On integrability and rigid-body systems

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Abstract

Higher dimensional generalizations of the classical Hess-Apel'rot rigid body system are constructed. The Lax representations are found. Integration procedures are performed in classical and algebro-geometric ways. The algebro-geometric integration is based on deep facts from the geometry of Prym varieties such as the Mumford relation for theta divisors of double unramified coverings and the Mumford - Dalalyan theory. This procedure is closely related to the procedure of integration of the Lagrange bitop, which was performed recently.

1 Poisson structures and completely integrable systems.

Algebra $C^\infty(M)$ of smooth functions on symplectic manifold (M, ω) admits a binary operation

$$\{f, g\} := \omega(X_f, X_g),$$

where X_f and X_g are Hamiltonian vector fields defined by Hamiltonians f and g. Its basic properties are

- bilinearity;
- antisymmetricity: $\{f, g\} = -\{g, f\}$.
- the Jacobi identity: $\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = 0$.
- the Leibnitz rule: $\{f, gh\} = \{f, g\}h + g\{f, h\}$.

The more general class of manifolds are Poisson manifolds.

Definition 1 *Poisson algebra is a commutative algebra with an antisymmetric bilinear operation* $\{\cdot,\cdot\}$ *satisfying the Jacobi identity and the Leibnitz rule. A manifold* M *is the Poisson manifold if there is an operation* $\{\cdot,\cdot\}$ *giving to the* $C^\infty(M)$ *the structure of the Poisson algebra.*

Let H be a smooth function on a Poisson manifold M. Then dynamical system

$$\dot{x} = \{x, H\}$$

is a Hamiltonian system with the Hamiltonian function H. The function F which is constant along the trajectories of the system is its first integral. For a Hamiltonian system with the Hamiltonian function H, a function F is a first integral if and only if $\{H, F\} = 0$.

Let us remind that for the functions F, H for which $\{H, F\} \equiv 0$ we say that they are *in involution*.

Specially, $\{H, H\} = 0$, so the Hamiltonian function is one first integral for the Hamiltonian system.

The following theorem is fundamental.

Theorem 1 (Liouvill–Arnol'd) *Let* M *be a symplectic manifold and assume* $n = \frac{1}{2}\dim(M)$ *functions in involution* $F_1, \cdots, F_n : M \to \mathbb{R}$ *are given. Denote* $c := (c_1, \cdots, c_n) \in \mathbb{R}^n$ *and* $M_c = \{x \in M \mid F_k(x) = c_k\}$. *If the functions* $F_1, \cdots, F_n$ *are independent on* M_c, *then*

1. M_c *is a smooth manifold, invariant with respect to the Hamiltonian diffeomorphism generated by functions* F_k.

2. If the manifold M_c *is compact and connected, then it is diffeomorphic to a torus* $\mathbb{T}^n = (\mathbb{S}^1)^n$.

3. There exist coordinates $(\varphi_1, \cdots, \varphi_n) \in \mathbb{T}^n$ *such that the Hamiltonian equations with the Hamiltonian* F_1 *have the form*

$$\dot{\varphi}_1 = \varpi_{1,c}, \ldots, \dot{\varphi}_n = \varpi_{n,c}$$

linearizing the flow.

Definition 2 *A Hamiltonian system* (M^{2n}, ω, H) *which has* n *independent first integrals in involution is completely integrable in Liouville sense.*

All Hamiltonian systems with one degree of freedom are obvious examples of completely integrable systems. Starting with two degrees of freedom, the situation is not simple at all any more.

Example 1 *The problem of geodesics on the surfaces of revolution in* $\mathbb{R}^3$ *is completely integrable.*

Example 2 *The problem of geodesics on ellipsoid in E^n is completely integrable, as a consequence of the Jacobi-Shal theorem.*

Completely integrable systems have, according to the Theorem 1, very regular dynamics. However, they are very rare. Although for completely integrable systems there exist action-angle coordinates where those systems could be explicitly integrated, the construction of those coordinates is not explicit. Thus, in the theory of completely integrable systems there exist two basic and usually difficult questions:

- *For a given system to show that it is completely integrable*;
- *For a given completely integrable system to perform explicit integration.*

For the systems given in the first two examples, integration is done by methods of separation of variables of Hamilton-Jacobi equation. After 1967 and discovery of infinite-dimensional completely integrable systems, such as Korteweg – de Vries equation, new techniques of solving such problems were found. These techniques are based on the inverse scattering methods, and some additional analytical, algebraic or algebraically-geometrical theories are used.

