LECTURES ON TWO-DIMENSIONAL NONCOMMUTATIVE GAUGE THEORY

1. Classical Aspects

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These notes comprise the first of two articles devoted to the construction of exact solutions of noncommutative gauge theory in two spacetime dimensions. This first part deals solely with the classical theory on a noncommutative torus. Topics covered include a mathematical introduction to the geometry of the noncommutative torus, the definition, properties and symmetries of noncommutative Yang-Mills theory, and the complete solution of the classical field equations.

1. Introduction

In these lecture notes we will describe how to explicitly construct exact, analytic solutions of noncommutative gauge theory in two dimensions. The analysis is naturally divided into two articles. In this first part the classical field theory will be studied in some detail. This is done in order to provide a pedagogical introduction to the relevant mathematical concepts which are presently at the forefront of the field. As such, most of the analysis in the following consists of material that has been known to mathematicians for well over ten years now. The second part of these lecture notes will analyse the quantum field theory in detail and will deal with more recent results. As we will see there, the quantum theory is semi-classically exact and hence is determined in large part by the classical aspects dealt with here.

1.1. Some Background and Motivation

Yang-Mills theory on the noncommutative torus was first studied in the mathematics literature in the late 1980's at the classical level [1, 2]. It represents one of the few instances in which a physical model has attained popularity among physicists, yet was first introduced and studied by mathematicians. Although field theories on noncommutative spaces have been widely studied in the past in a variety of different contexts, the recent surge of interest in the subject has primarily arisen from the fact that these models naturally arise in string theory [3, 4]. When a D-brane is placed in the background of certain non-vanishing supergravity fields, the dynamics of some low-energy excitations of the open strings which attach to them are described by a noncommutative field theory. As field theories, these models retain much of the non-locality which is present in string theory. It is hoped that noncommutative field theories will reveal many of the general features of string theory but within the simpler setting of quantum field theory. Recent reviews of the subject within this context, along with exhaustive lists of references, may be found in [5]–[7].

Besides being of interest from the point of view of string theory, noncommutative field theories challenge the conventional wisdom of ordinary quantum field theory. They are non-local and contain infinitely many derivative interactions which would appear to lead to non-renormalizability problems in a full quantum theory. There are indications though that these issues are dealt with by the specific structure of higher-derivative interactions introduced by noncommutative geometry. Unfortunately, answering such questions for a noncommutative field theory is complicated by the mixing of low and high momentum modes in loop diagrams which ruins the conventional Wilsonian renormalization scheme that requires a distinct separation of energy scales. This effect is commonly called "UV/IR mixing". It appears to make the renormalization of these theories a complete disaster, but it is not yet well understood if these effects are really artifacts of the way we are treating these models using perturbation theory, or if they also persist at a full non-perturbative level.

The non-locality of noncommutative field theories leads to many interesting phenomena which makes these models interesting in their own right as potentially well-defined, non-local quantum field theories. Furthermore, because of "Morita duality", it has even been suggested that noncommutative gauge theories may shed new light on non-perturbative aspects of *ordinary* Yang-Mills theory. Another exotic feature of the quantum theory is the property that the perturbation series does not reduce smoothly to its commutative version, but rather exhibits poles in the noncommutativity parameter. Again, this "θ-smoothness" behaviour may simply be a perturbative artifact of the theory and can disappear in the full quantum field theory.

These open questions motivate the search for models on noncommutative spaces which are exactly solvable in order that we may observe whether or not these unusual effects persist non-perturbatively. Until very recently, surprisingly little attention has been paid to finding such theories. In these lecture notes we will describe and analyse one such field theory, namely two-dimensional Yang-Mills theory on the noncommutative torus. Various aspects of this model and its analogue on the noncommutative plane have been the subject of recent investigation [8]–[20]. In the course of analysing this model we will exploit some of the beautiful mathematics behind noncommutative geometry and two-dimensional gauge theories. As such, these lecture notes also provide to some extent an introduction to the building blocks of noncommutative geometry. Even though we restrict our discussion to the simplest noncommutative space, this case nevertheless exhibits most of the rich features of noncommutative geometry. This setting also enables us to precisely define and describe the exotic properties of noncommutative gauge theories, and to show how noncommutative geometry may be exploited to explicitly solve the field theory.

1.2. Outline

The outline of material contained in these notes is as follows. In section 2 we summarize all the relevant mathematical details from noncommutative geometry that we shall need. A more general and detailed presentation may be found in [21]. We present a concise introduction to the geometry of the two-dimensional noncommutative torus, the construction of differential structures and bundles over it, and the corresponding notions of connection and curvature. In section 3 we then define classical Yang-Mills theory on the noncommutative torus and describe the rich geometrical structure underpinning the gauge group in this case. We also describe the integrability properties underlying the exact solvability features of this model. In section 4 we explicitly construct all solutions to the classical field equations of noncommutative Yang-Mills theory. Finally, in section 5 we introduce the important notion of Morita equivalence, discuss its physical implications, and show that noncommutative gauge theory is invariant under it.

2. Background from Noncommutative Geometry

This section comprises a quick introduction to the geometry of the noncommutative torus. We shall be fairly mathematical here in order to present the results in a precise and rigorous fashion. Later on this level of formality will become a true asset in arriving at the exact solution of the noncommutative field theory.

