–brane as well as all the lower dimensional D–branes.

The evidence that has been found for the above conjectures does not have a uniform degree of accuracy and reliability, but it is enough to conclude that they provide a correct description of tachyon condensation in SFT. Especially elegant is the proof of the existence of solitonic solutions in Vacuum String Field Theory (VSFT), the SFT version which is believed to represent the theory near the minimum.

The aim of this review is not of giving a full account of the entire subject of SFT and tachyon condensation. In this regards there are already several reviews, [3, 4, 10, 5]. We will rather concentrate on some specific topics which are not covered in other reviews. The first part of this article is devoted to the operator formulation of SFT. The reason for this is that the latest developments in VSFT and especially in supersymmetric SFT (see in particular [3, 4]), have brought up aspects of the theory that had not been analyzed in sufficient detail in the existing literature. We refer in particular to the ghost structure of SFT and the relation between the operator formulation and the (twisted) conformal field theory formulation. To clarify this issue we extensively use the CFT interpretation of SFT, advocated especially by [6, 7].

The second part of this review is a synopsis of D–branes in VSFT and noncommutative solitons. Our main purpose is finding families of tachyonic lumps that can consistently be interpreted as D–branes and studying their low energy limit. We do so by introducing a constant background B field, with the purpose of smoothing out some singularities that appear in the low energy limit when the B field is absent. The result is rewarding: we find a series of noncommutative solitons (the GMS solitons) that were found some time ago by studying noncommutative effective field theories of the tachyon.

2 A summary of String Field Theory

The open string field theory action proposed by E.Witten, [1], years ago is

$$\mathcal{S}(\Psi) = -\frac{1}{g_0^2}\int\left(\frac{1}{2}\Psi * Q\Psi + \frac{1}{3}\Psi * \Psi * \Psi\right) \tag{1}$$

In this expression Ψ is the string field, which can be understood either as a classical functional of the open string configurations or as a vector in the Fock space of states of the open string. We will consider in the following the second point of view. In the field theory limit it makes sense to represent it as a superposition of Fock space states with ghost number 1, with coefficient represented by local fields,

$$|\Psi\rangle = (\phi(x) + A_\mu(x)a_1^{\mu\dagger} + \ldots)c_1|0\rangle \tag{2}$$

The BRST charge Q has the same form as in the first quantized string theory. The star product of two string fields Ψ_1, Ψ_2 represents the process of identifying the right half of the first string with the left half of the second string and integrating over the overlapping degrees of freedom, to produce a third string which corresponds to $\Psi_1 * \Psi_2$. This can be done in various ways, either using the classical string functionals (as in the original formulation), or using the three string vertex (see below), or the conformal field theory language [6]. Finally the integration in (1) corresponds to bending the left half of the string over the right half and integrating over the corresponding degrees of freedom in such a way as to produce a number. The following rules are obeyed

$$\begin{aligned}
&Q^2 = 0\\
&\int Q\Psi = 0\\
&(\Psi_1 * \Psi_2) * \Psi_3 = \Psi_1 * (\Psi_2 * \Psi_3)\\
&Q(\Psi_1 * \Psi_2) = (Q\Psi_1) * \Psi_2 + (-1)^{|\Psi_1|}\Psi_1 * (Q\Psi_2)
\end{aligned} \tag{3}$$

where $|\Psi|$ is the Grassmannality of the string field Ψ, which, for bosonic strings, coincides with the ghost number. The action (1) is invariant under the BRST transformation

$$\delta\Psi = Q\Lambda + \Psi * \Lambda - \Lambda * \Psi \tag{4}$$

Finally, the ghost numbers of the various objects $Q, \Psi, \Lambda, *, \int$ are $1, 1, 0, 0, -3$, respectively.

Following these rules it is possible to explicitly compute the action (1). For instance, in the low energy limit, where the string field may be assumed to take the form (2), the action becomes an integrated function F of an infinite series of local polynomials (kinetic and potential terms) of the fields involved in (2):

$$\mathcal{S}(\Psi) = \int d^{26}x F(\varphi_i, \partial\varphi_i, ...) \tag{5}$$

2.1 Vacuum string field theory

The action (1) represents open string theory about the trivial unstable vacuum $|\Psi_0\rangle = c_1|0\rangle$. Vacuum string field theory (VSFT) is instead a version of Witten's open SFT which is conjectured to correspond to the minimum of the tachyon potential. As explained in the introduction at the minimum of the tachyon potential a dramatic change occurs in the theory, which, corresponding to the new vacuum, is expected to represent closed string theory rather that the open string theory we started with. In particular, this theory should host tachyonic lumps representing unstable D–branes of any dimension less than 25, beside the original D25–brane. Unfortunately we have been so far unable to find an exact classical solution, say $|\Phi_0\rangle$, representing the new vacuum. One can nevertheless guess the form taken by the theory at the new minimum, see [10]. The VSFT action has the same form as (1), where the new string field is still denoted by Ψ, the $*$ product is the same as in the previous theory, while the BRST operator Q is replaced by a new one, usually denoted $\mathcal{Q}$, which is characterized by universality and vanishing cohomology. Relying on such general arguments, one can even deduce a precise form of $\mathcal{Q}$ ([14],[16], see also [15, 17, 18, 19, 20, 21] and [22, 23, 24, 26, 27, 28, 29]),

$$\mathcal{Q} = c_0 + \sum_{n>0} (-1)^n \left(c_{2n} + c_{-2n}\right) \tag{6}$$

Now, the equation of motion of VSFT is

$$\mathcal{Q}\Psi = -\Psi * \Psi \tag{7}$$

and nonperturbative solutions are looked for in the factorized form

$$\Psi = \Psi_m \otimes \Psi_g \tag{8}$$

where Ψ_g and Ψ_m depend purely on ghost and matter degrees of freedom, respectively. Then eq.(7) splits into

$$\begin{aligned} \mathcal{Q}\Psi_g &= -\Psi_g * \Psi_g & (9)\\ \Psi_m &= \Psi_m * \Psi_m & (10) \end{aligned}$$

We will see later on how to compute solutions to both equations. A solution to eq.(9) was calculated in [14, 15]. Various solutions of the matter part have been found in the literature, [10, 16, 22, 23, 30, 49, 50].

2.2 Organization of the paper. First part

In the first part of this review we rederive the three strings vertex coefficients by relying on the definition and the methods introduced by [6, 7, 8, 9]. We do it first for the matter part (section 3). In section 4 we derive the ghost Neumann coefficients and in section 5 we concentrate on the equation of motion of VSFT and look for matter–ghost factorized solutions. We show how to rederive the solution for the ghost part with a new method. Finally section 6 is meant as an introduction to the second part of the paper.

3 Three strings vertex and matter Neumann coefficients

The three strings vertex [1, 44, 45] of Open String Field Theory is given by

$$|V_3\rangle = \int d^{26}p_{(1)}d^{26}p_{(2)}d^{26}p_{(3)}\delta^{26}(p_{(1)}+p_{(2)}+p_{(3)})\exp(-E)\,|0,p\rangle_{123} \quad (11)$$

where

$$\begin{aligned} E \;=\; \sum_{a,b=1}^{3}\Bigg(\frac{1}{2}\sum_{m,n\geq 1}\eta_{\mu\nu}a_m^{(a)\mu\dagger}V_{mn}^{ab}a_n^{(b)\nu\dagger} \\ +\sum_{n\geq 1}\eta_{\mu\nu}p_{(a)}^{\mu}V_{0n}^{ab}a_n^{(b)\nu\dagger}+\frac{1}{2}\eta_{\mu\nu}p_{(a)}^{\mu}V_{00}^{ab}p_{(b)}^{\nu}\Bigg) \end{aligned} \quad (12)$$

Summation over the Lorentz indices $\mu,\nu = 0,\ldots,25$ is understood and η denotes the flat Lorentz metric. The operators $a_m^{(a)\mu}, a_m^{(a)\mu\dagger}$ denote the non–zero modes matter oscillators of the a–th string, which satisfy

$$[a_m^{(a)\mu}, a_n^{(b)\nu\dagger}] = \eta^{\mu\nu}\delta_{mn}\delta^{ab}, \qquad m,n\geq 1 \quad (13)$$

$p_{(r)}$ is the momentum of the a–th string and $|0,p\rangle_{123}\equiv |p_{(1)}\rangle\otimes|p_{(2)}\rangle\otimes|p_{(3)}\rangle$ is the tensor product of the Fock vacuum states relative to the three strings.

$|p_{(a)}\rangle$ is annihilated by the annihilation operators $a_m^{(a)\mu}$ and it is eigenstate of the momentum operator $\hat{p}^\mu_{(a)}$ with eigenvalue $p^\mu_{(a)}$. The normalization is

$$\langle p_{(a)}|\, p'_{(b)}\rangle = \delta_{ab}\delta^{26}(p+p') \tag{14}$$

The symbols $V^{ab}_{nm}, V^{ab}_{0m}, V^{ab}_{00}$ will denote the coefficients computed in [44, 45]. We will use them in the notation of Appendix A and B of [46] and refer to them as the *standard* ones. The notation V^{rs}_{MN} for them will also be used at times (with $M(N)$ denoting the couple $\{0,m\}$ $(\{0,n\})$) .

An important ingredient in the following are the *bpz* transformation properties of the oscillators

$$bpz(a_n^{(a)\mu}) = (-1)^{n+1} a_{-n}^{(a)\mu} \tag{15}$$

Our purpose here is to discuss the definition and the properties of the three strings vertex by exploiting as far as possible the definition given in [6] for the Neumann coefficients. Remembering the description of the star product given in the previous section, the latter is obtained in the following way. Let us consider three unit semidisks in the upper half z_a $(a = 1,2,3)$ plane. Each one represents the string freely propagating in semicircles from the origin (world-sheet time $\tau = -\infty$) to the unit circle $|z_a| = 1$ $(\tau = 0)$, where the interaction is supposed to take place. We map each unit semidisk to a 120° wedge of the complex plane via the following conformal maps:

$$f_a(z_a) = \alpha^{2-a} f(z_a)\,,\; a = 1,2,3 \tag{16}$$

where

$$f(z) = \Big(\frac{1+iz}{1-iz}\Big)^{\frac{2}{3}} \tag{17}$$

Here $\alpha = e^{\frac{2\pi i}{3}}$ is one of the three third roots of unity. In this way the three semidisks are mapped to nonoverlapping (except at the interaction points $z_a = i$) regions in such a way as to fill up a unit disk centered at the origin. The curvature is zero everywhere except at the center of the disk, which represents the common midpoint of the three strings in interaction.

The interaction vertex is defined by a correlation function on the disk in the following way

$$\int \psi * \phi * \chi = \langle f_1 \circ \psi(0)\, f_2 \circ \phi(0)\, f_3 \circ \chi(0)\rangle = \langle V_{123}|\psi\rangle_1 |\phi\rangle_2 |\chi\rangle_3 \tag{18}$$

Now we consider the string propagator at two generic points of this disk. The Neumann coefficients N^{ab}_{NM} are nothing but the Fourier modes of the

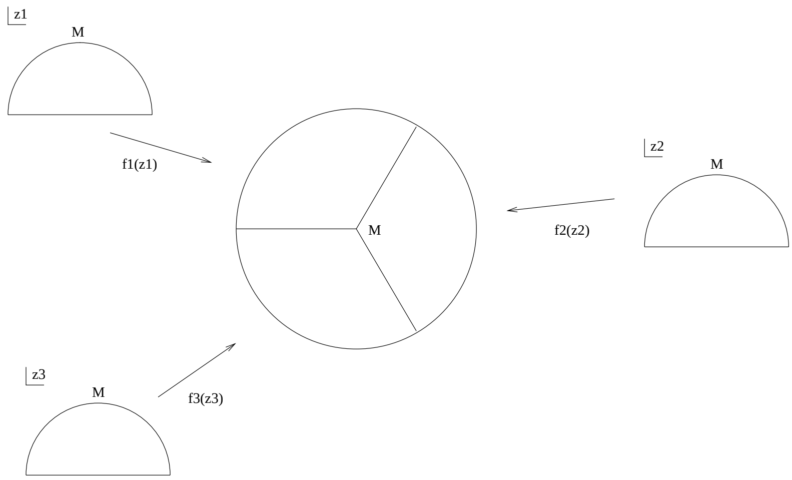

Figure 1: *The conformal maps from the three unit semidisks to the three-wedges unit disk*

propagator with respect to the original coordinates z_a. We shall see that such Neumann coefficients are related in a simple way to the standard three strings vertex coefficients.

Due to the qualitative difference between the $\alpha_{n>0}$ oscillators and the zero modes p, the Neumann coefficients involving the latter will be treated separately.

3.1 Non–zero modes

The Neumann coefficients N^{ab}_{mn} are given by [6]

$$\begin{aligned} N^{ab}_{mn} &= \langle V_{123}|\alpha^{(a)}_{-n}\alpha^{(b)}_{-m}|0\rangle_{123} = \\ &\quad -\frac{1}{nm}\oint\frac{dz}{2\pi i}\oint\frac{dw}{2\pi i}\frac{1}{z^n}\frac{1}{w^m}f'_a(z)\frac{1}{(f_a(z)-f_b(w))^2}f'_b(w) \end{aligned} \tag{19}$$

where the contour integrals are understood around the origin. It is easy to check that

$$\begin{aligned} N^{ab}_{mn} &= N^{ba}_{nm} \\ N^{ab}_{mn} &= (-1)^{n+m}N^{ba}_{mn} \\ N^{ab}_{mn} &= N^{a+1,b+1}_{mn} \end{aligned} \tag{20}$$

In the last equation the upper indices are defined mod 3.

Let us consider the decomposition

$$N^{ab}_{mn} = \frac{1}{3\sqrt{nm}}\Big(E_{nm} + \bar{\alpha}^{a-b}U_{nm} + \alpha^{a-b}\bar{U}_{nm}\Big) \tag{21}$$

After some algebra one gets

$$\begin{aligned}
E_{nm} &= \frac{-1}{\sqrt{nm}} \oint \frac{dz}{2\pi i} \oint \frac{dw}{2\pi i} \frac{1}{z^n} \frac{1}{w^m} K(z,w) \qquad (22)\\
U_{nm} &= \frac{-1}{3\sqrt{nm}} \oint \frac{dz}{2\pi i} \oint \frac{dw}{2\pi i} \frac{1}{z^n} \frac{1}{w^m} \Big(\frac{f^2(w)}{f^2(z)} + 2\frac{f(z)}{f(w)}\Big) K(z,w)\\
\bar{U}_{nm} &= \frac{-1}{3\sqrt{nm}} \oint \frac{dz}{2\pi i} \oint \frac{dw}{2\pi i} \frac{1}{z^n} \frac{1}{w^m} \Big(\frac{f^2(z)}{f^2(w)} + 2\frac{f(w)}{f(z)}\Big) K(z,w)\\
K(z,w) &= \frac{1}{(1+zw)^2} + \frac{1}{(z-w)^2}
\end{aligned}$$

By changing $z \to -z$ and $w \to -w$, it is easy to show that

$$(-1)^n U_{nm} (-1)^m = \bar{U}_{nm}, \qquad \text{or} \qquad CU = \bar{U}C, \qquad C_{nm} = (-1)^n \delta_{nm} \tag{23}$$

In the second part of this equation we have introduced a matrix notation which we will use throughout the paper.

The integrals can be directly computed in terms of the Taylor coefficients of f. The result is

$$\begin{aligned}
E_{nm} &= (-1)^n \delta_{nm} \qquad (24)\\
U_{nm} &= \frac{1}{3\sqrt{nm}} \sum_{l=1}^{m} l \Big[(-1)^n B_{n-l} B_{m-l} + 2 b_{n-l} b_{m-l} (-1)^m\\
&\quad -(-1)^{n+l} B_{n+l} B_{m-l} - 2 b_{n+l} b_{m-l} (-1)^{m+l}\Big] \qquad (25)\\
\bar{U}_{nm} &= (-1)^{n+m} U_{nm} \qquad (26)
\end{aligned}$$

where we have set

$$\begin{aligned}
f(z) &= \sum_{k=0}^{\infty} b_k z^k\\
f^2(z) &= \sum_{k=0}^{\infty} B_k z^k, \qquad \text{i.e.} \qquad B_k = \sum_{p=0}^{k} b_p b_{k-p} \qquad (27)
\end{aligned}$$

Eqs.(24, 25, 26) are obtained by expanding the relevant integrands in powers of z, w and correspond to the pole contributions around the origin. We notice that the above integrands have poles also outside the origin, but these poles either are not in the vicinity of the origin of the z and w plane, or, like the poles at $z = w$, simply give vanishing contributions.

One can use this representation for (25, 26) to make computer calculations. For instance it is easy to show that the equations

$$\sum_{k=1}^{\infty} U_{nk} U_{km} = \delta_{nm}, \qquad \sum_{k=1}^{\infty} \bar{U}_{nk} \bar{U}_{km} = \delta_{nm} \tag{28}$$

are satisfied to any desired order of approximation. Each identity follows from the other by using (23). It is also easy to make the identification

$$V^{ab}_{nm} = (-1)^{n+m} \sqrt{nm}\, N^{ab}_{nm} \tag{29}$$

of the Neumann coefficients with the standard three strings vertex coefficients[1]. Using (28), together with the decomposition (21), it is easy to establish the commutativity relation (written in matrix notation)

$$[CV^{ab}, CV^{a'b'}] = 0 \tag{30}$$

for any a, b, a', b'. This relation is fundamental for the next developments.

An analytic proof of eq.(28) is given in Appendix.