2 Rotations of a heavy rigid-body about a fixed point

Let us consider rotations of a rigid body about a fixed point O, under the gravitational field. Motion of the rigid-body is represented in two coordinate systems: *the fixed* $Oxyz$, and *the moving frame* $OXYZ$, which is attached to the body.

Traditionally, vectors in the fixed frame are denoted by small letters, and in the moving frame by capital letters. The vector $\Omega(t) = (p, q, r)$ will denote *angular velocity in the moving frame* and velocity V of a point Q is $V = \Omega \times Q$. Now the kinetic momentum G becomes $G = \int\int\int_\sigma Q \times (\Omega \times Q)dm = J(\Omega)$. where the operator J is symmetric and called inertia tensor of a rigid body.

The operator J defines quadratic form which gives *the ellipsoid of inertia* of the body $(JX, X) = 1$. The ellipsoid describes the mass distribution in the body. Choosing the basis $e = [i, j, k]$ where the operator J is diagonal, we get $[J]_e = I = \text{diag}\,(A, B, C)$. These three numbers A, B, C, the principal momenta of inertia, which describe the mass distribution, together with the coordinates of the mass center $\chi = (x_0, y_0, z_0)$, give complete description of the dynamical properties of the rigid body. (Instead of A, B, C we will also use $I_1,\ I_2,\ I_3$ as a notation for the principal momenta.)

In the same basis the vector of kinetic momentum becomes

$$G = Api + Bqj + Crk.$$

Denote by $\Gamma = (\gamma, \gamma', \gamma'')$ coordinates of the vertical ort in the moving frame. Gravitational force acts in direction of Γ, and assuming $mg = 1$, we get $L = \chi \times \Gamma$, where L is the principal momentum of forces. From the equation $\dot{G} = L$, the first group of the Euler - Poisson equations follow:

$$\dot{M} = M \times \Omega + \chi \times \Gamma, \tag{1}$$

where $M = I\Omega$.

The second group of Euler – Poisson equations follow from the fact that the vector Γ is fixed in the space:

$$\dot{\Gamma} = \Gamma \times \Omega. \tag{2}$$

The equations (1) and (2) are six differential equations of motion on Ω and Γ as functions of time.

2.1 The first integrals of motion. Integrable cases

The Euler – Poisson equations always have three first integrals of motion:

$$F_1 = \frac{1}{2}\langle I\Omega, \Omega\rangle + \langle \Gamma, \chi\rangle \text{ (energy integral)},$$
$$F_2 = \langle \Gamma, \Gamma\rangle (= 1), \quad F_3 = \langle I\Omega, \Gamma\rangle.$$

The Euler case (1751). It is defined by the condition $\chi = 0$. The additional first integral is $F_4 = \langle M, M\rangle$.

The Lagrange case (1788). This case is defined by the conditions $A = B, \quad \chi = (0, 0, z_0)$. So, the ellipsoid of inertia is symmetric, and mass-center is placed on the symmetry axis. Additional first integral, linear in impulses, is $F_4 = M_3$.

The Kowalevska case. It is well known that Kowalevska, in her celebrated 1889 paper [24], starting with a careful analysis of the solutions of the Euler and the Lagrange case of rigid-body motion, formulated a problem *to describe the parameters* (A, B, C, x_0, y_0, z_0), *for which the Euler – Poisson equations have a general solution in the form of uniform functions with only moving poles as singularities.* Here $I = \operatorname{diag}(A, B, C)$ represents the inertia operator, and $\chi = (x_0, y_0, z_0)$ is the center of mass of the rigid body.

Then, in §1 of [24], some necessary conditions were formulated and a new case was discovered, now known as Kowalevska case, as a unique possible beside the cases of Euler and Lagrange:

$$A = B = 2C, \quad \chi = (x_0, 0, 0).$$

Additional first integral found by Kowalevska is of the fourth degree in impulses

$$F_4 = (\Omega_1^2 - \Omega_2^2 + \frac{x_0}{I_3}\Gamma_1)^2 + (2\Omega_1\Omega_2 + \frac{x_0}{I_3}\Gamma_2)^2.$$

However, considering the situation where all momenta of inertia are different, Kowalevska came to the relation analogue to the following (see [Go]):

$$x_0\sqrt{A(B-C)} + y_0\sqrt{B(C-A)} + z_0\sqrt{C(A-B)} = 0.$$

And she concluded that in such a case has to be $x_0 = y_0 = z_0$, giving the Euler case.