2.1. The Noncommutative Torus

We shall begin by introducing the noncommutative torus as an abstract object and then below describe it from a more heuristic level. Let $\mathcal{A}=\mathcal{A}_\theta$ be the noncommutative associative $*$-algebra, with unit $1\!\!1$ and conjugation involution $\dagger$, generated by two elements $\hat{Z}_1$ and $\hat{Z}_2$ which obey the relations

$$\hat{Z}_1\,\hat{Z}_2=\mathrm{e}^{2\pi\,\mathrm{i}\,\theta}\;\hat{Z}_2\,\hat{Z}_1\;,\quad \hat{Z}_i^\dagger=\hat{Z}_i^{-1}\;,\tag{2.1}$$

where the real number $\theta\in(0,1)$ is called the noncommutativity parameter. We shall usually assume that it is irrational-valued. The "smooth" completion of this algebra consists of elements $\hat{f}$ which are formal power series of the form

$$\hat{f}=\sum_{\boldsymbol{m}\in\mathbb{Z}^2}f_{\boldsymbol{m}}\;\mathrm{e}^{\pi\,\mathrm{i}\,\theta\,m_1m_2}\;\hat{Z}_1^{m_1}\,\hat{Z}_2^{m_2}\;,\tag{2.2}$$

where $\boldsymbol{m}=(m_1,m_2)$. To ensure convergence of the expansion we assume that $f_{\boldsymbol{m}}$ are Schwartz sequences (of sufficiently rapid decrease at $m_1,m_2\to\pm\infty$), and the θ dependent phase factor is inserted for convenience to enforce the symmetric ordering of all noncommuting products. The hat notation here is used in order to distinguish these abstract algebraic objects from their corresponding spacetime fields that will be introduced shortly.

2.2. Derivatives

At this abstract level we can define derivatives as linear maps $\hat{\partial}:\mathcal{A}\to\mathcal{A}\otimes\mathcal{L}^*$, where $\mathcal{L}=\mathcal{L}_\phi$ is the Heisenberg Lie algebra with "Planck constant" ϕ. These maps define a Lie algebra homomorphism, $[\hat{\partial}_X,\hat{\partial}_Y]=\hat{\partial}_{[X,Y]}$ $\forall X,Y\in\mathcal{L}$, which in components are defined by the commutation relations

$$\Big[\hat{\partial}_1\,,\hat{\partial}_2\Big]=\mathrm{i}\,\phi\;,\quad\Big[\hat{\partial}_i\,,\,\hat{Z}_j\Big]=2\pi\,\mathrm{i}\;\delta_{ij}\;\hat{Z}_j\;.\tag{2.3}$$

In the first commutator the real number ϕ implicitly multiplies the unit element $1\!\!1$ of the algebra $\mathcal{A}$, which for brevity we will not indicate explicitly. We extend $\hat{\partial}$ to the whole of $\mathcal{A}$ by using the fact that they are linear derivations and employing the Leibnitz rule. The convention that the operators $\hat{\partial}_i$ generate a central extension of the two-dimensional translation group (with central element ϕ) is simply for convenience later on and will have no real bearing on the physical and geometrical objects that we construct. This is because $\hat{\partial}$ acts on $\mathcal{A}$ through infinitesimal automorphisms (i.e. commutators).

2.3. Trace

A trace on the algebra is a linear functional $\mathrm{Tr}\,:\mathcal{A}\to\mathbb{C}$ which is cyclic with respect to its product. Up to normalization there is a unique trace which is positive, $\mathrm{Tr}\,\hat f^\dagger\hat f\geq 0\ \ \forall\hat f\in\mathcal{A}$, and which is compatible with the $*$-conjugation on $\mathcal{A}$ in the sense that

$$\mathrm{Tr}\,\hat f^\dagger=\overline{\mathrm{Tr}\,\hat f}\ . \tag{2.4}$$

It is defined on elements (2.2) by picking out the zero mode in the power series expansion,

$$\mathrm{Tr}\,\hat f=f_{\mathbf{0}}\ . \tag{2.5}$$

This trace obeys the "integration by parts" property

$$\mathrm{Tr}\,\left[\hat\partial_i\,,\,\hat f\right]=0 \tag{2.6}$$

with respect to the derivations constructed in (2.3). This means that Tr is invariant under the infinitesimal action of the (centrally-extended) translation group generated by $\hat\partial$.

2.4. Fields

To make contact with the geometrical notion of a torus $\mathbf{T}^2=\mathbf{S}^1\times\mathbf{S}^1$, we will view $\mathcal{A}$ as a deformation of the algebra $C(\mathbf{T}^2)$ of functions on $\mathbf{T}^2\to\mathbb{C}$. For simplicity we assume that the torus is square and of unit area. Then there is a one-to-one correspondence between elements of the algebra $\mathcal{A}$ of the form (2.2) and functions $f:\mathbf{T}^2\to\mathbb{C}$ with the two-dimensional Fourier series expansions

$$f(x)=\sum_{\boldsymbol{m}\in\mathbb{Z}^2}f_{\boldsymbol{m}}\ \mathrm{e}^{\,2\pi\,\mathrm{i}\,m_ix^i}\ , \tag{2.7}$$

where $x=(x^1,x^2)$, $x^i\in[0,1]$ are local coordinates on $\mathbf{T}^2$. Throughout we shall use the Einstein summation convention for implicitly summing over repeated upper and lower indices. From this correspondence we see that, roughly speaking, the expansion coefficients $f_{\boldsymbol{m}}$ of (2.2) may be thought of as Fourier coefficients, and the generators of $\mathcal{A}$ as the plane wave basis $\hat Z_i=\mathrm{e}^{\,2\pi\,\mathrm{i}\,\hat x^i}$ for the Fourier expansion of functions on the torus. The algebraic relations (2.1) are then obtained by promoting the coordinates of $\mathbf{T}^2$ to Hermitian operators $\hat x^i$ which obey the commutation relations $[\hat x^1,\hat x^2]=\mathrm{i}\,\theta/2\pi$, and hence a noncommutative space appears.