3.2 Zero modes

The Neumann coefficients involving one zero mode are given by

$$N^{ab}_{0m} = -\frac{1}{m} \oint \frac{dw}{2\pi i} \frac{1}{w^m} f_b'(w) \frac{1}{f_a(0) - f_b(w)} \tag{31}$$

In this case too we make the decomposition

$$N^{ab}_{0m} = \frac{1}{3}\Big(E_m + \bar{\alpha}^{a-b} U_m + \alpha^{a-b} \bar{U}_m\Big) \tag{32}$$

where $E, U, \bar{U}$ can be given, after some algebra, the explicit expression

$$\begin{aligned} E_n &= -\frac{4i}{n} \oint \frac{dw}{2\pi i} \frac{1}{w^n} \frac{1}{1+w^2} \frac{f^3(w)}{1-f^3(w)} = \frac{2i^n}{n} \\ U_n &= -\frac{4i}{n} \oint \frac{dw}{2\pi i} \frac{1}{w^n} \frac{1}{1+w^2} \frac{f^2(w)}{1-f^3(w)} = \frac{\alpha_n}{n} \\ \bar{U}_n &= (-1)^n\, U_n = (-1)^n \frac{\alpha_n}{n} \end{aligned} \tag{33}$$

[1]The factor of $(-1)^{n+m}$ in (29) arises from the fact that the original definition of the Neumann coefficients (19) in [6] refers to the bra three strings vertex $\langle V_3|$, rather than to the ket vertex like in (11); therefore the two definitions differ by a *bpz* operation.

The numbers α_n are Taylor coefficients

$$\sqrt{f(z)} \quad = \quad \sum_0^\infty \alpha_n z^n$$

They are related to the A_n coefficients of Appendix B of [46] (see also [44]) as follows: $\alpha_n = A_n$ for n even and $\alpha_n = iA_n$ for n odd. N_{0n}^{ab} are not related in a simple way as (29) to the corresponding three strings vertex coefficients. The reason is that the latter satisfy the conditions

$$\sum_{a=1}^{3} V_{0n}^{ab} = 0 \tag{34}$$

These constraints fix the invariance $V_{0n}^{ab} \to V_{0n}^{ab} + B_n^b$, where B_n^b are arbitrary numbers, an invariance which arises in the vertex (11) due to momentum conservation. For the Neumann coefficients N_{0n}^{ab} we have instead

$$\sum_{a=1}^{3} V_{0n}^{ab} = E_n \tag{35}$$

It is thus natural to define

$$\hat{N}_{0n}^{ab} = N_{0n}^{ab} - \frac{1}{3} E_n \tag{36}$$

Now one can easily verify that[2]

$$V_{0n}^{ab} = -\sqrt{2n}\,\hat{N}_{0n}^{ab} \tag{37}$$

It is somewhat surprising that in this relation we do not meet the factor $(-1)^n$, which we would expect on the basis of the *bpz* conjugation (see footnote after eq.(29)). However eq.(37) is also naturally requested by the integrable structure found in [11]. The absence of the $(-1)^n$ factor corresponds to the exchange $V_{0n}^{12} \leftrightarrow V_{0n}^{21}$. This exchange does not seem to affect in any significant way the results obtained so far in this field.

To complete the discussion about the matter sector one should recall that beside eq.(28), there are other basic equations from which all the results about the Neumann coefficients can be derived. They concern the quantities

$$W_n = -\sqrt{2n}\,U_n = -\sqrt{\frac{2}{n}}\,\alpha_n, \qquad W_n^* = -\sqrt{2n}\,\bar{U}_n = -\sqrt{\frac{2}{n}}\,(-1)^n \alpha_n \tag{38}$$

[2]The $\sqrt{2}$ factor is there because in [46] the $\alpha' = 1$ convention is used

The relevant identities, [44, 46], are

$$\sum_{n=1}^{\infty} W_n \, U_{np} = W_p, \qquad \sum_{n\geq 1} W_n W_n^* = 2V_{00}^{aa} \tag{39}$$

These identities can easily be shown numerically to be correct at any desired approximation. An analytic proof can presumably be obtained with the same methods as in Appendix.

Finally let us concentrate on the Neumann coefficients N_{00}^{ab}. Although a formula for them can be found in [6], these numbers are completely arbitrary due to momentum conservation. The choice

$$V_{00}^{ab} = \delta_{ab} \ln \frac{27}{16} \tag{40}$$

is the same as in [44], but it is also motivated by one of the most surprising and mysterious aspects of SFT, namely its underlying integrable structure: the matter Neumann coefficients obey the Hirota equations of the dispersionless Toda lattice hierarchy. This was explained in [11] following a suggestion of [12]. On the basis of these equations the matter Neumann coefficients with nonzero labels can be expressed in terms of the remaining ones. The choice of (40) in this context is natural.

4 Ghost three strings vertex and bc Neumann coefficients

The three strings vertex for the ghost part is more complicated than the matter part due to the zero modes of the c field. As we will see, the latter generate an ambiguity in the definition of the Neumann coefficients. Such an ambiguity can however be exploited to formulate and solve in a compact form the problem of finding solutions to eq.(9)[3] .

4.1 Neumann coefficients: definitions and properties

To start with we define, in the ghost sector, the vacuum states $|\hat{0}\rangle$ and $|\dot{0}\rangle$ as follows

$$|\hat{0}\rangle = c_0 c_1 |0\rangle, \qquad |\dot{0}\rangle = c_1 |0\rangle \tag{41}$$

[3]An alternative treatment of the ghost three–strings vertex has been given recently in [60, 62]

where $|0\rangle$ is the usual $SL(2,\mathbb{R})$ invariant vacuum. Using *bpz* conjugation

$$c_n \to (-1)^{n+1}c_{-n}, \qquad b_n \to (-1)^{n-2}b_{-n}, \qquad |0\rangle \to \langle 0| \tag{42}$$

one can define conjugate states. It is important that, when applied to products of oscillators, the *bpz* conjugation does not change the order of the factors, but transforms rigidly the vertex (see eq.(44) below).

The three strings interaction vertex is defined, as usual, as a squeezed operator acting on three copies of the bc Hilbert space

$$\langle \tilde{V}_3| = {}_1\langle\hat{0}|\,{}_2\langle\hat{0}|\,{}_3\langle\hat{0}|e^{\tilde{E}}, \qquad \tilde{E} = \sum_{a,b=1}^{3}\sum_{n,m}^{\infty} c_n^{(a)}\tilde{N}_{nm}^{ab}b_m^{(b)} \tag{43}$$

Under *bpz* conjugation

$$|\tilde{V}_3\rangle = e^{\tilde{E}'}|\hat{0}\rangle_1|\hat{0}\rangle_2|\hat{0}\rangle_3, \qquad \tilde{E}' = -\sum_{a,b=1}^{3}\sum_{n,m}^{\infty}(-1)^{n+m}c_n^{(a)\dagger}\tilde{N}_{nm}^{ab}b_m^{(b)\dagger} \tag{44}$$

In eqs.(43, 44) we have not specified the lower bound of the m,n summation. This point will be clarified below.
The Neumann coefficients $\tilde{N}_{nm}^{ab}$ are given by the contraction of the bc oscillators on the unit disk (constructed out of three unit semidisks, as explained in section 3). They represent Fourier components of the $SL(2,\mathbb{R})$ invariant bc propagator (i.e. the propagator in which the zero mode have been inserted at fixed points ζ_i, $i = 1,2,3$):

$$\langle b(z)c(w)\rangle = \frac{1}{z-w}\prod_{i=1}^{3}\frac{w-\zeta_i}{z-\zeta_i} \tag{45}$$

Taking into account the conformal properties of the b,c fields we get

$$\begin{aligned}\tilde{N}_{nm}^{ab} &= \langle \tilde{V}_{123}|b_{-n}^{(a)}c_{-m}^{(b)}|\dot{0}\rangle_{123}\\ &= -\oint\frac{dz}{2\pi i}\oint\frac{dw}{2\pi i}\frac{1}{z^{n-1}}\frac{1}{w^{m+2}}\frac{(f_a'(z))^2}{f_a(z)-f_b(w)}\prod_{i=1}^{3}\frac{f_b(w)-\zeta_i}{f_a(z)-\zeta_i}(f_b'(w))^{-1}\end{aligned}$$

It is straightforward to check that

$$\tilde{N}_{nm}^{ab} = \tilde{N}_{nm}^{a+1,b+1} \tag{46}$$

and (by letting $z \to -z$, $w \to -w$)

$$\tilde{N}_{nm}^{ab} = (-1)^{n+m}\tilde{N}_{nm}^{ba} \tag{47}$$

Now we choose $\zeta_i = f_i(0) = \alpha^{2-i}$ so that the product factor in (46) nicely simplifies as follows

$$\prod_{i=1}^{3} \frac{f_b(w) - f_i(0)}{f_a(z) - f_i(0)} = \frac{f^3(w) - 1}{f^3(z) - 1}, \quad \forall\, a, b = 1, 2, 3 \tag{48}$$

Now, as in the matter case, we consider the decomposition

$$\tilde{N}^{ab}_{nm} = \frac{1}{3}(\tilde{E}_{nm} + \bar{\alpha}^{a-b}\tilde{U}_{nm} + \alpha^{a-b}\tilde{\bar{U}}_{nm}) \tag{49}$$

where

$$\begin{aligned}
\tilde{E}_{nm} &= \oint \frac{dz}{2\pi i} \oint \frac{dw}{2\pi i} \mathcal{N}_{nm}(z,w)\mathcal{A}(z,w) \\
\tilde{U}_{nm} &= \oint \frac{dz}{2\pi i} \oint \frac{dw}{2\pi i} \mathcal{N}_{nm}(z,w)\mathcal{U}(z,w) \\
\tilde{\bar{U}}_{nm} &= \oint \frac{dz}{2\pi i} \oint \frac{dw}{2\pi i} \mathcal{N}_{nm}(z,w)\bar{\mathcal{U}}(z,w)
\end{aligned} \tag{50}$$

and

$$\begin{aligned}
\mathcal{A}(z,w) &= \frac{3f(z)f(w)}{f^3(z) - f^3(w)} \\
\mathcal{U}(z,w) &= \frac{3f^2(z)}{f^3(z) - f^3(w)} \\
\bar{\mathcal{U}}(z,w) &= \frac{3f^2(w)}{f^3(z) - f^3(w)} \\
\mathcal{N}_{nm}(z,w) &= \frac{1}{z^{n-1}} \frac{1}{w^{m+2}} (f'(z))^2 (f'(w))^{-1} \frac{f^3(w) - 1}{f^3(z) - 1}
\end{aligned}$$

After some elementary algebra, using $f'(z) = \frac{4i}{3}\frac{1}{1+z^2} f(z)$, one finds

$$\begin{aligned}
\tilde{E}_{nm} &= \oint \frac{dz}{2\pi i} \oint \frac{dw}{2\pi i} \frac{1}{z^{n+1}} \frac{1}{w^{m+1}} \Big(\frac{1}{1+zw} - \frac{w}{w-z} \Big) \\
\tilde{U}_{nm} &= \oint \frac{dz}{2\pi i} \oint \frac{dw}{2\pi i} \frac{1}{z^{n+1}} \frac{1}{w^{m+1}} \frac{f(z)}{f(w)} \Big(\frac{1}{1+zw} - \frac{w}{w-z} \Big) \\
\tilde{\bar{U}}_{nm} &= \oint \frac{dz}{2\pi i} \oint \frac{dw}{2\pi i} \frac{1}{z^{n+1}} \frac{1}{w^{m+1}} \frac{f(w)}{f(z)} \Big(\frac{1}{1+zw} - \frac{w}{w-z} \Big)
\end{aligned} \tag{51}$$

Using the property $f(-z) = (f(z))^{-1}$, one can easily prove that

$$\tilde{\bar{U}}_{nm} = (-1)^{n+m}\tilde{U}_{nm} \tag{52}$$

4.2 Computation of the coefficients

In this section we explicitly compute the above integrals. We shall see that the presence of the three c zero modes induces an ambiguity in the $(0,0)$, $(-1,1)$, $(1,-1)$ components of the Neumann coefficients. This in turn arises from the ambiguity in the radial ordering of the integration variables z, w. While the result does not depend on what variable we integrate first, it does depend in general on whether $|z| > |w|$ or $|z| < |w|$.

If we choose $|z| > |w|$ we get

$$\tilde{E}^{(1)}_{nm} = \theta(n)\theta(m)(-1)^n\delta_{nm} + \delta_{n,0}\delta_{m,0} + \delta_{n,-1}\delta_{m,1} \tag{53}$$

while, if we choose $|z| < |w|$, we obtain

$$\tilde{E}^{(2)}_{nm} = \theta(n)\theta(m)(-1)^n\delta_{nm} - \delta_{n,1}\delta_{m,-1} \tag{54}$$

where $\theta(n) = 1$ for $n > 0$, $\theta(n) = 0$ for $n \leq 0$. We see that the result is ambiguous for the components $(0,0)$, $(-1,1)$, $(1,-1)$.

To compute $\tilde{U}_{nm}$ we expand $f(z)$ for small z, as in section 3,

$$f(z) \quad = \quad \sum_{k=0}^{\infty} b_k z^k$$

Since $f^{-1}(z) = f(-z)$ we get the relation

$$\sum_{k=0}^{n} (-1)^k b_k b_{n-k} = \delta_{n,0} \tag{55}$$

which is identically satisfied for n odd, while for n even it can be also rewritten as

$$b_n^2 = -2\sum_{k=1}^{n} (-1)^k b_{n-k} b_{n+k} \tag{56}$$

Taking $|z| > |w|$ and integrating z first, one gets

$$\tilde{U}^{(1a)}_{nm} = \delta_{n+m} + (-1)^m \sum_{l=1}^{n} (b_{n-l}b_{m-l} - (-1)^l b_{n-l}b_{m+l}) \tag{57}$$

If, instead, we integrate w first,

$$\tilde{U}^{(1b)}_{nm} = (-1)^m b_n b_m + (-1)^m \sum_{l=1}^{m} (b_{n-l}b_{m-l} + (-1)^l b_{n+l}b_{m-l}) \tag{58}$$

One can check that, due to (56),

$$\tilde{U}^{(1a)}_{nm} \quad = \quad \tilde{U}^{(1b)}_{nm} \equiv \tilde{U}^{(1)}_{nm} \tag{59}$$

Now we take $|z| < |w|$ and get similarly

$$\tilde{U}^{(2a)}_{nm} = (-1)^m \sum_{l=1}^{n} (b_{n-l} b_{m-l} - (-1)^l b_{n-l} b_{m+l})$$

$$\tilde{U}^{(2b)}_{nm} = -\delta_{n+m} + (-1)^m b_n b_m + (-1)^m \sum_{l=1}^{m} (b_{n-l} b_{m-l} + (-1)^l b_{n+l} b_{m-l})$$

Again, due to (56)

$$\tilde{U}^{(2a)}_{nm} \quad = \quad \tilde{U}^{(2b)}_{nm} = \tilde{U}^{(2)}_{nm} \tag{60}$$

Comparing $\tilde{U}^{(1)}$ with $\tilde{U}^{(2)}$, we see once more that the ambiguity only concerns the $(0,0)$, $(-1,1)$, $(1,-1)$ components. Using (49) we define

$$\tilde{N}^{ab,\,(1,2)}_{nm} \quad = \quad \frac{1}{3}(\tilde{E}^{(1,2)}_{nm} + \bar{\alpha}^{(a-b)} \tilde{U}^{(1,2)}_{nm} + \alpha^{a-b} (-1)^{n+m} \bar{\tilde{U}}^{(1,2)}_{nm})$$

The above ambiguity propagates also to these coefficients, but only when $a = b$. For later reference it is useful to notice that

$$\begin{aligned} \tilde{N}^{ab,\,(1,2)}_{-1,m} &= 0, \quad \text{except} \quad \text{perhaps} \quad \text{for} \quad a = b, \quad m = 1 \\ \tilde{N}^{ab,\,(1,2)}_{0,m} &= 0, \quad \text{except} \quad \text{perhaps} \quad \text{for} \quad a = b \quad m = 0 \end{aligned} \tag{61}$$

and, for $|n| \leq 1$,

$$\tilde{N}^{ab,\,(1,2)}_{n,1} = 0, \quad \text{except} \quad \text{perhaps} \quad \text{for} \quad a = b \quad n = -1 \tag{62}$$

We notice that, if in eq.(43,44) the summation over m, n starts from -1, the above ambiguity is consistent with the general identification proposed in [6]

$$\tilde{N}^{ab}_{nm} = \langle \tilde{V}_3 | b^{(a)}_{-n} c^{(b)}_{-m} |\dot{0}\rangle_1 |\dot{0}\rangle_2 |\dot{0}\rangle_3 \tag{63}$$

It is easy to see that the expression in the RHS is not *bpz* covariant when (m, n) take values $(0,0)$, $(-1,1)$, $(1,-1)$ and the lower bound of the m, n summation in the vertex (see above) is -1. Such *bpz* noncovariance corresponds exactly to the ambiguity we have come across in the explicit evaluation of the Neumann coefficients. We can refer to it as the *bpz* or *radial ordering anomaly.*

4.3 Two alternatives

It is clear that we are free to fix the ambiguity the way we wish, provided the convention we choose is consistent with *bpz* conjugation. We consider here two possible choices. The first consists in setting to zero all the components of the Neumann coefficients which are ambiguous, i.e. the $(0,0)$, $(-1,1)$, $(1,-1)$ ones. This leads to a definition of the vertex (43) in which the summation over n starts from 1 while the summation over m starts from 0. In this way any ambiguity is eliminated and the Neumann coefficients are *bpz* covariant. This is the preferred choice in the literature, [14, 16, 15, 17, 18]. In particular, it has led in [14] to a successful comparison of the operator formulation with a twisted conformal field theory one.