But, it was Apel'rot who noticed in the beginning of 1890's, that the last relation admits one more case, not mentioned by Kowalevska:

$$x_0\sqrt{A(B-C)} + z_0\sqrt{C(A-B)} = 0, \quad y_0 = 0,$$

under the assumption $A > B > C$. Such systems were considered also by Hess, even before Apel'rot, in 1890. But such intriguing position corresponding to the Kowalevska paper, made the Hess-Apel'rot systems very attractive for leading Russian mathematicians from the end of XIX century. After few years, Nekhrasov and Lyapunov meaneged to provide new arguments and they demonstrated that the Hess-Apel'rot systems didn't satisfy the condition investigated by Kowalevska, which means that conclusion of §1 of [24] *was correct.*

Few years ago, we constructed a Lax representation for it (see [13]). We provide the Lax representation for all new systems, generalizing the Lax pair from [13]. It appears that new systems belong to the class of *isoholomorphic systems.* This class of systems was introduced and studied in [14], in connection with the Lagrange bitop.

Such systems have specific distribution of zeroes in Lax matrices. Therefor standard integration technics of [15], [1] cannot be applied directly. Its integration requires more detailed analysis of geometry of the Prym varieties and it is based on Mumford's relation on theta - divisors of unramified double coverings.

The L operator, a quadratic polynomial in λ of the form $\lambda^2 C + \lambda M + \Gamma$, in the case $n = 4$ satisfies the condition

$$L_{12} = L_{21} = L_{34} = L_{43} = 0.$$

Such situation, explicitly excluded by Adler-van Moerbeke (see [1], Theorem 1) and implicitly by Dubrovin (see [12], Lemma 5 and Corollary) has been studied for the first time in [14].

Study of the spectral curve and the Baker-Akhiezer function for the four - dimensional Hess-Apel'rot systems shows that, similarly to [14], dynamics of the system is related to certain Prym variety Π. It is connected to the evolution of divisors of some meromorphic differentials Ω^i_j. From the condition on zeroes of the Lax matrix, it follows that differentials

$$\Omega^1_2,\ \Omega^2_1,\ \Omega^3_4,\ \Omega^4_3$$

are *holomorphic* during the whole evolution. Compatibility of this requirement with dynamics is based on Mumford's relation (see [14])

$$\Pi^- \subset \Theta,$$

where Π^- is a translation of the Prym variety Π.

Classical Hess-Apel'rot system. Let $J_1 < J_2 < J_3$ and $\chi = (x_0, y_0, z_0)$. Hess in [23] and Apel'rot in [2] found that if the inertia momenta and the radius vector of center of masses satisfy the conditions

$$y_0 = 0, \quad x_0\sqrt{J_2 - J_1} + z_0\sqrt{J_3 - J_2} = 0 \tag{3}$$

then, the surface

$$F_4 = M_1 x_0 + M_3 z_0 = 0 \tag{4}$$

is invariant. Integration of such system, using classical techniques can be found in [Go]. In [13], an L-A pair for the Hess-Apel'rot system is constructed:

$$\dot{L}(\lambda) = [L(\lambda), A(\lambda)],$$
$$L(\lambda) = \lambda^2 C + \lambda M + \Gamma, \quad A(\lambda) = \lambda\chi + \Omega, \quad C = \frac{J_1 + J_3}{J_1 J_3}\chi,$$

where the skew-symmetric matrices represent the vectors denoted by the same letter. Also, the basic steps in algebro-geometric integration procedure are given.

The Zhukovskii geometric interpretation of the conditions (3) [40, 26] Let us consider the ellipsoid

$$\frac{M_1^2}{J_1} + \frac{M_2^2}{J_2} + \frac{M_3^2}{J_3} = 1,$$

and the plane containing the middle axis and intersecting the ellipsoid through a circle. Denote by l corresponding normal to the plane, which passes through the fixed point O. Then the condition (3) means that the center of masses lies on the line l.