More precisely, when $\theta=0$, starting from the commutative algebra $C(\mathbf{T}^2)$ (with the usual pointwise multiplication of functions) it is possible to entirely reconstruct the torus as a topological space. In fact, the celebrated Gel'fand-Naimark theorem asserts that there is a one-to-one correspondence between the category of commutative C^*-algebras and the category of Hausdorff topological spaces. Thus commutative algebras correspond to ordinary geometrical spaces, which always admit an equivalent "dual" description in terms of their algebras of functions (i.e. once every function on a space is known, then so is the space itself). When $\theta \neq 0$, the algebra $\mathcal{A}$ is noncommutative, and the correspondence ceases to exist. The notion of "space" formally disappears in this case, but we may still keep the underlying torus through the one-to-one correspondence described above. Now, however, the noncommutativity of $\mathcal{A}$ is manifested through the property that, for any two functions f and g on $\mathbf{T}^2$, the function corresponding to the element of $\mathcal{A}$ obtained by multiplying the two associated operators is given through

$$\hat{f}\,\hat{g} = \widehat{f \star g}\ , \tag{2.8}$$

and the commutative pointwise multiplication of functions is replaced by the noncommutative star-product. There are different ways to define the star-product by using the Fourier series (2.7) and (2.2) [7], but the one we shall primarily refer back to is the integral kernel representation

$$(f \star g)(x) = \frac{4}{\theta^2} \iint \mathrm{d}^2y\ \mathrm{d}^2z\ f(x+y)\,g(x+z)\ \mathrm{e}^{\,4\pi\,\mathrm{i}\,y\wedge z/\theta}\ , \tag{2.9}$$

where we have introduced the two-dimensional cross-product

$$y \wedge z = y^1 z^2 - y^2 z^1 \tag{2.10}$$

and the integrations extend over $\mathbf{T}^2$. The star-product is associative and noncommutative, and reduces to ordinary pointwise multiplication when $\theta = 0$. The presence of the phase oscillations in (2.9) illustrates the true non-local nature of the result of multiplying together two fields with this product.

This is the picture that is usually employed in field theoretical considerations, because in this manner one can treat fields on noncommutative spaces as fields on ordinary spaces but multiplied together using the non-local star-product. Furthermore, with respect to this correspondence the linear derivations introduced in (2.3) induce ordinary derivatives through

$$\left[\hat{\partial}_i\,,\hat{f}\right] = \widehat{\partial_i f} \tag{2.11}$$

with $\partial_i \equiv \partial/\partial x^i$, while the trace defined in (2.5) corresponds to integration over $\mathbf{T}^2$,

$$\mathrm{Tr}\,\hat{f} = \int \mathrm{d}^2x\ f(x)\ , \tag{2.12}$$

since by periodicity averages over $\mathbf{T}^2$ also pick out zero modes of functions. The cyclic symmetry of the trace manifests itself in the identity

$$\mathrm{Tr}\,\hat{f}\hat{g} \equiv \int \mathrm{d}^2x\ (f \star g)(x) = \int \mathrm{d}^2x\ f(x)\,g(x) \tag{2.13}$$

which follows from an integration by parts (c.f. (2.6)). The property (2.13) also ensures that free field actions are unaltered by noncommutativity, which thereby only appears through the addition of interaction terms. While we shall have occasion to refer back to this picture for illustrative purposes, for the most part our analysis will be done in the abstract setting without recourse to considerations of fields on $\mathbf{T}^2$ and star-products. This will prove to be the most natural setting for solving noncommutative gauge theory in two-dimensions. We shall therefore drop the hat label distinction between elements of $\mathcal{A}$ and fields for notational simplicity in the following, the distinction always being clear from the context in which we will be working.

2.5. Projective Modules

In order to define gauge theories on the noncommutative torus we shall need the appropriate notion of a complex vector bundle over $\mathcal{A}$. Within the context described above, this is readily accomplished by looking at the representations of the algebra $\mathcal{A}$, which from an algebraic point of view is where the real interesting structures always lie. The reason for this characterization follows from a classic result in bundle theory known as the Serre-Swan theorem. Just as spaces are in a one-to-one correspondence with commutative algebras (of functions), so too are vector bundles in a one-to-one correspondence with finitely-generated projective modules over the algebras. For instance, the correspondence in the case of the ordinary torus associates to each vector bundle $E \to \mathbf{T}^2$ the module of sections $C(\mathbf{T}^2, E) \to C(\mathbf{T}^2)$ of the bundle. The space of sections clearly forms a representation of the algebra of functions, because given any section $s : \mathbf{T}^2 \to E$ the map $f \cdot s$ is also a section for any function $f : \mathbf{T}^2 \to \mathbb{C}$, i.e. the algebra $C(\mathbf{T}^2)$ acts on $C(\mathbf{T}^2, E)$.