We would like, now, to make some comments about this first choice, with the purpose of stressing the difference with the alternative one we will discuss next. In particular we would like to emphasize some aspects of the BRST cohomology in VSFT. In VSFT the BRST operator is conjectured [14, 15] to take the form

$$\mathcal{Q} = c_0 + \sum_{n=1}^{\infty} f_n (c_n + (-1)^n c_{-n}) \tag{64}$$

It is easy to show that the vertex is BRST invariant, i.e.

$$\sum_{a=1}^{3} \mathcal{Q}^{(a)} |\tilde{V}_3\rangle = 0 \tag{65}$$

Due to

$$\{\mathcal{Q}, b_0\} = 1 \tag{66}$$

it follows that the cohomology of $\mathcal{Q}$ is trivial. As was noted in [17], this implies that the subset of the string field algebra that solves (9) is the direct sum of $\mathcal{Q}$–closed states and b_0–closed states (i.e. states in the Siegel gauge).

$$|\Psi\rangle = \mathcal{Q}|\lambda\rangle + b_0|\chi\rangle \tag{67}$$

As a consequence of the BRST invariance of the vertex it follows that the star product of a BRST-exact state with any other is identically zero. This implies that the VSFT equation of motion can determine only the Siegel gauge part of the solution.

For this reason previous calculations were done with the use of the *reduced* vertex [15, 14] which consists of Neumann coefficients starting from the (1,1) component. The unreduced star product can be recovered by the midpoint insertion of $\mathcal{Q} = \frac{1}{2i}(c(i) - c(-i))$ as

$$|\psi * \phi\rangle = \mathcal{Q}|\psi *_{b_0} \phi\rangle \tag{68}$$

where $*_{b_0}$ is the reduced product.

In the alternative treatment given below, using an enlarged Fock space, we compute the star product and solve (9), without any gauge choice and any explicit midpoint insertion.

Motivated by the advantages it offers in the search of solutions to (9), we propose therefore a second option. It consists in fixing the ambiguity by setting

$$\tilde{N}^{aa}_{-1,1} = \tilde{N}^{aa}_{1,-1} = 0, \qquad \tilde{N}^{aa}_{0,0} = 1. \tag{69}$$

If we do so we get a fundamental identity, valid for $\tilde{U}_{nm} \equiv \tilde{U}^{(1)}_{nm}$ (for $n, m \geq 0$),

$$\sum_{k=0} \tilde{U}_{nk}\tilde{U}_{km} = \delta_{nm} \tag{70}$$

Defining

$$\tilde{X}^{ab} = C\tilde{V}^{ab}, \tag{71}$$

eq.(70) entails

$$[\tilde{X}^{ab}, \tilde{X}^{a'b'}] = 0 \tag{72}$$

One can prove eq.(70) numerically. By using a cutoff in the summation one can approximate the result to any desired order (although the convergence with increasing cutoff is less rapid than in the corresponding matter case, see section 3.1). A direct analytic proof of eq.(70) is given in Appendix.

The next subsection is devoted to working out some remarkable consequences of eq.(70).

4.4 Matrix structure

Once the convention (69) is chosen, we recognize that all the matrices $(\tilde{E}, \tilde{U}, \bar{\tilde{U}})$ have the $(0,0)$ component equal to 1, all the other entries of the upper row equal to 0, and a generally non vanishing zeroth column. More precisely

$$\begin{aligned}
&\tilde{U}_{00} = \tilde{E}_{00} = 1 \\
&\tilde{U}_{n0} = b_n \quad \tilde{E}_{n0} = 0, \qquad \tilde{U}_{0n} = \tilde{E}_{0n} = \delta_{n,0} \\
&\tilde{U}_{nm} \neq 0, \quad n, m > 0
\end{aligned} \tag{73}$$

This particular structure makes this kind of matrices simple to handle under a generic analytic map f. In order to see this, let us inaugurate a new notation, which we will use in this and the next section. We recall that the labels M, N indicate the couple $(0,m),(0,n)$. Given a matrix M, let us distinguish between the 'large' matrix M_{MN} denoted by the calligraphic symbol $\mathcal{M}$ and the 'small' matrix M_{mn} denoted by the plain symbol M. Accordingly, we will denote by $\mathcal{Y}$ a matrix of the form (73), $\vec{y}=(y_1,y_2,...)$ will denote the nonvanishing column vector and Y the 'small' matrix

$$\mathcal{Y}_{NM}=\delta_{N0}\delta_{M0}+y_n\delta_{M0}+Y_{mn}, \tag{74}$$

or, symbolically, $\mathcal{Y}=(1,\vec{y},Y)$.

Then, using a formal Taylor expansion for f, one can show that

$$f[\mathcal{Y}]_{NM}=f[1]\delta_{N0}\delta_{M0}+\Big(\frac{f[1]-f[Y]}{1-Y}\vec{y}\Big)_n\delta_{M0}+f[Y]_{mn} \tag{75}$$

Now let us define

$$\begin{aligned}\mathcal{Y} &\equiv \tilde{X}^{11}\\ \mathcal{Y}_+ &\equiv \tilde{X}^{12} &(76)\\ \mathcal{Y}_- &\equiv \tilde{X}^{21} &(77)\end{aligned}$$

These three matrices have the above form. Using (70) one can prove the following properties (which are well–known for the 'small' matrices)

$$\begin{gathered}\mathcal{Y}+\mathcal{Y}_++\mathcal{Y}_-=1\\ \mathcal{Y}^2+\mathcal{Y}_+^2+\mathcal{Y}_-^2=1\\ \mathcal{Y}_+^3+\mathcal{Y}_-^3=2\mathcal{Y}^3-3\mathcal{Y}^2+1\\ \mathcal{Y}_+\mathcal{Y}_-=\mathcal{Y}^2-\mathcal{Y}\\ [\mathcal{Y},\mathcal{Y}_\pm]=0\\ [\mathcal{Y}_+,\mathcal{Y}_-]=0\end{gathered} \tag{78}$$

Using (74, 75) we immediately obtain (we point out that, in particular for $\mathcal{Y}$, $y_{2n}=\frac{2}{3}\,b_{2n}$, $y_{2n+1}=0$ and $Y_{nm}=\tilde{X}_{nm}$ for $n,m>0$)

$$\begin{gathered}Y+Y_++Y_-=1\\ \vec{y}+\vec{y}_++\vec{y}_-=0\\ Y^2+Y_+^2+Y_-^2=1\\ (1+Y)\vec{y}+Y_+\vec{y}_++Y_-\vec{y}_-=0\end{gathered}$$

$$
\begin{aligned}
Y_+^3 + Y_-^3 &= 2Y^3 - 3Y^2 + 1 \\
Y_+^2 \vec{y}_+ + Y_-^2 \vec{y}_- &= (2Y^2 - Y - 1)\vec{y} \\
Y_+ Y_- &= Y^2 - Y \\
[Y, Y_\pm] &= 0 \\
[Y_+, Y_-] &= 0 \\
Y_+ \vec{y}_- &= Y\vec{y} = Y_- \vec{y}_+ \\
-Y_\pm \vec{y} &= (1 - Y)\vec{y}_\pm
\end{aligned}
\tag{79}
$$

These properties were shown in various papers, see [15, 18]. Here they are simply consequences of (78), and therefore of (70). In particular we note that the properties of the 'large' matrices are isomorphic to those of the 'small' ones. This fact allows us to work directly with the 'large' matrices, handling at the same time both zero and not zero modes.

4.5 Enlarged Fock space

We have seen in the last subsection the great advantages of introducing the convention (69). In this subsection we make a proposal as to how to incorporate this convention in an enlargement of the bc system's Fock space. In fact, in order for eq.(63) to be consistent, a modification in the RHS of this equation is in order. This can be done by, so to speak, 'blowing up' the zero mode sector. We therefore enlarge the original Fock space, while warning that our procedure may be far from unique. For each string, we split the modes c_0 and b_0:

$$
\eta_0 \leftarrow c_0 \rightarrow \eta_0^\dagger, \qquad \xi_0^\dagger \leftarrow b_0 \rightarrow \xi_0 \tag{80}
$$

In other words we introduce two additional couple of conjugate anticommuting creation and annihilation operators $\eta_0, \eta_0^\dagger$ and $\xi_0, \xi_0^\dagger$

$$
\{\xi_0, \eta_0\} = 1, \qquad \{\xi_0^\dagger, \eta_0^\dagger\} = 1 \tag{81}
$$

with the following rules on the vacuum

$$
\xi_0|0\rangle = 0, \qquad \langle 0|\xi_0^\dagger = 0 \tag{82}
$$

$$
\eta_0^\dagger|0\rangle = 0, \qquad \langle 0|\eta_0 = 0 \tag{83}
$$

while $\xi_0^\dagger, \eta_0$ acting on $|0\rangle$ create new states. The bpz conjugation properties are defined by

$$
bpz(\eta_0) = -\eta_0^\dagger, \qquad bpz(\xi_0) = \xi_0^\dagger \tag{84}
$$

The reason for this difference is that η_0 (ξ_0) is meant to be of the same type as c_0 (b_0). The anticommutation relation of c_0 and b_0 remain the same

$$\{c_0, b_0\} = 1 \tag{85}$$

All the other anticommutators among these operators and with the other bc oscillators are required to vanish. In the enlarged Fock space all the objects we have defined so far may get slightly changed. In particular the three strings vertex (43,44) is now defined by

$$\tilde{E}'_{(en)} = \sum_{n\geq 1, m\geq 0}^{\infty} c_n^{(a)\,\dagger} \tilde{V}_{nm}^{(ab)} b_m^{(b)\dagger} - \eta_0^{(a)} b_0^{(a)} \tag{86}$$

With this redefinition of the vertex any ambiguity is eliminated, as one can easily check. In a similar way we may have to modify all the objects that enter into the game.

The purpose of the Fock space enlargement is to make us able to evaluate vev's of the type

$$\langle\dot{0}|\exp\Big(cFb + c\mu + \lambda b\Big)\,\exp\Big(c^\dagger G b^\dagger + \theta b^\dagger + c^\dagger\zeta\Big)|\hat{0}\rangle \tag{87}$$

which are needed in star products. Here we use an obvious compact notation: F, G denotes matrices F_{NM}, G_{NM}, and $\lambda, \mu, \theta, \zeta$ are anticommuting vectors $\lambda_N, \mu_N, \theta_N, \zeta_N$. In $cFb + c\mu + \lambda b$ it is understood that the mode b_0 is replaced by ξ_0 and in $c^\dagger G b^\dagger + \theta b^\dagger + c^\dagger \zeta$ the mode c_0 is replaced by η_0. In this way the formula is unambiguous and we obtain

$$\langle\dot{0}|\exp\left(cFb + c\mu + \lambda b\right)\exp\left(c^\dagger G b^\dagger + \theta b^\dagger + c^\dagger\zeta\right)|\hat{0}\rangle \tag{88}$$

$$= \det(1+FG)\,\exp\Big(-\theta\tfrac{1}{1+FG}F\zeta - \lambda\tfrac{1}{1+GF}G\mu - \theta\tfrac{1}{1+FG}\mu + \lambda\tfrac{1}{1+GF}\zeta\Big) \tag{89}$$

Eventually, after performing the star products, we will return to the original Fock space.

5 Solving the ghost equation of motion in VSFT

We are now ready to deal with the problem of finding a solution to (9)

$$\mathcal{Q}|\psi\rangle + |\psi\rangle * |\psi\rangle = 0 \tag{90}$$

Since now we are operating in an enlarged the Fock space, $\mathcal{Q}$ must be modified, with respect to the conjectured form of the BRST operator (64) in VSFT, in the following way

$$\mathcal{Q} \to \mathcal{Q}_{(en)} = c_0 - \eta_0 + \eta_0^\dagger + \sum_{n=1}^{\infty} f_n(c_n + (-1)^n c_{-n}) \tag{91}$$

The first thing we would like to check is BRST invariance of the vertex, i.e.

$$\sum_{a=1}^{3} \mathcal{Q}_{(en)}^{(a)} |\tilde{V}_3\rangle_{(en)} = 0 \tag{92}$$

It is easy to verify that both equations are identically satisfied thanks to the first two eqs.(79), and thanks to addition of $-\eta_0$ in (91) ($\eta_0^\dagger$ passes through and annihilates the vacuum).

In order to solve equation (90) we proceed to find a solution to

$$|\hat{\psi}\rangle_3 =_1 \langle\dot{\psi}|_2\langle\dot{\psi}|V_{123}\rangle \tag{93}$$

where $\hat{\psi}$ and $\dot{\psi}$ are the same state on the ghost number 2 and 1 vacuum, respectively. We choose the following ansatz

$$|\hat{\psi}\rangle \quad = \quad |\hat{S}_{(en)}\rangle = \mathcal{N} \exp\left(\sum_{n,m\geq 1} c_n^\dagger S_{nm} b_m^\dagger + \sum_{N\geq 0} c_N^\dagger S_{N0} \xi_0^\dagger\right)|\hat{0}\rangle \tag{94}$$

$$|\dot{\psi}\rangle \quad = \quad |\dot{S}_{(en)}\rangle = \mathcal{N} \exp\left(\sum_{n,m\geq 1} c_n^\dagger S_{nm} b_m^\dagger + \sum_{N\geq 0} c_N^\dagger S_{N0} \xi_0^\dagger\right)|\dot{0}\rangle \tag{95}$$

Following now the standard procedure, [10, 30], from (93), using (88), we get

$$\mathcal{T} = \mathcal{Y} + (\mathcal{Y}_+, \mathcal{Y}_-)\frac{1}{1 - \Sigma\mathcal{V}}\Sigma \begin{pmatrix} \mathcal{Y}_- \\ \mathcal{Y}_+ \end{pmatrix} \tag{96}$$

In RHS of these equations

$$\Sigma = \begin{pmatrix} \mathcal{T} & 0 \\ 0 & \mathcal{T} \end{pmatrix}, \qquad \mathcal{V} = \begin{pmatrix} \mathcal{Y} & \mathcal{Y}_+ \\ \mathcal{Y}_- & \mathcal{Y} \end{pmatrix}.$$

where $\mathcal{T} = C\mathcal{S}$ and $\mathcal{Y}, \mathcal{Y}_\pm$ have been defined by eq.(77).

We repeat once more that the matrix equation (96) is understood for 'large' matrices, which include the zeroth row and column, i.e. $\mathcal{Y} = \tilde{X}^{11} =$

$C\tilde{N}^{11} = (1, \vec{y}, Y \equiv \tilde{X})$, $\mathcal{T} = (1, \vec{t}, \tilde{T})$ and $\mathcal{S} = (1, \vec{s}, \tilde{S})$. This is a novelty of our treatment. In fact, solving eq.(96), we obtain the algebraic equation

$$\mathcal{T} = C\mathcal{S} = \frac{1}{2\mathcal{Y}}\Big(1 + \mathcal{Y} - \sqrt{(1-\mathcal{Y})(1+3\mathcal{Y})}\Big) \tag{97}$$

which splits into the relations

$$\begin{aligned} \mathcal{T}_{00} &= \mathcal{S}_{00} = 1 \\ \tilde{T} &= \frac{1}{2\tilde{X}}\Big(1 + \tilde{X} - \sqrt{(1-\tilde{X})(1+3\tilde{X})}\Big) \\ \vec{t} &= \frac{1-\tilde{T}}{1-\tilde{X}}\vec{y} \end{aligned} \tag{98}$$

The normalization constant $\mathcal{N}$ is, formally, given by

$$\mathcal{N} = \frac{1}{\det(1 - \Sigma\mathcal{V})} \tag{99}$$

However we notice that the (0,0) entry of $\Sigma\mathcal{V}$ is 1, so the determinant vanishes. Therefore we have to introduce a regulator $\varepsilon \to 0$, and write

$$\mathcal{N}_\varepsilon = \frac{1}{\varepsilon}\frac{1}{\det'(1 - \Sigma\mathcal{V})} \tag{100}$$

where $\det'$ is the determinant of the 'small' matrix part alone. This divergence is not present in the literature, [14, 18]. It is in fact related to the 1 eigenvalue of $\mathcal{T}$ and $\mathcal{Y}$ in the twist even sector (i.e. in the eigenspace of C with eigenvalue 1). This is therefore an additional divergence with respect to the usual one due to the 1 eigenvalue of $\tilde{X}$ in the twist–odd sector (see below).