Having this interpretation in mind, we choose the basis of moving frame such that the third axis is l, the second one is directed as the middle axis of ellipsoid, and the first one is chosen according to the orientation of the orthogonal frame. In this basis (see [BM]), particular integral (4) becomes

$$F_4 = M_3 = 0,$$

the matrix J obtains the form:

$$J = \begin{pmatrix} J_1 & 0 & J_{13} \\ 0 & J_1 & 0 \\ J_{13} & 0 & J_3 \end{pmatrix},$$

and $\chi = (0, 0, z_0)$. This will serve us as a motivation for the definition of the four- dimensional Hess-Apel'rot system.

3 The definition of Lagrange bitop and its basic properties

The equations of motion of a heavy n-dimensional rigid body fixed at a point in the moving frame are:

$$\dot{M} = [M, \Omega] + [\Gamma, \chi], \quad \dot{\Gamma} = [\Gamma, \Omega], \tag{5}$$

where the moving frame is such that the matrix I is diagonal in it, $diag(I_1, \ldots, I_n)$. Here $M_{ij} = (I_i + I_j)\Omega_{ij} \in so(n)$ is the kinetic momentum, $\Omega \in so(n)$ is the angular velocity, $\chi \in so(n)$ is a given constant matrix (describing a generalized center of the mass), $\Gamma \in so(n)$. Then $I_i + I_j$ are the principal inertia momenta. These equations are on the semidirect product $so(n) \times so(n)$ and they were introduced in [25].

We are going to consider a four-dimensional case of these equations defined by

$$\begin{array}{l} I_1 = I_2 = a \\ I_3 = I_4 = b \end{array} \quad \text{and} \quad \chi = \begin{pmatrix} 0 & \chi_{12} & 0 & 0 \\ -\chi_{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & \chi_{34} \\ 0 & 0 & -\chi_{34} & 0 \end{pmatrix} \tag{6}$$

with the conditions $a \neq b$, $\chi_{12}, \chi_{34} \neq 0, |\chi_{12}| \neq |\chi_{34}|$.

We will call this system *the Lagrange bitop.*

Proposition 1 [13] *The equations of motion (5) under the conditions (6) have an $L-A$ pair representation $\dot{L}(\lambda) = [L(\lambda), A(\lambda)]$, where*

$$L(\lambda) = \lambda^2 C + \lambda M + \Gamma, \quad A(\lambda) = \lambda\chi + \Omega, \tag{7}$$

and $C = (a+b)\chi$.

One can observe that both leading terms in the operators L and A (matrices C and χ) are skewsymmetric, while in [12, 13, 15] one is always symmetric and another one is skewsymmetric.

Before analysing the spectral properties of the matrices $L(\lambda)$, we will change the coordinates in order to diagonalize the matrix C. In this new basis the matrices $L(\lambda)$ have the form $\tilde{L}(\lambda) = U^{-1}L(\lambda)U$,

$$\tilde{L}(\lambda) = \begin{pmatrix} -i\Delta_{34} & 0 & -\beta_3^* - i\beta_4^* & i\beta_3 - \beta_4 \\ 0 & i\Delta_{34} & -i\beta_3^* - \beta_4^* & -\beta_3 + i\beta_4 \\ \beta_3 - i\beta_4 & -i\beta_3 + \beta_4 & -i\Delta_{12} & 0 \\ i\beta_3^* + \beta_4^* & \beta_3^* + i\beta_4^* & 0 & i\Delta_{12} \end{pmatrix}$$

where $\Delta_{12} = \lambda^2 C_{12} + \lambda M_{12} + \Gamma_{12}$, $\quad \Delta_{34} = \lambda^2 C_{34} + \lambda M_{34} + \Gamma_{34}$, and

$$\begin{array}{ll} \beta_3 = x_3 + \lambda y_3, & x_3 = \frac{1}{2}\left(\Gamma_{13} + i\Gamma_{23}\right), \\ \beta_4 = x_4 + \lambda y_4, & x_4 = \frac{1}{2}\left(\Gamma_{14} + i\Gamma_{24}\right), \\ \beta_3^* = \bar{x}_3 + \lambda \bar{y}_3, & y_3 = \frac{1}{2}\left(M_{13} + iM_{23}\right), \\ \beta_4^* = \bar{x}_4 + \lambda \bar{y}_4, & y_4 = \frac{1}{2}\left(M_{14} + iM_{24}\right). \end{array}$$