With this algebraic characterization, we can define a vector bundle in the noncommutative case as a finitely-generated, projective module $\mathcal{E} \to \mathcal{A}$ over the algebra. For definiteness we always assume that $\mathcal{A}$ acts on $\mathcal{E}$ from the right (in the commutative case there is no distinction between left and right actions). The stated restrictions on the module mean that it is of the form $\mathcal{E} = \mathsf{P}\,\mathcal{A}^n$, where $\mathcal{A}^n = \mathcal{A} \oplus \cdots \oplus \mathcal{A}$ (n times) is the free module of rank n over $\mathcal{A}$, while $\mathsf{P} \in \mathbb{M}_n(\mathcal{A}) = \mathcal{A} \otimes \mathbb{M}_n$ (with $\mathbb{M}_n$ the algebra of $n \times n$ complex matrices) is a Hermitian projector on the algebra with $\mathsf{P}^2 = \mathsf{P} = \mathsf{P}^\dagger$. This condition ensures that $\mathcal{E}$ can be embedded as a summand in a trivial module, as then $\mathcal{A}^n = \mathcal{E} \oplus (\mathbb{1} - \mathsf{P})\mathcal{A}^n$ with $(\mathbb{1} - \mathsf{P})\mathcal{A}^n$ projective. This is the noncommutative version of the requirement spelled out in Swan's theorem, which asserts that any complex vector bundle $E \to \mathbf{T}^2$ can be embedded as a Whitney summand in a trivial bundle $\mathbf{T}^2 \times \mathbb{C}^n$ over the torus.

The connected components of the infinite-dimensional Grassmannian manifold of Hermitian projectors on $\mathcal{A}$ is parametrized by its K-theory [22]

$$\mathrm{K}_0(\mathcal{A}) = \mathbb{Z} \oplus \mathbb{Z} , \tag{2.14}$$

defined as the group of equivalence classes of projectors modulo stable isomorphism. The canonical trace defined in section 2.3 yields an isomorphism of ordered groups $\mathrm{K}_0(\mathcal{A}) \to \mathbb{Z}+\mathbb{Z}\,\theta \subset \mathbb{R}$ through $\mathrm{Tr} \otimes \mathrm{tr}_{\,n}\,\mathsf{P} = p - q\,\theta$, with $\mathrm{tr}_{\,n}$ the usual $n \times n$ matrix trace and $\mathsf{P} = \mathsf{P}_{p,q}$ the projector corresponding to the K-theory class labelled by $(p,q) \in \mathbb{Z}^2$. Since P acts as the identity on the module $\mathcal{E} = \mathsf{P}\,\mathcal{A}^n$, by positivity of the trace its dimension obeys

$$\dim \mathcal{E} = \ \mathrm{Tr} \ \otimes \ \mathrm{tr}_{\,n}\,\mathsf{P} = \ \mathrm{Tr} \ \otimes \ \mathrm{tr}_{\,n}\,\mathsf{P}^\dagger\mathsf{P} \geq 0 \ . \tag{2.15}$$

It follows that the stable projective modules are classified by the positive cone of the ordered K-theory group (2.14). As the module $\mathcal{E}$ will be taken to be a separable Hilbert space, its dimension (2.15) must be suitably defined in a regulated fashion. It is well-known how to do this in functional analysis, and using this it follows that to every pair of integers (p,q) we can associate a Heisenberg module $\mathcal{E} = \mathcal{E}_{p,q}$ of positive Murray-von Neumann dimension [23]

$$\dim \mathcal{E}_{p,q} = p - q\,\theta > 0 \ . \tag{2.16}$$

All finitely-generated projective modules over the algebra $\mathcal{A}$ are either free modules $\mathcal{A}^N$, Heisenberg modules, or combinations thereof [24]. The explicit representation of $\mathcal{A}$ furnished by the Heisenberg modules will be described later on. To understand geometrically the abstraction just presented, one should simply repeat the above analysis in the more familiar case $\theta = 0$. Then (2.14) coincides with the topological K-theory of the ordinary torus $\mathbf{T}^2$ (K-theory groups are stable under deformations of algebras), whose positive cone classifies complex vector bundles over $\mathbf{T}^2$ of rank p and magnetic charge $q \in \mathbb{Z}$. While the interpretation of p as rank generally ceases to hold when $\theta \neq 0$, the integer q still corresponds to the Chern character of the "gauge bundle" over $\mathcal{A}$ [25].

2.6. Gauge Fields

To introduce gauge fields in analogy with ordinary geometry, we need to specify the appropriate notion of connection on a module $\mathcal{E} \to \mathcal{A}$. Given the way we defined derivatives in section 2.2, we define a connection ∇ to be a "representation" of the linear derivations ∂ corresponding to the module $\mathcal{E}$, or more precisely as a vector space homomorphism $\nabla : \mathcal{E} \to \mathcal{E} \otimes \mathcal{L}^*$, where here and in the following the direct product is implicitly understood to be $\mathcal{A}$-linear. Its components satisfy the commutation relations

$$[\nabla_i, Z_j] = 2\pi\,\mathrm{i}\ \delta_{ij}\ Z_j \ , \tag{2.17}$$

which when represented on $\mathcal{E}$ corresponds to the usual Leibnitz rule. From (2.3) and (2.17) it follows that $\nabla - \partial$ commutes with all elements of $\mathcal{A}$, and as a consequence we may write the connection as a covariant derivative

$$\nabla = \partial + A \ . \tag{2.18}$$

More precisely, the derivations in (2.18) should be written as $\mathsf{P}\,\partial\,\mathsf{P}$ with ∂ extended to $\mathcal{A}^n$ in the obvious way. The gauge fields $A_i \in \mathrm{End}(\mathcal{E}) = \mathcal{E}^* \otimes \mathcal{E}$ are elements of the algebra of $\mathcal{A}$-linear endomorphisms of the module $\mathcal{E}$ (again with the $\mathcal{A}$-linearity not indicated explicitly in the notation for brevity), which is the commutant of the algebra $\mathcal{A}$ in $\mathcal{E}$ consisting of those elements of the algebra representation which commute with all elements of $\mathcal{A}$.