Now we prove that this solves (90). Indeed, after some elementary algebra, we arrive at the expression

$$\mathcal{Q}_{(en)}|\dot{S}\rangle + |\hat{S}\rangle = \Big(-c_n^\dagger\,[(\vec{s})_n - (C-\mathcal{S})_{nk}f_k] + c_0 - \eta_0\Big)|\dot{S}\rangle \tag{101}$$

We would like to find $\vec{f}$ so that the expression in square brackets in (101) vanishes. Using the last equation in (98) we see that this is true provided

$$\vec{y} = (1 - \tilde{X})\vec{f} \tag{102}$$

Now, by means of an explicit calculation, we verify that the solution to (102) is

$$f_n = \frac{1}{2}(1 + (-1)^n)(-1)^{\frac{n}{2}} \tag{103}$$

For inserting in the RHS of (102) both (103) and $\tilde{X}$ in the form

$$\tilde{X} \;=\; \frac{1}{3}(1 + C\tilde{U} + \tilde{U}C)$$

we see that the vanishing of f_n for n odd is consistent since $\vec{y}$ has no odd components, while for n even we have

$$y_{2n} = \sum_{k=1}^{\infty} \frac{2}{3}(-1)^k \Big(\delta_{2n,2k} - \tilde{U}_{2n,2k}\Big) \tag{104}$$

The second sum is evaluated with the use of the integral representation of $\mathcal{U}$ (51)

$$\begin{aligned}
&\sum_{k=1}^{\infty} (-1)^k \, \tilde{U}_{2n,2k} = \\
&= \oint \frac{dz}{2\pi i} \oint \frac{dw}{2\pi i} \frac{1}{z^{2n+1}} \sum_{k=1}^{\infty} (-1)^k \frac{1}{w^{2k+1}} \frac{f(z)}{f(w)} \Big(\frac{1}{1+zw} - \frac{w}{w-z}\Big) \\
&= -\oint \frac{dz}{2\pi i} \oint \frac{dw}{2\pi i} \frac{1}{z^{2n+1}} \frac{1}{w} \frac{1}{1+w^2} \frac{f(z)}{f(w)} \Big(\frac{1}{1+zw} - \frac{w}{w-z}\Big) \qquad (105) \\
&= -\oint \frac{dz}{2\pi i} \frac{1}{z^{2n+1}} f(z) \Big(1 - \frac{1}{f(z)} \frac{1}{1+z^2}\Big) \\
&= -b_{2n} + \sum_{k=1}^{\infty} (-1)^k \delta_{2n,2k}
\end{aligned}$$

The δ-piece cancels with the one in (104), while the remaining one is precisely y_{2n}.

The derivation in (105) requires some comments. In passing from the first to the second line we use $\sum_{k=1}^{\infty}(-1)^k \frac{1}{w^{2k+1}} = -\frac{1}{w}\frac{1}{1+w^2}$, which converges for $|w| > 1$. Therefore, in order to make sense of the operation, we have to move the w contour outside the circle of radius one. This we can do provided we introduce a regulator (see Appendix) to avoid the overlapping of the contour with the branch points of $f(w)$, which are located at $w = \pm i$. With the help of a regulator we move them far enough and eventually we will move them back to their original position. Now we can fully rely on the integrand in the second line of (105). Next we start moving the w contour back to its original position around the origin. In so doing we meet two poles (those referring to the $\frac{1}{1+w^2}$ factor), but it is easy to see that their contribution neatly vanishes due to the last factor in the integrand. The remaining contributions come from the poles at $w = z$ and at $w = 0$. Their evaluation leads to the third line in (105). The rest is obvious.

As a result of this calculation we find that eq.(101) becomes

$$\mathcal{Q}_{(en)}|\dot{S}_{(en)}\rangle + |\hat{S}_{(en)}\rangle = (c_0 - \eta_0)|\dot{S}_{(en)}\rangle \tag{106}$$

Finally, as a last step, we return to the original Fock space. A practical rule to do so is to drop all the double zero mode terms in the exponentials[4] (such as, for instance, $c_0\xi_0^\dagger$) and to impose the condition $c_0 - \eta_0 = 0$ on the states, i.e. by considering all the states that differ by $c_0 - \eta_0$ acting on some state as equivalent. The same has to be done also for $b_0 - \xi_0^\dagger$ (paying attention not to apply both constraints simultaneously, because they do not commute). These rules are enough for our purposes. In this context the RHS of eq.(106) is in the same class as 0.

Let us collect the results. In the original Fock space the three string vertex is defined by

$$\tilde{E}' = \sum_{n\geq 1, M\geq 0}^{\infty} c_n^{(a)\,\dagger}\tilde{V}_{nM}^{(ab)}b_M^{(b)\dagger} \tag{107}$$

eqs.(94,95) becomes

$$|\hat{S}\rangle = \mathcal{N}\exp\left(\sum_{n,m\geq 1} c_n^\dagger S_{nm} b_m^\dagger + \sum_{n\geq 1} c_n^\dagger S_{n0} b_0\right)|\hat{0}\rangle \tag{108}$$

$$|\dot{S}\rangle = \mathcal{N}\exp\left(\sum_{n,m\geq 1} c_n^\dagger S_{nm} b_m^\dagger\right)|\dot{0}\rangle \tag{109}$$

It is now easy to prove, as a check, that

$$\mathcal{Q}|\dot{S}\rangle + |\hat{S}\rangle = 0 \tag{110}$$

where

$$\mathcal{Q} = c_0 + \sum_{n=1}^{\infty}(-1)^n(c_{2n} + c_{-2n}) \tag{111}$$

The above computation proves in a very direct way that the BRST operator is nothing but the midpoint insertion ($z = i$) of the operator $\frac{1}{2i}(c(z) - c(\bar{z}))$ [14]. A different proof of this identification, which makes use of the continuous basis of the $*$–algebra [50], was given in [18].

[4] Which is equivalent to normal ordering these terms. We thank A.Kling and S.Uhlmann for this suggestion.

As an additional remark, we point out that the ghost action calculated in the enlarged and restricted Fock space are different, although they are both divergent due to the normalization (100).

A final warning to the reader: the method of 'large' matrices is extremely powerful and leads in a very straightforward way to results that are very laborious to be obtained by alternative methods; the incorporation of 'large' matrices in the Fock space formalism, on the other hand, is given here on an *ad hoc* basis and certainly needs some formal polishing.

5.1 A comment on the eigenvalues of Neumann coefficients matrices

The introduction of 'large' matrices gives us the opportunity to clarify an important point concerning the eigenvalues of the (twisted) matrices of three strings vertex coefficients. The diagonalization of the reduced $*$-product was performed in [18, 42, 43, 60], using a remarkable relation between matter coefficients and ghost coefficients [45]. Here we will make some comments about the singular role played by the midpoint, which turns out to be quite transparent in the diagonal basis. Our commuting vertex coefficients (including the $(0,0)$ component) are of the form

$$\mathcal{Y} = \begin{pmatrix} 1 & 0 \\ \vec{y} & Y \end{pmatrix} \tag{112}$$

$$\mathcal{Y}_{\pm} = \begin{pmatrix} 0 & 0 \\ \vec{y}_{\pm} & Y_{\pm} \end{pmatrix} \tag{113}$$

It is then evident that the eigenvalues are

$$eig[\mathcal{Y}] = 1 \oplus eig[Y] \tag{114}$$

$$eig[\mathcal{Y}_{\pm}] = 0 \oplus eig[Y_{\pm}] \tag{115}$$

We can easily put them in a block diagonal form

$$\hat{\mathcal{Y}} = \begin{pmatrix} 1 & 0 \\ 0 & Y \end{pmatrix} \tag{116}$$

$$\hat{\mathcal{Y}}_{\pm} = \begin{pmatrix} 0 & 0 \\ 0 & Y_{\pm} \end{pmatrix} \tag{117}$$

This is achieved by

$$\hat{\mathcal{Y}}_{(\pm)} = \mathcal{Z}^{-1}\mathcal{Y}_{(\pm)}\mathcal{Z} \tag{118}$$

The block-diagonalizing matrix is

$$\mathcal{Z} = \begin{pmatrix} 1 & 0 \\ \vec{f} & 1 \end{pmatrix} \tag{119}$$

$$\mathcal{Z}^{-1} = \begin{pmatrix} 1 & 0 \\ -\vec{f} & 1 \end{pmatrix} \tag{120}$$

where

$$\vec{f} = \frac{1}{1-Y}\vec{y} = -\frac{1}{Y_\pm}\vec{y}_\pm \tag{121}$$

At first sight one might think that, since (121) has the well known solution $f_n = \frac{1}{2}(1+(-1)^n)i^n$, it cannot be the case that either Y has eigenvalue 1, or $Y_\pm$ have the eigenvalue 0. However (121) is a statement in the twist–even sector of the Hilbert space of vectors $\{v_n\}, n = 1, 2, ...$ So we must conclude that small matrices do not have singular eigenvalues in this sector. The twist–odd sector on the contrary contains singular eigenvalues. This was noted in [42, 60], where the following spectrum for the 'small' matrices ,$(Y, Y_\pm)$ was found

$$y(\kappa) = \frac{1}{2\cosh x - 1} \tag{122}$$

$$y_\pm(\kappa) = \frac{\cosh x \pm \sinh x - 1}{2\cosh x - 1} \tag{123}$$

where $x \equiv \frac{\pi\kappa}{2}$ and κ is a continuous parameter in the range $(-\infty, \infty)$.

It was pointed out in [42, 60] that to any $\kappa \neq 0$ there correspond two eigenvectors of opposite twist–parity, while $\kappa = 0$ has only one twist–odd eigenvector. On the basis of the discussion in this paper we see that, when considering the 'large' matrix $\mathcal{Y}$, we have an additional 1 eigenvalue whose twist–even eigenvector is given by the first column of (119) [5] ..

In terms of the bc standard modes we can write

$$\tilde{c}_0 = c_0 + \sum_{n\geq 1} f_n(c_n + (-1)^n c_n^\dagger) = \mathcal{Q} \tag{124}$$

$$\tilde{c}_n = c_n \quad n \neq 0 \tag{125}$$

$$\tilde{b}_0 = b_0 \tag{126}$$

$$\tilde{b}_n = -f_n b_0 + b_n \quad n \neq 0 \tag{127}$$

[5] This is very similar to what happens with the Neumann coefficients in the matter sector with zero modes in [61].

where we have defined ($f_{-n} \equiv f_n$). As noted also in [17] this is an equivalent representation of the bc system[6]

$$\{\tilde{b}_N, \tilde{c}_M\} = \delta_{N+M} \quad N, M = -\infty, ..., 0, ..., \infty \tag{128}$$

This can possibly be viewed as a redefinition of the CFT fields $c(z), b(z)$.

If we exclude the $\tilde{c}_0, \tilde{b}_0$ zero modes we are left with the reduced $*$-product which turns out to be an associative product for states in the Siegel gauge. In particular we note that, since, in the reduced product, the value $\kappa = 0$ is related to only one eigenvector of Y and since the same value is intimately related to the midpoint (124), the Siegel gauge can be regarded as a split string description.

A possible suggestion to appropriately treat the additional singular 1 eigenvalue of $\mathcal{Y}$ may be to "blow up" the zero modes, as before, in two triples of conjugate operators. Then we can safely compute $*$-products in an enlarged Fock space and then return to the original space by appropriate restrictions. As an encouraging indication in this direction, we notice that

$$K_1 = L_1 + L_{-1} \tag{129}$$

where L_1 and L_{-1} are the ghost Virasoro generators, can be modified in the enlarged space by formally setting

$$c_0 \to c_0 - \eta_0 + \eta_0^\dagger \tag{130}$$

It is easy to see that the vertex is K_1-invariant. In fact, keeping for simplicity only the zero mode part which is the only one of interest in this aspect, we have

$$K_1 \mathrm{e}^{-\eta_0 b_0}|\hat{0}\rangle = \mathrm{e}^{-\eta_0 b_0}\Big((-c_0 - \eta_0 + \eta_0^\dagger + \eta_0)(b_1 + b_{-1})\Big)\, c_0|\dot{0}\rangle = 0 \tag{131}$$

6 Matter projectors and D–branes

In this second part of the review we will be concerned with the solutions of the matter part of the VSFT equations of motion, i.e. with solutions to (10). However instead of reviewing the well–known squeezed state solutions to [30, 46] as well as the related solutions discussed in [22, 23], our leading idea will be to find VSFT solutions in order to make a comparison with solutions in scalar noncommutative field theories. In particular we will

[6]In order to prove this, twist invarince of $\vec{f}$ is crucial ($C\vec{f} = \vec{f}$)

show that it is possible to establish a one-to-one correspondence between tachyonic lumps, i.e lower dimensional D–branes, in the former and solitonic solutions in the latter. This correspondence is fully exposed by introducing a constant background B field.

To start with we fix the solution of the ghost part in the form given in the previous section and concentrate on the matter part. The value of the action for such solutions is given by

$$\mathcal{S}(\Psi) = \mathcal{K}\langle\Psi_m|\Psi_m\rangle \tag{132}$$

where $\mathcal{K}$ contains the ghost contribution. We recall that $\mathcal{K}$ is infinite (see above) unless the action is suitably regularized. The choice of a regularization should be understood as a 'gauge' freedom, [22], in choosing the solutions to (7). So a coupled solution to (9) and (10), even if the action is naively infinite in its ghost component, is nevertheless a legitimate representative of the corresponding class of solutions.

6.1 Solitons in noncommutative field theories

Before we set out to discuss solutions to (10) and to better explain our aim, let us briefly describe solitons in noncommutative field theories, (see [31] for a beautiful review). Noncommutative field theories are effective low energy field theories which live on the world-volume of D–branes in the presence of a constant background B field. To be definite, let us think of a D25–brane in bosonic string theory. The simplest example of effective theories is a noncommutative theory of a scalar field ϕ, which is thought to represent the tachyon living on the brane (this is an oversimplified situation, in fact one could easily take the gauge field as well into account, while the massive string states are thought to having been integrated out). We concentrate on the case in which B is switched on along two space directions, which we denote by $x = q$ and $y = p$ (from now one, for simplicity, we will drop the other coordinates). The coordinates become noncommutative and the ensuing situation can be described by replacing the initial theory with a theory in which all products are replaced by the Moyal $\star$ product with deformation parameter θ (linked to B as explained below). Alternatively one can use the Weyl map and replace the noncommutative coordinates by two conjugate operators $\hat{q}$, $\hat{p}$, such that $[\hat{q},\hat{p}] = i\theta$. In the large θ limit, after a suitable rescaling of the coordinates, the kinetic part of the action becomes negligible, so only the potential part, $\int dx\, dy\, V_\star(\phi)$, is relevant. Using the Weyl correspondence, the action can be replaced by $2\pi\theta\, \mathrm{Tr}_{\mathcal{H}} V(\hat{\phi})$, where $\mathcal{H}$ is the Hilbert space constructed out of $\hat{q},\hat{p}$, and $\hat{\phi}$ is the operator corresponding to the noncommutative field ϕ. Solutions to the equations of motion take the form

$$\hat{\phi} = \lambda_i P, \qquad P^2 = P \tag{133}$$

where λ_i, $i = 1, \dots, n$ are the minima of the classical commutative potential V, which is assumed to be polynomial. The energy of such a solution is therefore given by $2\pi\theta\, V(\lambda_i)\, \mathrm{Tr}_{\mathcal{H}} P$.

On the basis of this discussion it is clear that, in order to know the finite energy solutions of the noncommutative scalar theory, we have to find the finite rank projectors in the space $\mathcal{H}$. The latter can be constructed in the following way. Define the harmonic oscillator $a = (\hat{q} + i\hat{p})/\sqrt{(2\theta)}$ and its hermitean conjugate $a^\dagger$: $[a, a^\dagger] = 1$. By a standard construction we can define the normalized harmonic oscillator eigenstates: $|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle$. Now, via the Weyl correspondence, we can map any operator $|n\rangle\langle m|$ to a classical function of the coordinates x, y. In particular $|n\rangle\langle n|$, which are rank one projectors, will be mapped to classical functions

$$\psi_n(x, y) = 2\,(-1)^n L_n\left(\frac{2r^2}{\theta}\right) e^{-\frac{r^2}{\theta}} \tag{134}$$

where $r^2 = x^2 + y^2$. Each of these solutions, by construction, satisfy $\psi_n \star \psi_n = \psi_n$. We refer to them as the GMS solitons [57]. They can be interpreted as D23–branes. We can of course consider any finite sum of these projectors. They will also be solutions. They are interpreted as collapsing D23–branes. Moreover, using the shift operator $S = \sum_{n=0}^{\infty} |n+1\rangle\langle n|$, one can set up a solution generating techniques, whereby a nontrivial soliton solution can be generated starting from a trivial one by repeated application of S (see ([31])).

Our purpose in this paper is to show how the soliton solutions just described arise in VSFT. The way they can be seen is by taking the low energy limit of tachyonic lumps representing D23–branes in VSFT [27, 28]. They are, so to speak, relics of VSFT branes in the $\alpha' \to 0$ limit.

6.2 The background B field

But before we turn to this, we will introduce a background field B in SFT. In ref. [32] and [33] it was shown that, when such a field is switched on, in the low energy limit the string field theory star product factorizes into the ordinary Witten $*$ product and the Moyal $\star$ product. A related result can be obtained in the following way. The string field theory action (1) can be explicitly calculated in terms of local fields, provided the string field is expressed itself in terms of local fields as in (2). Of course this makes sense only in the limit in which string theory can be approximated by a local field theory. In this framework (1) takes the form of an infinite series of integrated local polynomials (kinetic and potential terms) of the fields involved in (2) as explained in section 2. Now, it has been proven by [34, 35] that, when a B field is switched on, the kinetic term of (1) remains the same while the three string vertex changes, being multiplied by a (cyclically invariant) noncommutative phase factor (see ([34, 35]) and eq.(138) below). It is easy to see on a general basis that the overall effect of such noncommutative factor is to replace the ordinary product with the Moyal product in the RHS of the effective action (for a related approach see [36]).

Therefore, we know pretty well the effects of a B field when perturbative configurations are involved. What we wish to explore here are the effects of a B field on nonperturbative solutions. We find that a B field has the virtue of smoothing out some of the singularities that appear in VSFT. As for the overall star product in the presence of a background B field, it turns out that Witten's star product and Moyal product are completely entangled in nonperturbative configurations. Nevertheless, in the low energy limit, we can again witness the factorization into Witten's star product and the Moyal one. It is exactly this factorization that allows us to recover the noncommutative field theory solitons.

We also remark that switching on a B field in VSFT is consistent with the interpretation of VSFT. The latter is thought to describe closed string theory, and the antisymmetric field B belongs in the massless sector of a bosonic closed string theory.