The spectral polynomial $p(\lambda,\mu) = det\left(\tilde{L}(\lambda) - \mu \cdot 1\right)$ has the form

$$p(\lambda,\mu) = \mu^4 + P(\lambda)\mu^2 + [Q(\lambda)]^2, \tag{8}$$

where

$$P(\lambda) = \Delta_{12}^2+\Delta_{34}^2+4\beta_3\beta_3^*+4\beta_4\beta_4^*, \quad Q(\lambda) = \Delta_{12}\Delta_{34}+2i(\beta_3^*\beta_4-\beta_3\beta_4^*). \tag{9}$$

We can rewrite it in terms of M_{ij} and Γ_{ij}:

$$P(\lambda) = A\lambda^4 + B\lambda^3 + D\lambda^2 + E\lambda + F, \quad Q(\lambda) = G\lambda^4 + H\lambda^3 + I\lambda^2 + J\lambda + K. \tag{10}$$

Their coefficients

$$\begin{aligned}
A &= C_{12}^2 + C_{34}^2 = \langle C_+, C_+\rangle + \langle C_-, C_-\rangle,\\
B &= 2C_{34}M_{34} + 2C_{12}M_{12} = 2\left(\langle C_+, M_+\rangle + \langle C_-, M_-\rangle\right),\\
D &= M_{13}^2 + M_{14}^2 + M_{23}^2 + M_{12}^2 + M_{34}^2 + 2C_{12}\Gamma_{12} + 2C_{34}\Gamma_{34}\\
&= \langle M_+, M_+\rangle + \langle M_-, M_-\rangle + 2\left(\langle C_+, \Gamma_+\rangle + \langle C_-, \Gamma_-\rangle\right),\\
E &= 2\Gamma_{12}M_{12} + 2\Gamma_{13}M_{13} + 2\Gamma_{14}M_{14} + 2\Gamma_{23}M_{23} + 2\Gamma_{24}M_{24} + 2\Gamma_{34}M_{34}\\
&= 2\left(\langle \Gamma_+, M_+\rangle + \langle \Gamma_-, M_-\rangle\right),\\
F &= \Gamma_{12}^2 + \Gamma_{13}^2 + \Gamma_{14}^2 + \Gamma_{23}^2 + \Gamma_{24}^2 + \Gamma_{34}^2 = \langle \Gamma_+, \Gamma_+\rangle + \langle \Gamma_-, \Gamma_-\rangle,\\
G &= C_{12}C_{34} = \langle C_+, C_-\rangle,\\
H &= C_{34}M_{12} + C_{12}M_{34} = \langle C_+, M_-\rangle + \langle C_-, M_+\rangle,\\
I &= C_{34}\Gamma_{12} + \Gamma_{34}C_{12} + M_{12}M_{34} + M_{23}M_{14} - M_{13}M_{24}\\
&= \langle C_+, \Gamma_-\rangle + \langle C_-, \Gamma_+\rangle + \langle M_+, M_-\rangle,\\
J &= M_{34}\Gamma_{12} + M_{12}\Gamma_{34} + M_{14}\Gamma_{23} + M_{23}\Gamma_{14} - \Gamma_{13}M_{24} - \Gamma_{24}M_{13}\\
&= \langle M_+, \Gamma_-\rangle + \langle M_-, \Gamma_+\rangle,\\
K &= \Gamma_{34}\Gamma_{12} + \Gamma_{23}\Gamma_{14} - \Gamma_{13}\Gamma_{24} = \langle \Gamma_+, \Gamma_-\rangle.
\end{aligned} \tag{11}$$

are integrals of motion of the system (3, 4). We used two vectors $M_+, M_- \in R^3$ which correspond to $M_{ij} \in so(4)$ according to

$$(M_+, M_-) \to \begin{pmatrix} 0 & -M_+^3 & M_+^2 & -M_-^1 \\ M_+^3 & 0 & -M_+^1 & -M_-^2 \\ -M_+^2 & M_+^1 & 0 & -M_-^3 \\ M_-^1 & M_-^2 & M_-^3 & 0 \end{pmatrix} \tag{12}$$

Here M_+^j are the j-th coordinates of the vector M_+. The system (3, 4) is Hamiltonian with the Hamiltonian function

$$\mathcal{H} = \frac{1}{2}(M_{13}\Omega_{13} + M_{14}\Omega_{14} + M_{23}\Omega_{23} + M_{12}\Omega_{12} + M_{34}\Omega_{34}) + \chi_{12}\Gamma_{12} + \chi_{34}\Gamma_{34}$$

The algebra $so(4) \times so(4)$ is 12 dimensional. The general orbits of the coadjoint action are 8 dimensional. According to [25], the Casimir functions are coefficients of $\lambda^0, \lambda, \lambda^4$ in the polynomials $[det\tilde{L}(\lambda)]^{1/2}$ and $-\frac{1}{2}Tr(\tilde{L}(\lambda))^2$.