As in the commutative case, we will work only with compatible connections. The compatibility condition may be stated by introducing an $\mathcal{A}$-valued inner product on $\mathcal{E}$ which is compatible with its $\mathcal{A}$-module structure [21]. We will not write this condition explicitly, as it will not be required in the following. All we shall need to know is that compatibility implies that the curvature of the connection ∇ commutes with all elements of $\mathcal{A}$, and therefore that $[\nabla_1, \nabla_2] \in \mathrm{End}(\mathcal{E})$. From a physical standpoint, it is this restriction that allows the construction of gauge field theory actions from the algebraic objects above. From this curvature one defines the noncommutative field strength F_A of A through

$$[\nabla_1, \nabla_2] = [\partial_1, A_2] - [\partial_2, A_1] + [A_1, A_2] + \phi \equiv F_A + \phi \ . \tag{2.19}$$

The space of compatible connections on an $\mathcal{A}$-module $\mathcal{E}$ will be denoted $\mathcal{C} = \mathcal{C}(\mathcal{E})$.

2.7. Differential Forms

Starting from the commutant of the algebra $\mathcal{A}$ in a given module $\mathcal{E}$, and the exterior products of the dual Heisenberg Lie algebra, we can define a graded differential algebra

$$\Omega(\mathcal{E}) = \bigoplus_{n\geq 0} \Omega^n(\mathcal{E}) \ , \quad \Omega^n(\mathcal{E}) = \mathrm{End}(\mathcal{E}) \otimes \bigwedge^n \mathcal{L}^* \ . \tag{2.20}$$

Elements of (2.20) are (left) translationally-invariant differential forms on the Lie group $\exp \mathcal{L}$ with coefficients in $\mathrm{End}(\mathcal{E})$. For example, the curvature form $[\nabla, \nabla] \in \Omega^2(\mathcal{E})$. The space of compatible connections $\mathcal{C}$ is an affine space over the vector space $\mathrm{End}(\mathcal{E}) \otimes \mathcal{L}^*$ of linear maps $\mathcal{L} \to \mathrm{End}(\mathcal{E})$, whose tangent space may thereby be identified with

$$\Omega^1(\mathcal{E}) = \mathrm{End}(\mathcal{E}) \otimes \bigwedge^1 \mathcal{L}^* \ . \tag{2.21}$$

Given this notion of cotangent vector, we may then define functional differentiation of any functional $f[A]$ at a point $\nabla = \partial + A \in \mathcal{C}$ through

$$\frac{\delta}{\delta A} f[a] = \frac{\partial}{\partial t} f[A + t\,a] \bigg|_{t=0} \ , \quad a \in \Omega^1(\mathcal{E}) \ . \tag{2.22}$$

2.8. Constant Curvature Connections

As in the case of vector bundles over the torus $\mathbf{T}^2$, in the noncommutative case every Heisenberg module $\mathcal{E} = \mathcal{E}_{p,q}$ admits a constant curvature connection $\nabla^c \in \mathcal{C}$, for which

$$[\nabla^c_1\,,\,\nabla^c_2] = \mathrm{i}\,f \tag{2.23}$$

with f a real constant. At the algebraic level, by constant we mean that the curvature (2.23) is proportional to identity operator on $\mathcal{E} \to \mathcal{E}$, while at the level of fields we mean that it is independent of the coordinates of $\mathbf{T}^2$. The constant curvature connections are the natural extensions of the derivations defined by (2.3). They allow us to explicitly construct the corresponding representation Hilbert spaces.

For $q = 0$ we define $\mathcal{E}$ to be the (L^2-completion of the) free module of rank p over $C(\mathbf{T}^2)$,

$$\mathcal{E}_{p,0} = L^2\left(\mathbf{T}^2\right) \otimes \mathbb{C}^p \ , \tag{2.24}$$

with the action of $\mathcal{A}$ defined as multiplication with the star-product. For $q \neq 0$ we define

$$\mathcal{E}_{p,q} = L^2(\mathbb{R}) \otimes \mathbb{C}^q \ , \quad q \neq 0 \ . \tag{2.25}$$

The Hilbert space $L^2(\mathbb{R})$ is the Schrödinger representation of the Heisenberg commutation relations (2.23), which by the Stone-von Neumann theorem is the unique irreducible representation of the Lie algebra $\mathcal{L}_f$. The finite dimensional vector space $\mathbb{C}^q$ carries the $q \times q$ representation of the Weyl-'t Hooft algebra

$$\Gamma_1\,\Gamma_2 = \mathrm{e}^{\,2\pi\,\mathrm{i}\,p/q}\;\Gamma_2\,\Gamma_1 \tag{2.26}$$

defined by

$$\Gamma_1 = \left(W_q^\dagger\right)^{2p} \ , \quad \Gamma_2 = \left(V_q\right)^p \ . \tag{2.27}$$