Finally we would like to stress that the Moyal star product referred to here has nothing to do with the Moyal representation of Witten's star product which was suggested in [37, 38]. This representation can be seen as a confirmation of an old theorem [39] concerning the uniqueness of the Moyal product in the class of noncommutative associative products, however it is realized in an (unphysical) auxiliary space, see [40, 41, 42], therefore it cannot affect the physical space–time.

6.3 Organization of the paper. Second part

In section 7 we derive the new Neumann coefficients for the three string vertex in the presence of a background B field. In section 8 we solve the projector equation (10) for a 23–dimensional tachyonic lump and justify its D23–brane interpretation. In section 9 we start examining the effect of the B field on such solution. In section 10 we generalize the lump solution of section 8. We construct a series of solutions to the matter projector equation, which we denote by $|\Lambda_n\rangle$ for any natural number n. $|\Lambda_n\rangle$ is generated by acting on a tachyonic lump solution $|\Lambda_0\rangle$ with $(-\kappa)^n L_n(\mathbf{x}/\kappa)$, where L_n is the n-th Laguerre polynomial, $\mathbf{x}$ is a quadratic expression in the string creation operators, see below eqs.(74, 197), and κ is an arbitrary real constant. These states satisfy the remarkable properties

$$|\Lambda_n\rangle * |\Lambda_m\rangle = \delta_{n,m}|\Lambda_n\rangle \tag{135}$$

$$\langle\Lambda_n|\Lambda_m\rangle = \delta_{n,m}\langle\Lambda_0|\Lambda_0\rangle \tag{136}$$

Each $|\Lambda_n\rangle$ represents a D23–brane, parallel to all the others. In section 11 we show that the field theory limit of $|\Lambda_n\rangle$ factors into the sliver state (D25–brane) and the n-th GMS soliton. Section 12 describes related results.

7 The three string vertex in the presence of a constant background B field

The three string vertex [1, 44, 45] of the Open String Field Theory was given in eq.(11). The notation V_{MN}^{rs} for the vertex coefficients will often be used from now on, where $M(N)$ will denote the couple $\{0,m\}$ ($\{0,n\}$).

Our first goal is to find the new form of the coefficients V_{MN}^{rs} when a constant B field is switched on. We start from the simplest case, i.e. when B is nonvanishing in two space directions, say the 24–th and 25–th ones. Let us denote these directions with the Lorentz indices α and β. Then, as is well–known [47, 34, 35], in these two directions we have a new effective metric $G_{\alpha\beta}$, the open string metric, as well as an effective antisymmetric parameter $\theta_{\alpha\beta}$, given by

$$\begin{aligned} G^{\alpha\beta} &= \left(\frac{1}{\eta+2\pi\alpha' B}\,\eta\,\frac{1}{\eta-2\pi\alpha' B}\right)^{\alpha\beta}, \\ \theta^{\alpha\beta} &= -(2\pi\alpha')^2\left(\frac{1}{\eta+2\pi\alpha' B}\,B\,\frac{1}{\eta-2\pi\alpha' B}\right)^{\alpha\beta} \end{aligned}$$

Henceforth we set $\alpha' = 1$, unless otherwise specified.

The three string vertex is modified only in the 24-th and 25-th direction, which, in view of the subsequent D–brane interpretation, we call the transverse directions. We split the three string vertex into the tensor product of the perpendicular part and the parallel part

$$|V_3\rangle = |V_{3,\perp}\rangle \otimes |V_{3,_{||}}\rangle \tag{137}$$

The parallel part is the same as in the ordinary case and will not be rediscussed here. On the contrary we will describe in detail the perpendicular part of the vertex. We rewrite the exponent E as $E = E_{||} + E_\perp$, according to the above splitting. $E_\perp$ will be modified as follows

$$\begin{aligned} E_\perp \quad \to \quad E'_\perp = \sum_{r,s=1}^{3} \Bigg(\frac{1}{2} \sum_{m,n\geq 1} G_{\alpha\beta} a_m^{(r)\alpha\dagger} V_{mn}^{rs} a_n^{(s)\beta\dagger} + \sum_{n\geq 1} G_{\alpha\beta} p_{(r)}^{\alpha} V_{0n}^{rs} a_n^{(s)\beta\dagger} \\ + \frac{1}{2} G_{\alpha\beta} p_{(r)}^{\alpha} V_{00}^{rs} p_{(s)}^{\beta} + \frac{i}{2} \sum_{r<s} p_\alpha^{(r)} \theta^{\alpha\beta} p_\beta^{(s)} \Bigg) \end{aligned} \tag{138}$$

Next, as far as the zero modes are concerned, we pass from the momentum to the oscillator basis, [44, 45]. We define

$$a_0^{(r)\alpha} = \frac{1}{2}\sqrt{b}\hat{p}^{(r)\alpha} - i\frac{1}{\sqrt{b}}\hat{x}^{(r)\alpha}, \qquad a_0^{(r)\alpha\dagger} = \frac{1}{2}\sqrt{b}\hat{p}^{(r)\alpha} + i\frac{1}{\sqrt{b}}\hat{x}^{(r)\alpha}, \tag{139}$$

where $\hat{p}^{(r)\alpha}, \hat{x}^{(r)\alpha}$ are the zero momentum and position operator of the r–th string, and we have kept the 'gauge' parameter b of ref.[46] ($b \sim \alpha'$). From now on Lorentz indices are raised and lowered by means of the effective open string metric, for instance $p^{(r)\alpha} = G^{\alpha\beta} p_\beta^{(r)}$. We have

$$[a_M^{(r)\alpha}, a_N^{(s)\beta\dagger}] = G^{\alpha\beta}\delta^{rs}\delta_{MN}, \qquad N, M \geq 0 \tag{140}$$

Denoting by $|\Omega_{b,\theta}\rangle$ the oscillator vacuum ($a_N^\alpha|\Omega_{b,\theta}\rangle = 0$, for $N \geq 0$), the relation between the momentum basis and the oscillator basis is defined by

$$\begin{aligned} &|p^{24}\rangle_{123} \otimes |p^{25}\rangle_{123} \equiv |\{p^\alpha\}\rangle_{123} = \left(\frac{b}{2\pi\sqrt{\det G}} \right)^{\frac{3}{2}} \cdot \\ &\cdot \exp\left[\sum_{r=1}^{3} \left(-\frac{b}{4} p_\alpha^{(r)} G^{\alpha\beta} p_\beta^{(r)} + \sqrt{b} a_0^{(r)\alpha\dagger} p_\alpha^{(r)} - \frac{1}{2} a_0^{(r)\alpha\dagger} G_{\alpha\beta} a_0^{(r)\beta\dagger} \right) \right] |\Omega_{b,\theta}\rangle \end{aligned}$$

Now we insert this equation inside $E'_\perp$ and try to eliminate the momenta along the perpendicular directions by integrating them out. To this end we rewrite $E'_\perp$ in the following way and, for simplicity, drop all the labels α, β

and r,s:

$$E'_\perp = \frac{1}{2}\sum_{m,n\geq 1} a^\dagger_m G V_{mn} a^\dagger_n + \sum_{n\geq 1} p V_{0n} a^\dagger_n + \\ + \frac{1}{2} p \left[G^{-1}(V_{00} + \frac{b}{2}) + \frac{i}{2}\theta\epsilon\chi \right] p - \sqrt{b} p a^\dagger_0 + \frac{1}{2} a^\dagger_0 G a^\dagger_0$$

where we have set $\theta^{\alpha\beta} = \epsilon^{\alpha\beta}\theta$ and introduced the matrices ϵ with entries $\epsilon^{\alpha\beta}$ (which represent the 2×2 antisymmetric symbol with $\epsilon^1{}_2 = 1$) and χ with entries

$$\chi^{rs} = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \tag{141}$$

At this point we impose momentum conservation. There are three distinct ways to do that and eventually one has to (multiplicatively) symmetrize with respect to them. Let us start by setting $p_3 = -p_1 - p_2$ in $E'_\perp$ and obtain an expression of the form

$$p\, X_{00}\, p + \sum_{N\geq 0} p\, Y_{0N}\, a^\dagger_N + \sum_{M,N\geq 0} a^\dagger_M\, Z_{MN}\, a^\dagger_N \tag{142}$$

where, in particular, X_{00} is given by

$$X_{00}^{\alpha\beta,rs} = G^{\alpha\beta}\,(V_{00} + \frac{b}{2})\,\eta^{rs} + i\frac{\theta}{4}\,\epsilon^{\alpha\beta}\,\epsilon^{rs} \tag{143}$$

Here the indices r,s take only the values 1,2, and $\eta = \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}$.

Now, as usual, we redefine p so as eliminate the linear term in (142). At this point we can easily perform the Gaussian integration over $p_{(1)}, p_{(2)}$, while the remnant of (142) will be expressed in terms of the inverse of X_{00}:

$$\left(X_{00}^{-1}\right)^{\alpha\beta,rs} = \frac{2A^{-1}}{4a^2+3}\left(\frac{3}{2}\,G^{\alpha\beta}\,(\eta^{-1})^{rs} - 2i\,a\,\epsilon^{\alpha\beta}\,\epsilon^{rs}\right) \tag{144}$$

where

$$A = V_{00} + \frac{b}{2}, \qquad a = \frac{\theta}{4A}. \tag{145}$$

Let us use henceforth for the B field the explicit form

$$B_{\alpha\beta} = \begin{pmatrix} 0 & B \\ -B & 0 \end{pmatrix} \tag{146}$$

so that

$$\mathrm{DetG} = \left(1 + (2\pi B)^2\right)^2, \qquad \theta = -(2\pi)^2 B, \qquad a = -\frac{\pi^2}{A} B \tag{147}$$

Now one has to symmetrize with respect to the three possibilities of imposing the momentum conservation. Remembering the factors due to integration over the momenta and collecting the results one gets for the three string vertex in the presence of a B field

$$|V_3\rangle' = |V_{3,\perp}\rangle' \otimes |V_{3,||}\rangle \tag{148}$$

$|V_{3,||}\rangle$ is the same as in the ordinary case (without B field), while

$$|V_{3,\perp}\rangle' = K_2 \, e^{-E'} |\tilde{0}\rangle \tag{149}$$

with

$$K_2 = \frac{\sqrt{2\pi b^3}}{A^2(4a^2+3)} (\mathrm{Det}G)^{1/4}, \tag{150}$$

$$E' = \frac{1}{2} \sum_{r,s=1}^{3} \sum_{M,N\geq 0} a_M^{(r)\alpha\dagger} \mathcal{V}_{\alpha\beta,MN}^{rs} a_N^{(s)\beta\dagger} \tag{151}$$

and $|\tilde{0}\rangle = |0\rangle \otimes |\Omega_{b,\theta}\rangle$. The coefficients $\mathcal{V}_{MN}^{\alpha\beta,rs}$ are given by

$$\mathcal{V}_{00}^{\alpha\beta,rs} = G^{\alpha\beta}\delta^{rs} - \frac{2A^{-1}b}{4a^2+3}\left(G^{\alpha\beta}\phi^{rs} - ia\epsilon^{\alpha\beta}\chi^{rs}\right) \tag{152}$$

$$\mathcal{V}_{0n}^{\alpha\beta,rs} = \frac{2A^{-1}\sqrt{b}}{4a^2+3}\sum_{t=1}^{3}\left(G^{\alpha\beta}\phi^{rt} - ia\epsilon^{\alpha\beta}\chi^{rt}\right) V_{0n}^{ts} \tag{153}$$

$$\mathcal{V}_{mn}^{\alpha\beta,rs} = G^{\alpha\beta}V_{mn}^{rs} - \frac{2A^{-1}}{4a^2+3}\sum_{t,v=1}^{3} V_{m0}^{rv}\left(G^{\alpha\beta}\phi^{vt} - ia\epsilon^{\alpha\beta}\chi^{vt}\right) V_{0n}^{ts} \tag{154}$$

where, by definition, $V_{0n}^{rs} = V_{n0}^{sr}$, and

$$\phi = \begin{pmatrix} 1 & -1/2 & -1/2 \\ -1/2 & 1 & -1/2 \\ -1/2 & -1/2 & 1 \end{pmatrix} \tag{155}$$

while the matrix χ has been defined above (141). These two matrices satisfy the algebra

$$\chi^2 = -2\phi, \qquad \phi\chi = \chi\phi = \frac{3}{2}\chi, \qquad \phi^2 = \frac{3}{2}\phi \tag{156}$$

Next, let us notice that the above results can be easily extended to the case in which the transverse directions are more than two (i.e. the 24–th and 25–th ones) and even. The canonical form of the transverse B field is

$$B_{\alpha\beta} = \begin{pmatrix} \begin{matrix} 0 & B_1 \\ -B_1 & 0 \end{matrix} & 0 & \dots \\ 0 & \begin{matrix} 0 & B_2 \\ -B_2 & 0 \end{matrix} & \dots \\ \dots & \dots & \dots \end{pmatrix} \tag{157}$$

It is not hard to see that each couple of conjugate transverse directions under this decomposition, can be treated in a completely independent way. The result is that each couple of directions $(26-i, 25-i)$, corresponding to the eigenvalue B_i, will be characterized by the same formulas (152, 153, 154) above with B replaced by B_i.

The properties of the new Neumann coefficients $\mathcal{V}^{rs}_{NM}$ have been analyzed in [58]. Here we write down the results.

To start with, let us quote

- (i) $\mathcal{V}^{\alpha\beta,rs}_{NM}$ are symmetric under simultaneous exchange of the three couples of indices;
- (ii) they are endowed with the property of cyclicity in the r, s indices, i.e. $\mathcal{V}^{rs} = \mathcal{V}^{r+1,s+1}$, where $r, s = 4$ is identified with $r, s = 1$.

Next let us extend the twist matrix C by $C_{MN} = (-1)^M \delta_{MN}$ and define

$$\mathcal{X}^{rs} \equiv C\mathcal{V}^{rs}, \qquad r, s = 1, 2, \qquad \mathcal{X}^{11} \equiv \mathcal{X} \tag{158}$$

These matrices commute

$$[\mathcal{X}^{rs}, \mathcal{X}^{r's'}] = 0 \tag{159}$$

and

$$(\mathcal{X}^{rs})^* = \tilde{\mathcal{X}}^{rs}, \quad \text{i.e.} \quad (\mathcal{X}^{rs})^\dagger = \mathcal{X}^{rs}$$

Moreover we have the following properties, which mark a difference with the $B = 0$ case,

$$C\mathcal{V}^{rs} = \tilde{\mathcal{V}}^{sr}C, \qquad C\mathcal{X}^{rs} = \tilde{\mathcal{X}}^{sr}C \tag{160}$$

where we recall that tilde denotes transposition with respect to the α, β indices. Finally one can prove that

$$\begin{aligned} &\mathcal{X}^{11} + \mathcal{X}^{12} + \mathcal{X}^{21} = \mathbb{I} \\ &\mathcal{X}^{12}\mathcal{X}^{21} = (\mathcal{X}^{11})^2 - \mathcal{X} \\ &(\mathcal{X}^{12})^2 + (\mathcal{X}^{21})^2 = \mathbb{I} - (\mathcal{X}^{11})^2 \\ &(\mathcal{X}^{12})^3 + (\mathcal{X}^{21})^3 = 2(\mathcal{X}^{11})^3 - 3(\mathcal{X}^{11})^2 + \mathbb{I} \end{aligned} \tag{161}$$

In the matrix products of these identities, as well as throughout the paper, the indices α,β must be understood in alternating up/down position: $\mathcal{X}^{\alpha}{}_{\beta}$. For instance, in (161) $\mathbb{I}$ stands for $\delta^{\alpha}{}_{\beta}\,\delta_{MN}$.

8 The squeezed state solution

In this section we wish to find a solution to the equation of motion $|\Psi\rangle * |\Psi\rangle = |\Psi\rangle$ in the form of squeezed states [48, 49, 50, 29]. A squeezed state in the present context is written as

$$|S\rangle = |S_{\perp}\rangle \otimes |S_{||}\rangle \tag{162}$$

where $|S_{||}\rangle$ has the ordinary form, see [30, 46], and is treated in the usual way, while

$$|S_{\perp}\rangle \;=\; \mathcal{N}^2 \exp\left(-\frac{1}{2}\sum_{M,N\geq 0} a_M^{\alpha\dagger}\,\mathcal{S}_{\alpha\beta,MN}\,a_N^{\beta\dagger}\right)|\Omega_{b,\theta}\rangle \tag{163}$$

The $*$ product of two such states, labeled $_1$ and $_2$, is

$$\begin{aligned}|S'_{\perp}\rangle &= |S_{1,\perp}\rangle \,*\, |S_{2,\perp}\rangle\\ &= \frac{K_2\,(\mathcal{N}_1\mathcal{N}_2)^2}{\mathrm{DET}(\mathbf{I}-\Sigma\mathcal{V})^{1/2}}\,\exp\left(-\frac{1}{2}\sum_{M,N\geq 0} a_M^{\alpha\dagger}\mathcal{S}'_{\alpha\beta,MN}a_N^{\beta\dagger}\right)|\tilde{0}\rangle\end{aligned} \tag{164}$$

where, in matrix notation which includes both the indices N,M and α,β,

$$\mathcal{S}' = \mathcal{V}^{11} + (\mathcal{V}^{12},\mathcal{V}^{21})(\mathbf{I}-\Sigma\mathcal{V})^{-1}\Sigma\begin{pmatrix}\mathcal{V}^{21}\\ \mathcal{V}^{12}\end{pmatrix} \tag{165}$$

In RHS of these equations

$$\Sigma = \begin{pmatrix} C\mathcal{S}_1C & 0\\ 0 & C\mathcal{S}_2C\end{pmatrix}, \qquad \mathcal{V} = \begin{pmatrix}\mathcal{V}^{11} & \mathcal{V}^{12}\\ \mathcal{V}^{21} & \mathcal{V}^{22}\end{pmatrix}, \tag{166}$$

and $\mathbf{I}^{\alpha,rs}_{\beta,MN} = \delta^{\alpha}_{\beta}\,\delta_{MN}\,\delta^{rs}$, $r,s=1,2$. DET is the determinant with respect to all indices. To reach the form (165) one has to use cyclicity of $\mathcal{V}^{rs}$ under $r\to r+1, s\to s+1$, see above.