Since

$$\left[det\tilde{L}(\lambda)\right]^{1/2} = G\lambda^4+H\lambda^3+I\lambda^2+J\lambda+K, \quad -\frac{1}{2}Tr\left(\tilde{L}(\lambda)\right)^2 = A\lambda^4+E\lambda+F,$$

the Casimir functions are J, K, E, F. Nontrivial integrals of motion are B, D, H, I. They are in involution. Nontrivial integrals of motion are B, D, H, I are independent in the case $\chi_{12} \neq \pm\chi_{34}$. When $|\chi_{12}| = |\chi_{34}|$, then $2H = B$ or $2H = -B$ and there are only 3 independent integrals in involution. So we have

Proposition 2 [11] *For $|\chi_{12}| \neq |\chi_{34}|$, the system (3, 4) is completely integrable in the Liouville sense.*

There are two families of integrable Euler-Poisson equations introduced by Ratiu in [25]. *The generalized symmetric case* is defined by the conditions

$$I_1 = \ldots = I_n, \quad \chi \text{ arbitrary;}$$

and *the generalized Lagrange case* which is defined by

$$I_1 = I_2 = a, \quad I_3 = \ldots = I_n = b, \quad \chi_{ij} = 0 \text{ if } (i,j) \notin \{(1,2),(2,1)\}.$$

The system (3, 4) doesn't fall in any of those families and together with them it makes the complete list of systems with the L operator of the form

$$L(\lambda) = \lambda^2 C + \lambda M + \Gamma.$$

Proposition 3 [13] *If $\chi_{12} \neq 0$ then the Euler-Poisson equations (3) could be written in the form (5) (with arbitrary C) if and only if the equations (3) describe the generalized symmetric case, the generalized Lagrange case or the Lagrange bitop, including the case $\chi_{12} = \pm\chi_{34}$.*

One can compare this with [30] (Theorem 15, ch. 53). The proofs of the Propositions 1-3 could be found in [13].

The $L(\lambda)$ matrix is a quadratic polynomial in the spectral parameter λ with matrix coefficients. The general theories describing the isospectral deformations for polynomials with matrix coefficients were developed by Dubrovin [15, 16] in the midle of 70's and by Adler, van Moerbeke [1] few years later. Dubrovin's approach was based on the Baker-Akhiezer function. Both approaches were applied in rigid body problems (see [27, 1] respectively).

But, as it was shown in [14], non of these two theories can be directly applied in cases like this. Necessary modifications were suggested in [14], where the procedure of algebro-geometric integration was presented. It is based on some nontrivial facts from the theory of Prym varieties, such as the Mumford relation on theta divisors of unramified double coverings and the Mumford-Dalalyan theory (see [14, 30, 29, 12, 35, 3]).

Here we are going to follow closely the procedure from [14], with necessary changes, calculations and comments.

As usual, we start with the spectral curve

$$\Gamma : \, det\left(\tilde{L}(\lambda) - \mu \cdot 1\right) = 0.$$

So, we have

$$\Gamma : \, \mu^4 + \mu^2 \left(\Delta_{12}^2 + \Delta_{34}^2 + 4\beta_3\beta_3^* + 4\beta_4\beta_4^*\right) + [\Delta_{12}\Delta_{34} + 2i(\beta_3^*\beta_4 - \beta_3\beta_4^*)]^2 = 0. \tag{13}$$

There is an involution $\sigma : (\lambda, \mu) \to (\lambda, -\mu)$ on the curve Γ, which corresponds to the skew- symmetry of the matrix $L(\lambda)$. Denote the factor-curve by $\Gamma_1 = \Gamma/\sigma$.