The traceless $SU(q)$ shift and clock matrices

$$V_q = \begin{pmatrix} 0 & 1 & & & 0 \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ 1 & & & & 0 \end{pmatrix} ,$$

$$W_q = \begin{pmatrix} 1 & & & & 0 \\ & \mathrm{e}^{\,2\pi\,\mathrm{i}\,/q} & & & \\ & & \mathrm{e}^{\,4\pi\,\mathrm{i}\,/q} & & \\ & & & \ddots & \\ 0 & & & & \mathrm{e}^{\,2\pi\,\mathrm{i}\,(q-1)/q} \end{pmatrix} \tag{2.28}$$

obey the commutation relation

$$V_q\,W_q = \mathrm{e}^{\,2\pi\,\mathrm{i}\,/q}\;W_q\,V_q \ . \tag{2.29}$$

We may then represent the generators of the noncommutative torus on (2.25) as

$$Z_i = \mathrm{e}^{\,2\pi\,\nabla_i^{\mathrm{c}}/f}\otimes\Gamma_i \ . \tag{2.30}$$

By working out the commutation relation of the operators (2.30) using (2.23) and (2.26), and comparing with (2.1), we arrive at an expression for θ as a function of f, p and q. This determines the constant curvature (2.23) of the given Heisenberg module $\mathcal{E}_{p,q}$ in terms of its topological numbers and the noncommutativity parameter as

$$f = \frac{2\pi\,q}{p-q\,\theta} \ . \tag{2.31}$$

As we did in section 2.4, it is possible to alternatively construct the Heisenberg modules in terms of fields on the ordinary torus $\mathbf{T}^2$ [5, 7]. This gives a description of $\mathcal{E}_{p,q}$ as a deformation of the $C(\mathbf{T}^2)$-module of sections of a complex vector bundle $E_{p,q}\to\mathbf{T}^2$ of rank p, topological charge q, and constant curvature $2\pi\,q/p$. However, in what follows we shall find it more convenient to work in the Hilbert space picture above.

Up to $SU(q)$ equivalence, the Weyl-'t Hooft algebra (2.26) is known to possess a unique irreducible representation of dimension $q/\mathrm{gcd}(p,q)$ [26, 27]. It follows that the Heisenberg module (2.25) decomposes into irreducible $\mathcal{A}$-modules as

$$\mathcal{E}_{p,q} \;=\; \underbrace{\mathcal{E}'_{p,q}\oplus\cdots\oplus\mathcal{E}'_{p,q}}_{N} \ , \qquad \mathcal{E}'_{p,q} \;=\; L^2(\mathbb{R})\otimes\mathbb{C}^{q/\mathrm{gcd}\,(p,q)} \ . \tag{2.32}$$

This can be used to define the rank of the module $\mathcal{E}_{p,q}$ as

$$N = \mathrm{gcd}(p,q) \ . \tag{2.33}$$

By an explicit construction [1] it is possible to show that the commutant of the algebra $\mathcal{A}=\mathcal{A}_\theta$ in the irreducible Heisenberg module is another noncommutative torus $\mathrm{End}(\mathcal{E}'_{p,q})\cong\mathcal{A}_{\theta'}$, where

$$\theta' = \frac{n-s\,\theta}{p-q\,\theta}\,N \tag{2.34}$$

is a *dual* noncommutativity parameter, with the integers $n,s\in\mathbb{Z}$ satisfying the Diophantine property

$$p\,s-q\,n = N \ . \tag{2.35}$$

We shall describe this duality in more detail in section 5. It follows that the endomorphism algebra of (2.32) is given by

$$\mathrm{End}\,(\mathcal{E}_{p,q}) = \mathbb{M}_N\,(\mathcal{A}_{\theta'}) \ . \tag{2.36}$$

This fact will enable us to identify gauge fields of the $\mathcal{A}_\theta$-module as ordinary gauge fields on $\mathbf{T}^2$ multiplied together with a star-product defined by the dual noncommutativity parameter (2.34). Furthermore, the canonical trace Tr on $\mathcal{A}$ thereby induces a trace $\mathrm{Tr} \otimes \mathrm{tr}_N$ on (2.36). With a slight abuse of notation, we shall also denote this trace by Tr as it will be contextually clear which one we mean.

3. Gauge Theory on the Noncommutative Torus

Having dispensed with the mathematical formalities, we will now define classical Yang-Mills theory on the noncommutative torus. We will argue in this section, at the classical level, that this noncommutative field theory is an exactly solvable model. We fix a Heisenberg module $\mathcal{E} = \mathcal{E}_{p,q}$ over the algebra $\mathcal{A}$ and consider the affine space $\mathcal{C}$ of all compatible connections on $\mathcal{E}$. In string theory, this field theory describes the low-energy dynamics of a bound state configuration of p coincident D2-branes which carry q units of D0-brane vortex charge, in a background B-field [4].

3.1. Definition

For any $\nabla = \partial + A \in \mathcal{C}$, we define the noncommutative Yang-Mills action

$$S[A] = S[\nabla] = \frac{1}{2g^2}\ \mathrm{Tr}\,[\nabla_1, \nabla_2]^2 \ , \tag{3.1}$$

where g is the (dimensionless) Yang-Mills coupling constant. We will regard (3.1) as a functional $S : \mathcal{C} \to \mathbb{R}_+$. By using the operator-field correspondence described in section 2.4, the formula (2.19) and the discussion at the end of section 2.8, we may write (3.1) in a more conventional field theoretic form as

$$S[A] = \frac{1}{2g^2} \int \mathrm{d}^2x\ \ \mathrm{tr}_N\Big(F_A(x) + \phi\Big)^2 \ , \tag{3.2}$$

where

$$F_A = \partial_1 A_2 - \partial_2 A_1 + A_1 \star A_2 - A_2 \star A_1 \tag{3.3}$$

is the noncommutative field strength of an anti-Hermitian $U(N)$ gauge field $A_i(x)$ on the torus $\mathbf{T}^2$. The star-product in (3.3) is the tensor product of the associative star-product corresponding to the dual noncommutativity parameter (2.34) with ordinary matrix multiplication. This extended star-product is still associative.