Let us now discuss the squeezed state solution to the equation $|\Psi\rangle*|\Psi\rangle = |\Psi\rangle$ in the matter sector. In order for this to be satisfied with the above states $|S\rangle$, we must first impose

$$\mathcal{S}_1 = \mathcal{S}_2 = \mathcal{S}' \equiv \mathcal{S}$$

and then suitably normalize the resulting state. Then (165) becomes an equation for $\mathcal{S}$, i.e.

$$\tilde{\mathcal{S}} = \mathcal{V}^{11} + (\mathcal{V}^{12}, \mathcal{V}^{21})(\mathbf{I} - \Sigma\mathcal{V})^{-1}\Sigma \begin{pmatrix} \mathcal{V}^{21} \\ \mathcal{V}^{12} \end{pmatrix} \tag{167}$$

where $\Sigma, \mathcal{V}$ are the same as above with $\mathcal{S}_1 = \mathcal{S}_2 = \mathcal{S}$. Eq.(167) has an obvious (formal) solution by iteration. However in ref. [30] it was shown that it is possible to obtain the solution in compact form by 'abelianizing' the problem. Notwithstanding the differences with that case, it is possible to reproduce the same trick on eq.(167), thanks to (159). One denotes $C\mathcal{V}^{rs}$ by $\mathcal{X}^{rs}$ and $C\mathcal{S}$ by $\mathcal{T}$, and assumes that $[\mathcal{X}^{rs}, \mathcal{T}] = 0$ (of course this has to be checked *a posteriori*). Notice however that we cannot assume that C commutes with $\mathcal{S}$, but we assume that $C\mathcal{S} = \tilde{\mathcal{S}}C$. By multiplying (167) from the left by C we get:

$$\mathcal{T} = \mathcal{X}^{11} + (\mathcal{X}^{12}, \mathcal{X}^{21})(\mathbf{I} - \Sigma\mathcal{V})^{-1} \begin{pmatrix} \mathcal{T}\mathcal{X}^{21} \\ \mathcal{T}\mathcal{X}^{12} \end{pmatrix} \tag{168}$$

For instance $\tilde{\mathcal{S}}\mathcal{V}^{12} = \tilde{\mathcal{S}}CC\mathcal{V}^{12} = \mathcal{T}\mathcal{X}^{12}$, etc. In the same way,

$$(\mathbf{I} - \Sigma\mathcal{V})^{-1} = \begin{pmatrix} \mathbb{I} - \mathcal{T}\mathcal{X}^{11} & -\mathcal{T}\mathcal{X}^{12} \\ -\mathcal{T}\mathcal{X}^{21} & \mathbb{I} - \mathcal{T}\mathcal{X}^{11} \end{pmatrix}^{-1}$$

where $\mathbb{I}^{\alpha}_{\beta,MN} = \delta^{\alpha}_{\beta}\, \delta_{MN}$. Now all the entries are commuting matrices, so the inverse can be calculated straight away.

¿From now on everything is the same as in [30, 46], therefore we limit ourselves to a quick exposition. One arrives at an equation only in terms of $\mathcal{T}$ and $\mathcal{X} \equiv \mathcal{X}^{11}$:

$$(\mathcal{T} - \mathbb{I})(\mathcal{X}\mathcal{T}^2 - (\mathbb{I} + \mathcal{X})\mathcal{T} + \mathcal{X}) = 0 \tag{169}$$

This gives two solutions:

$$\mathcal{T} = \mathbb{I} \tag{170}$$

$$\mathcal{T} = \frac{1}{2\mathcal{X}}\left(\mathbb{I} + \mathcal{X} - \sqrt{(\mathbb{I} + 3\mathcal{X})(\mathbb{I} - \mathcal{X})}\right) \tag{171}$$

The third solution, with a + sign in front of the square root, is not acceptable, as explained in [46]. In both cases we see that the solution commutes with $\mathcal{X}^{rs}$. The squeezed state solution we are looking for is, in both cases, $\mathcal{S} = C\mathcal{T}$. As for (170), it is easy to see that it leads to the identity state. Therefore, from now on we will consider (171) alone.

Now, let us deal with the normalization of $|S_\perp\rangle$. Imposing $|S_\perp\rangle * |S_\perp\rangle = |S_\perp\rangle$ we find

$$\mathcal{N}^2 = K_2^{-1}\,\mathrm{DET}\,(\mathbf{I} - \Sigma\mathcal{V})^{1/2}$$

Replacing the solution in it one finds

$$\mathrm{DET}(\mathbf{I} - \Sigma\mathcal{V}) = \mathrm{Det}\;((\mathbb{I} - \mathcal{X})(\mathbb{I} + \mathcal{T})) \tag{172}$$

Det denotes the determinant with respect to the indices α, β, M, N. Using this equation and (150), and borrowing from [46] the expression for $|S_{\|}\rangle$, one finally gets for the 23–dimensional tachyonic lump

$$\begin{aligned} |S\rangle \;=\; & \left\{\det(1-X)^{1/2}\det(1+T)^{1/2}\right\}^{24} \cdot \\ & \cdot \exp\left(-\frac{1}{2}\eta_{\bar\mu\bar\nu}\sum_{m,n\geq 1} a_m^{\bar\mu\dagger} S_{mn} a_n^{\bar\nu\dagger}\right)|0\rangle \otimes \\ & \frac{A^2(3+4a^2)}{\sqrt{2\pi b^3}(\mathrm{Det}G)^{1/4}}\left(\mathrm{Det}(\mathbb{I}-\mathcal{X})^{1/2}\mathrm{Det}(\mathbb{I}+\mathcal{T})^{1/2}\right) \cdot \\ & \cdot \exp\left(-\frac{1}{2}\sum_{M,N\geq 0} a_M^{\alpha\dagger}\mathcal{S}_{\alpha\beta,MN} a_N^{\beta\dagger}\right)|\tilde{0}\rangle, \end{aligned} \tag{173}$$

where $\mathcal{S} = C\mathcal{T}$ and $\mathcal{T}$ is given by (171). The quantities in the first two lines are defined in ref.[46] with $\bar\mu, \bar\nu = 0, \ldots 23$ denoting the parallel directions to the lump.

The value of the action corresponding to (173) is easily calculated

$$\begin{aligned} \mathcal{S}_{\mathcal{S}} \;=\; & \mathcal{K}\frac{V^{(24)}}{(2\pi)^{24}}\left\{\det(1-X)^{3/4}\det(1+3X)^{1/4}\right\}^{24} \cdot \\ & \cdot \frac{A^4(3+4a^2)^2}{2\pi b^3(\mathrm{Det}G)^{1/2}}\;\mathrm{Det}(\mathbb{I}-\mathcal{X})^{3/4}\mathrm{Det}(\mathbb{I}+3\mathcal{X})^{1/4} \end{aligned} \tag{174}$$

where $V^{(24)}$ is the volume along the parallel directions and $\mathcal{K}$ is the constant of eq.(132).

Finally, let $\mathfrak{e}$ denote the energy per unit volume, which coincides with the brane tension when $B = 0$. Then one can compute the ratio of the D23–brane energy density $\mathfrak{e}_{23}$ to the D25-brane energy density $\mathfrak{e}_{25}$;

$$\frac{\mathfrak{e}_{23}}{\mathfrak{e}_{25}} = \frac{(2\pi)^2}{(\mathrm{Det}G)^{1/4}}\cdot\mathcal{R} \tag{175}$$

$$\mathcal{R} = \frac{A^4(3+4a^2)^2}{2\pi b^3(\mathrm{Det}G)^{1/4}}\frac{\mathrm{Det}(\mathbb{I}-\mathcal{X})^{3/4}\mathrm{Det}(\mathbb{I}+3\mathcal{X})^{1/4}}{\det(1-X)^{3/2}\det(1+3X)^{1/2}} \tag{176}$$

If the quantity $\mathcal{R}$ equals 1, this equation is exactly what is expected for the ratio of a flat static D25–brane action and a D23–brane action per unit volume in the presence of the B field (146)[51, 36]. In fact the DBI Lagrangian for a flat static Dp–brane is, [47],

$$\mathcal{L}_{DBI} = \frac{1}{g_s (2\pi)^p} \sqrt{\mathrm{Det}(1 + 2\pi B)} \tag{177}$$

where g_s is the closed string coupling. Substituting (146) and taking the ratio the claim follows. By extending the methods of [52] (see also [53, 54]) to the present case, we have indeed being able to prove in [55] that

$$\mathcal{R} = 1 \tag{178}$$

thus adding evidence to the interpretation of $|S\rangle$, given by (173), as a D23–brane in the presence of a background B field. A further confirmation of this interpretation could be obtained from the study of the spectrum of modes leaving on the brane, which can presumably be done along the same lines as [15, 16, 53, 54, 51].

To end this section let us briefly discuss the generalization of the above results to lower dimensional lumps. As remarked at the end of section 2, every couple of transverse directions corresponding to an eigenvalue B_i of the field B can be treated in the same way as the 24–th and 25–th directions. One has simply to replace in the above formulas B with B_i. The derivation of the above formulas for the case of $25 - 2i$ dimensional lumps is straightforward.

9 Some effects of the B field

In this section we would like to show that what we have obtained so far is not merely a formal replica of the same calculation without B field, but that it significantly affects the lumps solutions. Precisely we would like to show that a B field has the effect of smoothing out some of the singularities that appear in the VSFT, in particular in the low energy limit.

In [56] it was shown that the geometry of the lower–dimensional lump states representing Dp-branes is singular. This can be seen both in the zero slope limit $\alpha' \to 0$ and as an exact result. It can be briefly stated by saying that the midpoint of the string is confined on the hyperplane of vanishing transverse coordinates. It is therefore interesting to see whether the presence of a B field modifies this situation. Moreover, as explained in the introduction, soliton solutions of field theories defined on a noncommutative space describe Dp-branes ([57], [31]). It is then interesting to see if we can recover the simplest GMS soliton, using the particular low energy limit, i.e. the limit of [47], that gives a noncommutative field theory from a string theory with a B field turned on.

We start with the limit of [47], $\alpha' B \gg g$, in such a way that G, θ and B are kept fixed, which we represent by means of a parameter ϵ going to 0 as in [56] ($\alpha' \sim \epsilon^2$). We write the closed string metric $g_{\alpha\beta}$ as $g\,\delta_{\alpha\beta}$. We could also choose to parametrize the $\alpha' B \gg g$ condition by sending B to infinity, keeping g and α' fixed and operating a rescaling of the string modes as in [33], of course at the end we get identical results. By looking at the exponential of the 3-string field theory vertex in the presence of a B field

$$\sum_{r,s=1}^{3}\left(\frac{1}{2}\sum_{m,n\geq 1} G_{\alpha\beta}a_m^{(r)\alpha\dagger}V_{mn}^{rs}a_n^{(s)\beta\dagger}+\sqrt{\alpha'}\sum_{n\geq 1}G_{\alpha\beta}p_{(r)}^{\alpha}V_{0n}^{rs}a_n^{(s)\beta\dagger}\right.$$
$$\left.+\,\alpha'\,\frac{1}{2}G_{\alpha\beta}p_{(r)}^{\alpha}V_{00}^{rs}p_{(s)}^{\beta}+\frac{i}{2}\sum_{r<s}p_{\alpha}^{(r)}\theta^{\alpha\beta}p_{\beta}^{(s)}\right) \tag{179}$$

we see that the limit is characterized by the rescalings

$$\begin{aligned} V_{mn} &\to V_{mn}\\ V_{m0} &\to \epsilon V_{m0}\\ V_{00} &\to \epsilon^2 V_{00}\end{aligned} \tag{180}$$

The dependence of $G_{\alpha\beta}$ and $\theta^{\alpha\beta}$ on g, α' and B is understood. We will make it explicit at the end of our calculations in the form

$$G_{\alpha\beta}=\frac{(2\pi\alpha' B)^2}{g}\delta_{\alpha\beta},\qquad \theta=\frac{1}{B} \tag{181}$$

Substituting the leading behaviors of V_{MN} in eqs.(154), and keeping in mind that $A=V_{00}+\frac{b}{2}$, the coefficients $\mathcal{V}_{MN}^{\alpha\beta,rs}$ become

$$\mathcal{V}_{00}^{\alpha\beta,rs} \;\to\; G^{\alpha\beta}\delta^{rs}-\frac{4}{4a^2+3}\left(G^{\alpha\beta}\phi^{rs}-ia\epsilon^{\alpha\beta}\chi^{rs}\right) \tag{182}$$
$$\mathcal{V}_{0n}^{\alpha\beta,rs} \;\to\; 0 \tag{183}$$
$$\mathcal{V}_{mn}^{\alpha\beta,rs} \;\to\; G^{\alpha\beta}V_{mn}^{rs} \tag{184}$$

We see that the squeezed state (173) factorizes in two parts: the coefficients $\mathcal{V}_{mn}^{\alpha\beta,11}$ reconstruct the full 25 dimensional sliver, while the coefficients $\mathcal{V}_{00}^{\alpha\beta,11}$ take a very simple form

$$\mathcal{S}_{00}^{\alpha\beta}=\frac{2|a|-1}{2|a|+1}\,G^{\alpha\beta}\equiv s\,G^{\alpha\beta} \tag{185}$$

In the $\epsilon \to 0$ limit we also have

$$\mathrm{Det}(\mathbb{I} - \mathcal{X})^{1/2}\mathrm{Det}(\mathbb{I} + \mathcal{T})^{1/2} \to \frac{4}{4a^2+3}\,\det(1-X)\;\frac{4a}{2a+1}\,\det(1+T)$$

The complete lump state in this limit will be denoted by $|\hat{\mathcal{S}}\rangle$, and as a consequence of eq.(173) and these equations, it will take the form

$$\begin{aligned} |\hat{\mathcal{S}}\rangle \;=\; & \left\{\det(1-X)^{1/2}\det(1+T)^{1/2}\right\}^{26} \cdot \\ & \cdot \exp\left(-\frac{1}{2}G_{\mu\nu}\sum_{m,n\geq 1} a_m^{\mu\dagger} S_{mn} a_n^{\nu\dagger}\right)|0\rangle \otimes \\ & \frac{4a}{2a+1}\;\frac{b^2}{\sqrt{2\pi b^3}(\det G)^{1/4}}\;\exp\left(-\frac{1}{2}s\,a_0^{\alpha\dagger}G_{\alpha\beta}a_0^{\beta\dagger}\right)|\Omega_{b,\theta}\rangle, \end{aligned} \tag{186}$$

where $\mu,\nu = 0,\dots 25$ and $G_{\mu\nu} = \eta_{\bar\mu\bar\nu} \otimes G_{\alpha\beta}$. The first line of the RHS of this equation is nothing but the sliver state $|\Xi\rangle$, which represents the D25–brane. The norm of the lump is now regularized by the presence of a which is directly proportional to B: $a = -\frac{\pi^2}{A}B$. Using

$$\begin{aligned} |x\rangle = & \sqrt{\frac{2\sqrt{\det G}}{b\pi}} \cdot \\ & \cdot \exp\left[-\frac{1}{b}x^\alpha G_{\alpha\beta}x^\beta - \frac{2}{\sqrt{b}}i a_0^{\alpha\dagger}G_{\alpha\beta}x^\beta + \frac{1}{2}a_0^{\alpha\dagger}G_{\alpha\beta}a_0^{\beta\dagger}\right]|\Omega_{b,\theta}\rangle \end{aligned}$$

we can calculate the projection onto the basis of position eigenstates of the transverse part of the lump state

$$\begin{aligned} \langle x|e^{-\frac{s}{2}(a_0^\dagger)^2}|\Omega_{b,\theta}\rangle &= \sqrt{\frac{2\sqrt{\det G}}{b\pi}}\frac{1}{1+s}\,e^{-\frac{1-s}{1+s}\frac{1}{b}x^\alpha x^\beta G_{\alpha\beta}} \\ &= \sqrt{\frac{2\sqrt{\det G}}{b\pi}}\frac{1}{1+s}\,e^{-\frac{1}{2|a|b}x^\alpha x^\beta G_{\alpha\beta}} \end{aligned} \tag{187}$$

The transverse part of the lump state in the x representation is then

$$\langle x|\hat{\mathcal{S}}_\perp\rangle = \frac{1}{\pi}\,e^{-\frac{1}{2|a|b}x^\alpha x^\beta G_{\alpha\beta}}|\Xi_\perp\rangle \tag{188}$$

Using now the form (181) of $G_{\alpha\beta}$ and $\theta^{\alpha\beta}$ and the explicit expression for a in terms of g and α', [58]

$$a = \frac{\theta}{4A}\sqrt{\det G} = -\frac{2\pi^2(\alpha')^2 B}{b\,g} \tag{189}$$

we obtain the simplest soliton solution of [57] (see also [31] and references therein):

$$e^{-\frac{1}{2|a|b}x^\alpha x^\beta G_{\alpha\beta}} \quad \to \quad e^{-\frac{x^\alpha x^\beta \delta_{\alpha\beta}}{|\theta|}} \tag{190}$$

which corresponds to the $|0\rangle\langle 0|$ projector in the harmonic oscillator Hilbert space (see Introduction), and is a projector on a space endowed with a Moyal product.