Lemma 1 *The curve Γ_1 is a smooth hyperelliptic curve of the genus $g(\Gamma_1) = 3$. The arithmetic genus of the curve Γ is $g_a(\Gamma) = 9$.*

Proof. The curve:

$$\Gamma_1 : \, u^2 + P(\lambda)u + [Q(\lambda)]^2 = 0,$$

is hyperelliptic, and its equation in the canonical forme is:

$$u_1^2 = \frac{[P(\lambda)]^2}{4} - [Q(\lambda)]^2, \tag{14}$$

where $u_1 = u + P(\lambda)/2$. Since $\frac{[P(\lambda)]^2}{4} - [Q(\lambda)]^2$ is a polynomial of the degree 8, the genus of the curve Γ_1 is $g(\Gamma_1) = 3$. The curve Γ is a double covering of Γ_1, and the ramification divisor is of the degree 8. According to the Riemann-Hurwitz formula, the arithmetic genus of Γ is $g_a(\Gamma) = 9$.

Lemma 2 *The spectral curve Γ has four ordinary double points $S_i, i = 1, \ldots, 4$. The genus of its normalization $\tilde{\Gamma}$ is five.*

Lemma 3 *The singular points S_i of the curve Γ are fixed points of the involution σ. The involution σ exchanges the two branches of Γ at S_i.*

Together with the curve Γ_1, one can consider curves $\mathcal{C}_1$ and $\mathcal{C}_2$ defined by the equations

$$\mathcal{C}_1 : v^2 = \frac{P(\lambda)}{2} + Q(\lambda), \quad \mathcal{C}_2 : v^2 = \frac{P(\lambda)}{2} - Q(\lambda). \tag{15}$$

Since the curve Γ_1 is hyperelliptic, in a study of the Prym variety Π the Mumford-Dalalyan theory can be applied (see [28, 24, 10]). Thus, using the previous Lemma, we come to

Theorem 2 *a) The Prymian Π is isomorphic to the product of the curves E_i:*

$$\Pi = Jac(\mathcal{C}_1) \times Jac(\mathcal{C}_2).$$

b) The curve $\tilde{\Gamma}$ is the desingularization of $\Gamma_1 \times_{\mathbb{P}^1} \mathcal{C}_2$ and $\mathcal{C}_1 \times_{\mathbb{P}^1} \Gamma_1$.

c) The canonical polarization divisor Ξ of Π satisfies

$$\Xi = E_1 \times \Theta_2 + \Theta_1 \times E_2, \tag{16}$$

where Θ_i is the theta - divisor of E_i.

3.1 Equallysplitting double hyperelliptic coverings

According to the Mumford - Dalalyan theory (see [10, 24, 28]), double unramifide coverings over a hyperelliptic curve $y^2 = P_{2g+2}(x)$ of genus g are in the correspondence with the divisions of the set of the zeroes of the polynomial P_{2g+2} on two disjoint nonempty subsets with even number of elements. We will consider those coverings which correspond to the divisions on subsets with *equal number of elements* and we can call them *equallysplitting*, since the Prym variety splits then as a sum of two varieties of equal dimension.

Now, let us consider with the fixed operator A from (5) the whole hierarchy of systems defined by the Lax equations

$$\dot{L}_B^{(N)} = [L_B^{(N)}, A], \qquad L_B^{(N)}(\lambda) = \lambda^N B + \lambda^{N-1} M_1 + \ldots + M_N.$$

So $L_B^{(N)}(\lambda)$ is a polynomial in λ of degree $N \geq 2$, and the matrix B is proportional to the matrix χ: $B = d\chi$.

Generalizing the situation from the Section 4, we see that the spectral curve Γ_N is a singular curve of the form

$$p_N(\lambda, \mu) = \mu^4 + P_N(\lambda)\mu^2 + [Q_N(\lambda)]^2 = 0,$$

where the polynomials P_N, Q_N have degree $\deg P_N = \deg Q_N = 2N$. So, its normalization is a double covering over the hyperelliptic curve

$$\mu_1^2 = \frac{P_N^2(\lambda)}{4} - Q_N^2(\lambda)$$

of genus $g_N = 2N-1$. This covering corresponds to the division of the set of zeroes on subsets of zeroes of the polynomials $P_N/2 - Q_N$ and $P_N/2 + Q_N$. This is an equallysplitting covering under the assumption $|\chi_{12}| \neq |\chi_{34}|$ we fixed at the beginning. It is easy to see that all equallysplitted coverings can be realized in such a way. So we have

Theorem 3 *The Lagrange bitop hierarchy realizes all equallysplitting coverings over the hyperelliptic curves of odd genus.*

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