Due to the isomorphism (2.36), in the case of gauge theory on a Heisenberg module the action may be represented as in (3.2) in terms of star-products of fields on $\mathbf{T}^2$. This is *not* possible on a generic projective module over the noncommutative torus, in which case one can generally only use the more abstract definition (3.1) of the field theory. Note that the action (3.2) is quadratic in the field strength and so the non-local interactions due to the star-product appear explicitly only in (3.3). This is one of the properties that will enable an exact solution of the field theory later on. In (3.2) we also see that the central extension in (2.3) has the physical interpretation of a constant background magnetic flux in the noncommutative field theory, and thereby plays no role in the local dynamics.

3.2. Gauge Symmetry

The action (3.1) is invariant under any covariant transformation of the form

$$\nabla \longmapsto U \, \nabla \, U^\dagger \ , \tag{3.4}$$

where $U \in \mathrm{End}(\mathcal{E})$ is a unitary operator on $\mathcal{E}$,

$$U^\dagger \, U = U \, U^\dagger = \mathbb{1} \ . \tag{3.5}$$

The set of transformations (3.4) form the gauge group $\mathcal{G} = \mathcal{G}(\mathcal{E})$ of the noncommutative gauge theory. Infinitesimally, it follows from (3.4) that the gauge group acts on gauge potentials through

$$A \longmapsto A + \delta_\lambda A \ , \quad \delta_\lambda A = [\nabla, \lambda] \ , \tag{3.6}$$

where $\lambda \in \mathrm{End}(\mathcal{E})$ is anti-Hermitian. The commutator of two such gauge transformations gives

$$[\delta_\lambda \, , \, \delta_{\lambda'}] \, A = \delta_{[\lambda \, , \, \lambda']} A \ , \tag{3.7}$$

and as a consequence the gauge group $\mathcal{G}$ acts on $\mathcal{C}$. Furthermore, the curvature transforms covariantly under the action of $\mathcal{G}$ as

$$\delta_\lambda F_A = [\lambda \, , \, F_A] \ . \tag{3.8}$$

In fact, any element of the differential algebra (2.20) obeys this homogeneous transformation law,

$$\delta_\lambda a = [\lambda, a] \quad \forall a \in \Omega^n(\mathcal{E}), n \geq 0 \ , \tag{3.9}$$

because of the lifting of the space of connections $\mathcal{C}$ to the vector spaces of differential forms through the endomorphism algebra $\mathrm{End}(\mathcal{E})$.

Under the operator-field correspondence, we can write (3.6) in terms of functions on the torus $\mathbf{T}^2$ as

$$\delta_\lambda A_i = \partial_i \lambda + \lambda \star A_i - A_i \star \lambda \ , \tag{3.10}$$

where $\lambda(x)$ is a smooth, anti-Hermitian $N \times N$ matrix-valued field on $\mathbf{T}^2$. From (3.10) it follows that noncommutative gauge transformations mix colour and spacetime degrees of freedom together in a non-trivial way. This is a simple way to see the fact that there is no notion of a structure group in noncommutative gauge theory, only the gauge group $\mathcal{G}$. In the commutative case, the gauge group would be the direct product of the finite-dimensional structure group with the group of $\mathbf{S}^1$-valued functions, and the separation of spacetime and internal gauge symmetries would be clear. This is not so in the noncommutative case, and we are forced to deal with spacetime and colour symmetries on the same level.

This fact suggests that certain transformations (3.10) could induce spacetime transformations which manifest themselves as internal gauge symmetries of the noncommutative field theory. In fact, noncommutative gauge transformations have a deep geometrical interpretation [28]. The gauge parameters λ live in a Lie algebra that generates the gauge group. A basis for the spacetime part of this Lie algebra is provided by the generators Z_i' of the algebra $\mathcal{A}_{\theta'}$, with the λ's given as appropriate anti-Hermitian combinations. Motivated by the Fourier series expansion (2.2), we introduce the basis of operators $T_{\boldsymbol{n}}$, $\boldsymbol{n} = (n_1, n_2) \in \mathbb{Z}^2$ defined by

$$T_{\boldsymbol{n}} = \frac{1}{\theta'}\ \mathrm{e}^{\,\mathrm{i}\, n_1 n_2 \theta'/2}\ Z_1'^{\,n_1}\, Z_2'^{\,n_2}\ , \tag{3.11}$$

which satisfy the conjugation relation $T_{\boldsymbol{n}}^\dagger = T_{-\boldsymbol{n}}$. The Fourier coefficients of a gauge parameter λ thereby obey $\lambda_{-\boldsymbol{n}} = -\,\overline{\lambda_{\boldsymbol{n}}}$. By using (2.1) it is straightforward to compute that the elements (3.11) close the infinite-dimensional Lie algebra

$$\left[T_{\boldsymbol{n}}\,,\,T_{\boldsymbol{m}}\right] = \frac{2\,\mathrm{i}}{\theta'}\ \sin\left(\theta'\,\boldsymbol{n}\wedge\boldsymbol{m}\right)\ T_{\boldsymbol{n}+\boldsymbol{m}}\ , \tag{3.12}$$