In this way the B field provides a regularization of (188), as compared to [56]. This beneficial effect of the B field is confirmed by the fact that the projector (186) is no longer annihilated by x_0

$$\begin{aligned}
&x_0 \,\exp\left(-\frac{1}{2}s a_0^{\alpha\dagger} G_{\alpha\beta} a_0^{\beta\dagger}\right)|\Omega_{b,\theta}\rangle = \\
=\;& i\,\frac{\sqrt{b}}{2}\,(a_0 - a_0^\dagger)\exp\left(-\frac{1}{2}s a_0^{\alpha\dagger} G_{\alpha\beta} a_0^{\beta\dagger}\right)|\Omega_{b,\theta}\rangle \\
=\;& -i\,\frac{\sqrt{b}}{2}\left[\frac{4a}{2a+1}\right] a_0^\dagger \exp\left(-\frac{1}{2}s a_0^{\alpha\dagger} G_{\alpha\beta} a_0^{\beta\dagger}\right)|\Omega_{b,\theta}\rangle
\end{aligned}$$

Therefore, in the low energy limit, the singular structure found in [56] has disappeared in the presence of a nonvanishing B field. This is actually not true only in the low energy limit, but is an exact result, as was shown in [55].

10 More lumps in VSFT

In the two previous sections we have constructed a 23–dimensional lump solution, which we have interpreted as a D23–brane. In the low energy limit this solution in the coordinate basis, turned out to be the simplest (two–dimensional) GMS soliton multiplied by a translational invariant solution which represents the D25–brane. The question we want to deal with here is whether there are other lump solutions that correspond to the higher order GMS solitons. The answer is affirmative. We will construct an infinite sequence of them, denoted $|\Lambda_n\rangle$. These new star algebra projectors are D23–branes, constructed out of (173) and parallel to it. In the low energy limit they give rise to the full series of GMS solitons. We will construct them and prove that they satisfy the remarkable identities (135,136).

In order to construct these new solutions we need a new ingredient, given by the Fock space projectors similar to those introduced in [22]. We define them only along the transverse directions

$$\rho_1 = \frac{1}{(\mathbb{I}+\mathcal{T})(\mathbb{I}-\mathcal{X})}\left[\mathcal{X}^{12}(\mathbb{I}-\mathcal{T}\mathcal{X})+\mathcal{T}(\mathcal{X}^{21})^2\right] \tag{191}$$

$$\rho_2 = \frac{1}{(\mathbb{I}+\mathcal{T})(\mathbb{I}-\mathcal{X})}\left[\mathcal{X}^{21}(\mathbb{I}-\mathcal{T}\mathcal{X})+\mathcal{T}(\mathcal{X}^{12})^2\right] \tag{192}$$

They satisfy

$$\rho_1^2 = \rho_1, \qquad \rho_2^2 = \rho_2, \qquad \rho_1 + \rho_2 = \mathbb{I} \tag{193}$$

i.e. they project onto orthogonal subspaces. Moreover, if we use the superscript ^{T} to denote transposition with respect to the indices N, M and α, β, we have

$$\rho_1^T = \tilde{\rho}_1 = C\rho_2 C, \qquad \rho_2^T = \tilde{\rho}_2 = C\rho_1 C. \tag{194}$$

and

$$\begin{aligned} &\rho_i^\dagger = \rho_i, \quad \text{i.e.} \quad \rho_i^* = \tilde{\rho}_i, \quad i = 1,2 \\ &\tau\rho_i = \tilde{\rho}_i\tau, \qquad i = 1,2 \end{aligned}$$

where * denote complex conjugation and $^\dagger = ^{*T}$. Moreover τ is the matrix $\tau = \{\tau_\alpha{}^\beta\} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. We recall that in the absence of the B field, it has been shown that ρ_1, ρ_2 projects out half the string modes, [22, 26].

With all these ingredients we can now move on, give a precise definition of the $|\Lambda_n\rangle$ states and demonstrate the properties announced above.

To define the states $|\Lambda_n\rangle$ we start from the lump solution (173). I.e. we take $|\Lambda_0\rangle = |\mathcal{S}\rangle$. However, in the following, we will limit ourselves only to the transverse part of it, the parallel one being universal and irrelevant for our construction. We will denote the transverse part by $|\mathcal{S}_\perp\rangle$.

First we introduce two 'vectors' $\xi = \{\xi_{N\alpha}\}$ and $\zeta = \{\zeta_{N\alpha}\}$, which are chosen to satisfy the conditions

$$\rho_1\xi = 0, \qquad \rho_2\xi = \xi, \qquad \text{and} \quad \rho_1\zeta = 0, \qquad \rho_2\zeta = \zeta, \tag{195}$$

Next we define

$$\mathbf{x} = (a^\dagger\tau\xi)\,(a^\dagger C\zeta) = (a_N^{\alpha\dagger}\tau_\alpha{}^\beta\xi_{N\beta})(a_N^{\alpha\dagger}C_{NM}\zeta_{M\alpha}) \tag{196}$$

and introduce the Laguerre polynomials $L_n(z)$, of the generic variable z. The definition of $|\Lambda_n\rangle$ is as follows

$$|\Lambda_n\rangle = (-\kappa)^n L_n\left(\frac{\mathbf{x}}{\kappa}\right)|\mathcal{S}_\perp\rangle \tag{197}$$

As part of the definition of $|\Lambda_n\rangle$ we require the two following conditions to be satisfied

$$\xi^T \tau \frac{1}{\mathbb{I} - \mathcal{T}^2} \zeta = -1, \qquad \xi^T \tau \frac{\mathcal{T}}{\mathbb{I} - \mathcal{T}^2} \zeta = -\kappa \tag{198}$$

Hermiticity for $|\Lambda_n\rangle$ requires that

$$(a\tau\xi^*)(aC\zeta^*) = (a\tau C\xi)(a\zeta) \tag{199}$$

This condition admits the solution

$$\zeta = \tau\xi^* \tag{200}$$

which we will assume throughout the rest of the paper, even though it will be left implicit for notational simplicity. Eq.(200) is compatible with the conditions (195) and (198), see [59]. As a consequence of (200), the LHS's of both equations (198) are real, so κ must be real too. Let us show this for instance for the first equation, since for the second no significant modification is needed:

$$\begin{aligned}\left(\xi^T \tau \frac{1}{\mathbb{I} - \mathcal{T}^2} \zeta\right)^* &= \xi^{T*} \tau \frac{1}{\mathbb{I} - (\mathcal{T}^*)^2} \zeta^* = \\ &= \zeta^T \frac{1}{\mathbb{I} - (\mathcal{T}^*)^2} \tau\xi = \xi^T \tau \frac{1}{\mathbb{I} - (\mathcal{T}^\dagger)^2} \zeta = \xi^T \tau \frac{1}{\mathbb{I} - \mathcal{T}^2} \zeta\end{aligned}$$

where the second equality is obtained by replacement of (200), and the third by transposition.

The proof that (135,136) are satisfied was given in [59].

Before we pass to the low energy limit, let us make a comment on the definition of $|\Lambda_n\rangle$, wherein a central role is played by the Laguerre polynomials. While the true rationale of this role eludes us, it is possible to prove that the form of the definition $|\Lambda_n\rangle$ (together with (195,198) is not only sufficient for (135,136) to be true, but also necessary. The case $|\Lambda_1\rangle = (\mathbf{x} - \kappa)|\mathcal{S}_\perp\rangle$ was discussed in [22]. The next most complicated state is

$$(\alpha + \beta\mathbf{x} + \gamma\mathbf{x}^2)|\mathcal{S}_\perp\rangle \tag{201}$$

The conditions this state has to satisfy in order to define a $|\Lambda_2\rangle$ that obeys (135,136) for $n = 0, 1, 2$ are, given (195,198), by the following relations

$$-2(\alpha)^{1/2} = \beta, \qquad \gamma = \frac{1}{2} \tag{202}$$

Then, putting $\alpha = \kappa$

$$|\mathcal{P}'\rangle = \Big(\kappa^2 - 2\kappa\mathbf{x} + \frac{1}{2}\mathbf{x}^2\Big)|\mathcal{S}_\perp\rangle \tag{203}$$

The polynomial in the RHS is nothing but the second Laguerre polynomial of $\mathbf{x}/\kappa$ multiplied by κ^2. In fact using Mathematica it is easy to extend this analysis for n as large as one wishes.

Finally let us remark that the relations demonstrated in this section, in particular (135,136), are true for any value of B, therefore also for $B = 0$.

11 The GMS solitons

In order to analyze the same limit as in section (9) for a generic $|\Lambda_n\rangle$, first of all we have to find the low energy limit of the projectors ρ_1, ρ_2. In this limit these two projectors factorize into the zero mode and non–zero mode part. The former is given by

$$(\rho_1)_{00}^{\alpha\beta} \to \frac{1}{2}\Big[G^{\alpha\beta} + i\epsilon^{\alpha\beta}\Big], \qquad (\rho_2)_{00}^{\alpha\beta} \to \frac{1}{2}\Big[G^{\alpha\beta} - i\epsilon^{\alpha\beta}\Big], \tag{204}$$

Now, we take, in the definition (196), $\xi = \hat{\xi} + \bar{\xi}$ and $\zeta = \hat{\zeta} + \bar{\zeta}$, where $\bar{\xi}, \bar{\zeta}$ are such that they vanish in the limit $\alpha' \to 0$. Then we make the choice $\hat{\xi}_n = \hat{\zeta}_n = 0, \ \forall n > 0$ and determine $\hat{\xi}$ and $\hat{\zeta}$ in such a way that eqs.(195, 199) and (198) are satisfied in the limit $\alpha' \to 0$. We are assuming here that there exist solutions of the problem at $\alpha' \neq 0$ that take precisely this specific form when $\alpha' \to 0$. This is a plausible assumption since the α' dependence is smooth in all the involved quantities. In any case it is not hard to construct examples of this fact: for instance $\xi = \rho_2\hat{\xi}$ satisfies the above requirements to zeroth and first order of approximation in ϵ. More complete examples are provided in [59].

Now, in the field theory limit the conditions (195) become

$$\hat{\xi}_{0,24} + i\hat{\xi}_{0,25} = 0, \qquad \hat{\zeta}_{0,24} + i\hat{\zeta}_{0,25} = 0, \tag{205}$$

¿From now on we set $\hat{\xi}_0 = \hat{\xi}_{0,25} = -i\hat{\xi}_{0,24}$ and, similarly, $\hat{\zeta}_0 = \hat{\zeta}_{0,25} = -i\hat{\zeta}_{0,24}$. The conditions (198) become

$$\xi^T \tau \frac{1}{\mathbb{I} - \mathcal{T}^2}\zeta \to -\frac{1}{1-s^2}\frac{2}{\sqrt{\det G}}\hat{\xi}_0\hat{\zeta}_0 = -1 \tag{206}$$

$$\xi^T \tau \frac{\mathcal{T}}{\mathbb{I} - \mathcal{T}^2}\zeta \to -\frac{s}{1-s^2}\frac{2}{\sqrt{\det G}}\hat{\xi}_0\hat{\zeta}_0 = -\kappa \tag{207}$$

Compatibility requires

$$\frac{2\hat{\xi}_0\hat{\zeta}_0}{\sqrt{\det G}} = 1 - s^2, \qquad \kappa = s \tag{208}$$

At the same time

$$(\xi\tau a^\dagger)(\zeta C a^\dagger) \to -\hat{\xi}_0\hat{\zeta}_0((a_0^{24\dagger})^2 + (a_0^{25\dagger})^2) = -\frac{\hat{\xi}_0\hat{\zeta}_0}{\sqrt{\det G}} a_0^{\alpha\dagger} G_{\alpha\beta} a_0^{\beta\dagger} \tag{209}$$

Hermiticity requires that the product $\hat{\xi}_0\hat{\zeta}_0 = |\hat{\xi}_0|^2$, in accordance with (206,207). The solutions found in this way can be referred to as the *factorized solutions*, since, as will become clear in a moment, they realize the factorization of the star product into the Moyal $\star$ product and Witten's $*$ product. In order to be able to compute $\langle x|\Lambda_n\rangle$ in the field theory limit, we have to evaluate first

$$\begin{aligned} &\langle x| \left(a_0^{\alpha\dagger} G_{\alpha\beta} a_0^{\beta\dagger}\right)^k e^{-\frac{s}{2} a_0^{\alpha\dagger} G_{\alpha\beta} a_0^{\beta\dagger}} |\Omega_{b,\theta}\rangle \\ = \;& (-2)^k \frac{d^k}{ds^k} \left(\langle x| e^{-\frac{s}{2} a_0^{\alpha\dagger} G_{\alpha\beta} a_0^{\beta\dagger}} |\Omega_{b,\theta}\rangle\right) \\ = \;& (-2)^k \frac{d^k}{ds^k} \left(\sqrt{\frac{2\sqrt{\det G}}{b\pi}} \frac{1}{1+s} e^{-\frac{1-s}{1+s}\frac{1}{b} x^\alpha G_{\alpha\beta} x^\beta}\right) \end{aligned} \tag{210}$$

An explicit calculation gives

$$\begin{aligned} &\frac{d^k}{ds^k}\left(\frac{1}{1+s} e^{-\frac{1-s}{1+s}\frac{1}{b} x^\alpha x^\beta G_{\alpha\beta}}\right) = \\ &= \sum_{l=0}^{k} \sum_{j=0}^{k-l} \frac{(-1)^{k+j}}{(1-s)^j (1+s)^{k+1}} \frac{k!}{j!} \begin{pmatrix} k-l-1 \\ j-1 \end{pmatrix} \langle x, x\rangle^j e^{-\frac{1}{2}\langle x,x\rangle} \end{aligned} \tag{211}$$

where it must be understood that, by definition, the binomial coefficient $\begin{pmatrix} -1 \\ -1 \end{pmatrix}$ equals 1. Moreover we have set

$$\langle x, x\rangle = \frac{1}{ab} x^\alpha G_{\alpha\beta} x^\beta = \frac{2r^2}{\theta} \tag{212}$$

with $r^2 = x^\alpha x^\beta \delta_{\alpha\beta}$.

Now, inserting (211) in the definition of $|\Lambda_n\rangle$, we obtain after suitably reshuffling the indices:

$$\langle x|(-\kappa)^n L_n\left(\frac{\mathbf{x}}{\kappa}\right) e^{-\frac{1}{2} s a_0^{\alpha\dagger} G_{\alpha\beta} a_0^{\beta\dagger}} |\Omega_{b,\theta}\rangle \to$$

$$
\begin{aligned}
\to\quad & \langle x|(-s)^n L_n\Big(-\frac{1-s^2}{2s}\; a_0^{\alpha\dagger}G_{\alpha\beta}a_0^{\beta\dagger}\Big)e^{-\frac{1}{2}sa_0^{\alpha\dagger}G_{\alpha\beta}a_0^{\beta\dagger}}|\Omega_{b,\theta}\rangle\\
&= \frac{(-s)^n}{(1+s)}\sum_{j=0}^{n}\sum_{k=j}^{n}\sum_{l=j}^{k}\binom{n}{k}\binom{l-1}{j-1}\frac{1}{j!}\frac{(1-s)^k}{(1+s)^j s^k}\cdot\\
&\cdot(-1)^j\langle x,x\rangle^j\, e^{-\frac{1}{2}\langle x,x\rangle}\sqrt{\frac{2\sqrt{\det G}}{b\pi}} \qquad (213)
\end{aligned}
$$

The expression can be evaluated as follows. First one uses the result

$$
\sum_{l=j}^{k}\binom{l-1}{j-1}=\binom{k}{j} \qquad (214)
$$

Inserting this into (213) one is left with the following summation, which contains an evident binomial expansion,

$$
\sum_{k=j}^{n}\binom{n}{k}\binom{k}{j}\left(\frac{1-s}{s}\right)^k=\binom{n}{j}\frac{(1-s)^j}{s^n} \qquad (215)
$$

Replacing this result into (213) we obtain

$$
\begin{aligned}
&\langle x|(-\kappa)^n L_n\left(\frac{\mathbf{x}}{\kappa}\right)e^{-\frac{1}{2}sa_0^{\alpha\dagger}G_{\alpha\beta}a_0^{\beta\dagger}}|\Omega_{b,\theta}\rangle\to\\
&\to\frac{2|a|+1}{4|a|}\sqrt{\frac{2\sqrt{\det G}}{b\pi}}(-1)^n\sum_{j=0}^{n}\binom{n}{j}\frac{1}{j!}\left(-\frac{2r^2}{\theta}\right)^j e^{-\frac{r^2}{\theta}}
\end{aligned}
$$

Recalling now that the definition of $|\hat{\mathrm{S}}\rangle$ includes an additional numerical factor (see eq.(186)), we finally obtain

$$
\begin{aligned}
\langle x|\Lambda_n\rangle\to\langle x|\hat{\Lambda}_n\rangle &= \frac{1}{\pi}(-1)^n\sum_{j=0}^{n}\binom{n}{j}\frac{1}{j!}\left(-\frac{2r^2}{\theta}\right)^j e^{-\frac{r^2}{\theta}}|\Xi\rangle\\
&= \frac{1}{\pi}(-1)^n L_n\left(\frac{2r^2}{\theta}\right)e^{-\frac{r^2}{\theta}}|\Xi\rangle \qquad (216)
\end{aligned}
$$

as announced in section 6. The coefficient in front of the sliver $|\Xi\rangle$ is the $n-th$ GMS solution. Strictly speaking there is a discrepancy between these coefficients and the corresponding GMS soliton, given by the normalizations which differ by a factor of 2π. This can be traced back to the traditional normalizations used for the eigenstates $|x\rangle$ and $|p\rangle$ in the SFT theory context and in the Moyal context, respectively. This discrepancy can be easily dealt with a simple redefinition.

12 VSFT star product and Moyal product

In the previous section we have shown that the low energy limit of $\langle x|\hat{\Lambda}_n\rangle$ factorizes into the product of the sliver state and $\psi_n(x,y)$, see (134). This means, on the one hand, that the GMS solitons are the low energy remnants of corresponding D–branes in VSFT, and, on the other hand, that, for this type of solutions, the VSFT star product factorizes into Witten's star product and the Moyal $\star$ product. But, actually, much more can be said about the correspondence between the states $|\hat{\Lambda}_n\rangle$ and the solitons of noncommutative field theories with polynomial interaction.

We recall from section 6 that the latter are very elegantly constructed in terms of harmonic oscillators eigenstates $|n\rangle$. In particular the $\psi_n(x,y)$ solutions correspond to projectors $P_n = |n\rangle\langle n|$, via the Weyl transform. The correspondence is such that the operator product in the Hilbert space corresponds to the Moyal product in (x,y) space. Therefore we can formalize the following correspondence

$$\begin{array}{ccccc} |\Lambda_n\rangle & \longleftrightarrow & P_n & \longleftrightarrow & \psi_n(x,y) \\ |\Lambda_n\rangle * |\Lambda_{n'}\rangle & \longleftrightarrow & P_n P_{n'} & \longleftrightarrow & \psi_n \star \psi_{n'} \end{array} \tag{217}$$

where $\star$ denotes the Moyal product. Moreover

$$\langle \Lambda_n|\Lambda_{n'}\rangle \quad \longleftrightarrow \quad \mathrm{Tr}(P_n P_{n'}) \longleftrightarrow \int dxdy\, \psi_n(x,y)\psi_{n'}(x,y) \tag{218}$$

up to normalization (see (136)). This correspondence seems to indicate that the Laguerre polynomials hide a universal structure of these noncommutative algebras.

This parallelism can actually be pushed still further. In fact we can easily construct the correspondents of the operators $|n\rangle\langle m|$. Let us first define

$$X = a^\dagger \tau \xi \qquad Y = a^\dagger C \zeta \tag{219}$$

so that $\mathbf{x} = XY$. The definitions we are looking for are as follows

$$|\Lambda_{n,m}\rangle = \sqrt{\frac{n!}{m!}}(-\kappa)^n\, Y^{m-n} L_n^{m-n}\left(\frac{\mathbf{x}}{\kappa}\right)|\mathcal{S}_\perp\rangle, \qquad n \le m \tag{220}$$

$$|\Lambda_{n,m}\rangle = \sqrt{\frac{m!}{n!}}(-\kappa)^m\, X^{n-m} L_m^{n-m}\left(\frac{\mathbf{x}}{\kappa}\right)|\mathcal{S}_\perp\rangle, \qquad n \ge m \tag{221}$$

where $L_n^{m-n}(z) = \sum_{k=0}^m \binom{m}{n-k}(-z)^k/k!$. With the same techniques as in the previous sections one can prove that

$$|\Lambda_{n,m}\rangle * |\Lambda_{r,s}\rangle = \delta_{m,r}|\Lambda_{n,s}\rangle \tag{222}$$

for all natural numbers n, m, r, s. It is clear that the previous states $|\Lambda_n\rangle$ coincide with $|\Lambda_{n,n}\rangle$. In view of (222), we can extend the correspondence (217) to $|n\rangle\langle m| \leftrightarrow |\Lambda_{n,m}\rangle$. Therefore, following [57], [31], we can apply to the construction of projectors in the VSFT star algebra the solution generating technique, in the same way as in the harmonic oscillator Hilbert space $\mathcal{H}$. Naturally in this case we do not have any guarantee that all the projectors are recovered in this way.

A Appendix

This appendix is devoted to a direct analytic proof of eqs.(28) and (70). Let us start from the latter.

Proof of eq.(70). It is convenient to rewrite it as follows

$$\sum_{k=0}^{\infty} \tilde{U}_{nk}\, \tilde{U}_{km} = \delta_{n0}\delta_{m0} + \sum_{k=0}^{\infty} \tilde{U}^{(2)}_{nk}\, \tilde{U}^{(1)}_{km} \tag{223}$$

since, in the range $0 \leq n, m < \infty$, we have $\tilde{U}^{(1)}_{km} = \delta_{n0}\delta_{m0} + \tilde{U}^{(2)}_{km}$ and $\tilde{U}^{(1)}_{0m} = \delta_{m0}$. Therefore we have to compute

$$\sum_{k=0}^{\infty} \tilde{U}^{(2)}_{nk}\, \tilde{U}^{(1)}_{km} = \oint \frac{dz}{2\pi i} \frac{1}{z^{n+1}} \oint \frac{d\zeta}{2\pi i} \oint \frac{d\theta}{2\pi i} \oint \frac{dw}{2\pi i} \frac{1}{w^{m+1}} \cdot \tag{224}$$
$$\cdot \sum_{k=0}^{\infty} \frac{1}{(\zeta\theta)^{k+1}} \frac{f(z)}{f(\zeta)} \frac{f(\theta)}{f(w)} \left(\frac{1}{1+z\zeta} - \frac{\zeta}{\zeta - z}\right) \left(\frac{1}{1+\theta w} - \frac{w}{w-\theta}\right)$$

Here we have already exchanged the summation over k with integrals, which is allowed only under definite convergence conditions. The latter are guaranteed if $|\zeta\theta| > 1$, in which case

$$\sum_{k=0}^{\infty} \frac{1}{(\zeta\theta)^{k+1}} = \frac{1}{\theta\zeta - 1} \tag{225}$$

Now, we recall that, from the definition of $\tilde{U}^{(1)}, \tilde{U}^{(2)}$, we have $|z| < |\zeta|, |\theta| > |w|$. In order to comply with the condition $|\zeta\theta| > 1$ we choose to deform the θ contour while keeping the ζ contour fixed. In doing so we have to be careful to avoid possible singularities in θ. These are poles at $\theta = w, -\frac{1}{w}$ and branch cuts at $\theta = \pm i$, due to the $f(\theta)$ factor. One can deform the θ contour in such a way as to keep the pole at $-\frac{1}{w}$ external to the contour, since the w contour is as small as we wish around the origin. But, of course, one cannot avoid the branch points at $\theta = \pm i$. To make sense of the operation we introduce a regulator $K > 1$ and modify the integrand by modifying $f(\theta)$

$$f(\theta) \to f_K(\theta) = \left(\frac{K+i\theta}{K-i\theta}\right)^{\frac{2}{3}} \tag{226}$$

We will take K as large as needed and eventually move back to $K = 1$. Under these conditions we can safely perform the summation over k in (224) and make the replacement (225) in the integral.

As the next step we carry out the θ integration, which reduces to the contribution from the simple poles at $\theta = w$ and $\theta = \frac{1}{\zeta}$. The RHS of (224) becomes

$$\begin{aligned} = \quad & \oint \frac{dz}{2\pi i}\frac{1}{z^{n+1}} \oint \frac{d\zeta}{2\pi i} \oint \frac{dw}{2\pi i}\frac{1}{w^{m+1}} \cdot \\ & \cdot \left[\frac{f(z)}{f(\zeta)}\frac{f_K(1/\zeta)}{f(w)}\left(\frac{1}{1+z\zeta} - \frac{\zeta}{\zeta - z}\right)\left(\frac{1}{w+\zeta} - \frac{w}{\zeta w - 1}\right)\right. \\ & \qquad \left. + \frac{f(z)}{f(\zeta)}\frac{w}{\zeta w - 1}\left(\frac{1}{1+z\zeta} - \frac{\zeta}{\zeta - z}\right)\right] \end{aligned} \tag{227}$$

The first line corresponds to the contribution from the pole at $\theta = \frac{1}{\zeta}$, while the second comes from the pole at $\theta = w$.

Next we wish to integrate with respect to ζ. The singularities trapped within the ζ contour of integration are the poles at $\zeta = z, -w$ (not the poles at $\zeta = \frac{1}{w}, -\frac{1}{z}$). Since above we had $K > |\theta| > \frac{1}{|\zeta|}$, it follows that $|\zeta| > \frac{1}{K}$. Therefore also the branch points at $\zeta = \pm\frac{i}{K}$ of $f_K(1/\zeta)$ are trapped inside the ζ contour and we have to compute the relevant contribution to the integral. In the integrand of (227) we have two cuts in ζ. One is the cut we have just mentioned, let us call it $\mathfrak{c}_{1/K}$ and let us fix it to be the semicircle of radius $1/K$ at the RHS of the imaginary axis; the contour that surrounds it excluding all the other singularities will be denoted $C_{1/K}$. The other cut, due to $f(\zeta)$, with branch points at $\zeta = \pm i$, will be denoted $\mathfrak{c}_1$; the contour that surrounds it excluding all the other singularities will be denoted C_1.

After these lengthy preliminaries let us carry out the integration over ζ. We get

$$\begin{aligned} = \quad & \oint \frac{dz}{2\pi i}\frac{1}{z^{n+1}} \oint \frac{dw}{2\pi i}\frac{1}{w^{m+1}} \left[\frac{f(1/z)}{f(w)}\left(\frac{zw}{zw-1} - \frac{z}{z+w}\right) + \right. \\ & \left. + \frac{f(z)}{f(1/w)}\left(\frac{1}{1-zw} - \frac{w}{w+z}\right) + \oint_{C_{1/K}} \frac{d\zeta}{2\pi i}(\ldots) + \frac{zw}{1-zw}\right] \end{aligned} \tag{228}$$

The first two terms in square brackets come from the contribution of the poles at $\zeta = z$ and $\zeta = -w$ from the first line in (227), respectively. The symbol $(\ldots)$ represents the integrand contained within the square brackets in the first line of (227). Finally the last term in (228) is the contribution coming from the second line of (227) due to the pole at $\zeta = z$. We notice that

$$\frac{zw}{zw-1} - \frac{z}{z+w} = \frac{w}{z+w} - \frac{1}{1-zw} \tag{229}$$

but of course the problem here is how to evaluate the integral around the cut. Fortunately this can be reduced to an evaluation of contributions from poles. To see this, we first recall the properties of $f(z)$. It is easy to see that

$$f(1/z) = f(-z) \quad \text{and} \quad f(-z) = 1/f(z) \tag{230}$$

Therefore, in the limit $K \to 1$, the factor $f_K(1/\zeta)/f(\zeta)$ tends to $(f(-\zeta))^2$. As a consequence, in the same limit, the integral of $(\ldots)$ around the cut $\mathfrak{c}_{1/K}$ is the same as the integral around the cut $\mathfrak{c}_1$, and each equals one–half the integral around both contours, in other words each equals one–half the integral about a contour that surrounds both cuts and exclude all the other singularities (which are poles). By a well-known argument, the latter integral equals the negative of the integral of $(\ldots)$ about all the remaining singularities in the complex ζ–plane. This is easy to compute. The remaining singularities are poles around $\zeta = z, -w, -1/z, 1/w$. Notice that there is no singularity at $\zeta = \infty$. Carrying out this calculation explicitly we get

$$\begin{aligned}
= & \oint \frac{dz}{2\pi i}\frac{1}{z^{n+1}} \oint \frac{dw}{2\pi i}\frac{1}{w^{m+1}} \left\{ \frac{f(1/z)}{f(w)}\left(\frac{w}{z+w} - \frac{1}{1-zw}\right) \right. \\
& + \frac{f(z)}{f(1/w)}\left(\frac{1}{1-zw} - \frac{w}{w+z}\right) \\
& - \frac{1}{2}\left[\frac{f(1/z)}{f(w)}\left(\frac{w}{z+w} - \frac{1}{1-zw}\right) + \frac{f(z)}{f(1/w)}\left(\frac{1}{1-zw} - \frac{w}{w+z}\right)\right. \\
& \left. + \frac{f(1/z)}{f(w)}\left(\frac{w}{z+w} - \frac{1}{1-zw}\right) + \frac{f(z)}{f(1/w)}\left(\frac{1}{1-zw} - \frac{w}{w+z}\right)\right] \\
& \left. + \frac{zw}{1-zw} \right\}
\end{aligned} \tag{231}$$

The terms in square brackets represent the contribution from the cut $\mathfrak{c}_{1/K}$ and come from the simple poles at $\zeta = z, -w, -1/z, 1/w$, respectively. All the terms cancel out except the last. So the RHS of (224) reduces to

$$= \oint \frac{dz}{2\pi i}\frac{1}{z^{n+1}} \oint \frac{dw}{2\pi i}\frac{1}{w^{m+1}} \sum_{k=1}^{\infty} (zw)^k = \delta_{nm}, \qquad n, m \geq 1 \tag{232}$$

This complete the proof of (28). We remark that we could have integrated first with respect to ζ and then with respect to θ. The procedure is somewhat different, but the final result is the same. We also point out that there may be other equivalent ways to derive (28).

Proof of eq.(28). It is convenient to rewrite U_{nm} in an alternative form compared to (23). We start by replacing in eq.(19)

$$f'_a(z)\frac{1}{(f_a(z)-f_b(w))^2}f'_b(w) = -\partial_z \frac{1}{f_a(z)-f_b(w)}f'_b(w) \tag{233}$$

and integrating by part. We decompose the resulting expression as in eq.(21). After some algebra one gets

$$U_{nm} = \sqrt{\frac{n}{m}}\oint \frac{dz}{2\pi i}\frac{1}{z^{n+1}}\oint \frac{dw}{2\pi i}\frac{1}{w^{m+1}}\frac{g(z)}{g(w)}\left(\frac{1}{1+zw}-\frac{w}{w-z}\right) \tag{234}$$

where

$$g(z) = \frac{1}{z}(1+iz)^{\frac{2}{3}}(1-iz)^{\frac{4}{3}} \tag{235}$$

This function satisfies

$$g(1/z) = g(-z) \tag{236}$$

which corresponds to the first of eqs.(230). There is no analog of the second.

In order to prove eq.(28) we have to evaluate

$$\sqrt{\frac{m}{n}}\sum_{k=1}^{\infty} U_{nk}U_{km} = \oint \frac{dz}{2\pi i}\frac{1}{z^{n+1}}\oint\frac{d\zeta}{2\pi i}\oint\frac{d\theta}{2\pi i}\oint\frac{dw}{2\pi i}\frac{1}{w^{m+1}}\cdot$$
$$\cdot\sum_{k=1}^{\infty}\frac{1}{(\zeta\theta)^{k+1}}\frac{g(z)}{g(\zeta)}\frac{g(\theta)}{g(w)}\left(\frac{1}{1+z\zeta}-\frac{\zeta}{\zeta-z}\right)\left(\frac{1}{1+\theta w}-\frac{w}{w-\theta}\right)$$

The structure is the same as in (224), except for the substitution $f\to g$ and for the fact that now the summation over k starts from 1. We will thus proceed as above while paying attention to the differences. Using

$$\sum_{k=1}^{\infty}\frac{1}{(\zeta\theta)^{k+1}} = \frac{1}{\zeta\theta}\frac{1}{\theta\zeta-1} \tag{237}$$

instead of (225), we see that, when integrating over θ we have to take into account the pole at $\theta=0$. The result is

$$= \oint \frac{dz}{2\pi i}\frac{1}{z^{n+1}}\oint\frac{d\zeta}{2\pi i}\oint\frac{dw}{2\pi i}\frac{1}{w^{m+1}}\cdot$$

$$
\begin{aligned}
\cdot & \left[\frac{g(z)}{g(\zeta)}\frac{g_K(1/\zeta)}{g(w)}\left(\frac{1}{1+z\zeta}-\frac{\zeta}{\zeta-z}\right)\left(\frac{1}{w+\zeta}-\frac{w}{\zeta w-1}\right)\right.\\
& +\frac{f(z)}{f(\zeta)}\frac{w}{\zeta w-1}\left(\frac{1}{1+z\zeta}-\frac{\zeta}{\zeta-z}\right)\\
& \left.+\frac{g(z)}{\zeta g(\zeta)g(w)}\left(\frac{1}{1+z\zeta}-\frac{\zeta}{\zeta-z}\right)\frac{1+w^2}{w}\right]
\end{aligned}
\tag{238}
$$

The last contribution comes precisely from the double pole at $\theta = 0$.

Next let us integrate over ζ. There is no singularity at $\zeta = 0$ or $\zeta = \infty$, as one may have suspected. Let us deal first with the first line in eq.(238). This is exactly the first line of (227), except for the substitution $f \to g$. We proceed in the same way as above, but with some additional care because we cannot use the analog of the second eq.(230). However we remark that

$$
\frac{g_K(1/\zeta)}{g(\zeta)} = \frac{f_K(1/\zeta)}{f(\zeta)}\frac{(\zeta K - i)^2}{(1-i\zeta)^2}
\tag{239}
$$

Now we have recovered the same structure as in (227) except for the last factor in the RHS of (239), i.e. at the price of bringing into the game a double pole at $\zeta = -i$. Fortunately the residue of this pole vanishes. All is well what ends well. We can now safely repeat the same argument that leads from eq.(227) to eq.(231), and conclude that the various contributions from the first line of eq.(238) add up to zero. The second line is easy to compute, the only contribution comes from the simple pole at $\zeta = z$:

$$
= \oint \frac{dz}{2\pi i}\frac{1}{z^{n+1}}\oint \frac{d\zeta}{2\pi i}\oint \frac{dw}{2\pi i}\frac{1}{w^{m+1}}\left[\frac{1}{1-zw} - \frac{1}{g(w)}\frac{1+w^2}{w}\right] = \delta_{nm}
\tag{240}
$$

for $n, m \geq 1$. This completes the proof of (28).

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