where the two-dimensional cross product of vectors in $\mathbb{Z}^2$ is defined as in (2.10). The commutation relations (3.12) are familiar from condensed matter physics, where they correspond to the algebra of magnetic translation operators for electrons moving in two-dimensions under the influence of a constant perpendicularly applied magnetic field, projected onto the lowest Landau level [29]. This coincidence is not an accident, because noncommutative gauge theory arises most naturally as the low-energy effective field theory of open string modes on a D-brane, whose worldvolume is subjected to a constant background magnetic field $B \sim \theta'^{\,-1}$ [4]. The geometrical significance of this algebra may be seen by working close to the commutative limit $\theta' \to 0$ of the noncommutative gauge theory. With the gauge generators (3.11) denoted $T_{\boldsymbol{n}}^0$ in this limit, (3.12) reduces to

$$\left[T_{\boldsymbol{n}}^0\,,\,T_{\boldsymbol{m}}^0\right] = 2\,\mathrm{i}\,\boldsymbol{n}\wedge\boldsymbol{m}\ T_{\boldsymbol{n}+\boldsymbol{m}}^0\ . \tag{3.13}$$

This Lie algebra is recognized as the classical W_∞ algebra of area-preserving diffeomorphisms of $\mathbf{T}^2$.

We conclude that the Lie algebra (3.12) of noncommutative gauge transformations is a trigonometric deformation of the algebra $w_\infty(\mathbf{T}^2)$ of area-preserving diffeomorphisms of the torus in two-dimensions [28, 30]. This leads to the first indication that noncommutative Yang-Mills theory in two spacetime dimensions is an exactly solvable model. While the area-preserving transformations do not exhaust all two-dimensional diffeomorphisms, they kill enough of the local degrees of freedom to eliminate all propagating modes, and there are no gluons in this gauge theory. In other words, noncommutative gauge theory in two dimensions is "almost" a topological field theory, because its gauge symmetry "almost" includes general covariance. While a similar argumentation also follows through in the case of ordinary Yang-Mills theory on $\mathbf{T}^2$ [31], the situation here is more drastic. While in the commutative case the area-preserving symmetry is manifested through the geometrical invariance properties of the integrated two-form field strength and as such defines an outer automorphism of the algebra of functions $C(\mathbf{T}^2)$, here the symmetry manifests itself as an inner automorphism of the algebra $\mathcal{A}_{\theta'}$.

This argument also illustrates the crucial difference between two dimensions and higher spacetime dimensions. While the analogous argument in d even dimensions would lead one to the conclusion that the gauge symmetries contain symplectic diffeomorphisms [28], for $d>2$ the algebra of symplectomorphisms is much smaller than that of volume-preserving diffeomorphisms, and only for $d=2$ do the canonical transformations coincide with area-perserving diffeomorphisms. These same conclusions can also be reached by noting that locally one can fix an axial gauge to completely eliminate the star-product interaction terms from the action (3.2), and naively one would conclude that theory is just a trivial, free field Gaussian model. Insofar as the computation of vacuum amplitudes is concerned, this would certainly be true on the plane $\mathbb{R}^2$, but on the torus such a gauge choice is incompatible with topologically non-trivial gauge transformations which wind around its cycles. Because of this fact though, the characteristics of the theory will depend only on topological quantities associated with global degrees of freedom of the gauge theory. We will see this explicitly throughout these notes.

We can give the noncommutative gauge group another interpretation which is somewhat more algebraic in nature. Suppose that we approximate the noncommutativity parameter by a sequence of rational numbers m/n, with $m,n\to\infty$ relatively prime positive integers such that $\theta'=\lim_{m,n} m/n$ is finite. Upon substituting $\theta'=m/n$ into (3.12) we observe that this Lie algebra has a finite-dimensional, $n\times n$ unitary representation in terms of clock and shift matrices (c.f. (2.26)–(2.29)). These operators span the Lie algebra of traceless Hermitian $n\times n$ matrices and form the Fairlie-Fletcher-Zachos trigonometric basis for $su(n)$ [32]. The limit $n\to\infty$ thereby identifies the gauge group of noncommutative Yang-Mills theory as a certain completion of the infinite unitary group $U(\infty)$ [28]. Physically, this infinite-dimensional symmetry arises from the infinitely-many image D-branes associated with open string modes terminating on a toroidal worldvolume in a B-field [33]–[37]. The rigorous definition of the $n \to \infty$ limit may be found in [38]. We shall see how this result can be derived in another way in the second part of these lecture notes. We remark that this conclusion is only valid locally on $\mathbf{T}^2$, because the torus W_∞ algebra requires a central extension in order to make contact with its group theoretical description, and the torus unitary group has only a semi-infinite Dynkin diagram [28].

In the present case, the completed unitary symmetry group $\overline{U(\infty)}$ consists of those gauge transformations (3.4) whereby $U = 1\!\!1 + K$, with K an element of the algebra of compact endomorphisms of the module $\mathcal{E}$ which may be defined as the operator norm closure of the algebra of finite-rank endomorphisms [16],[39]–[41]. The latter algebra forms a self-adjoint two-sided ideal in $\mathrm{End}(\mathcal{E})$ which, since $\mathcal{E}$ is separable, is isomorphic to the infinite matrix algebra $\mathbb{M}_\infty$. What is particularly interesting in this respect is that, by Palais' theorem [42], the completed group $\overline{U(\infty)}$ has the same homotopy type as the finite rank group $U(\infty)$, all of whose homotopy groups are known. We can thereby write down explicitly the homotopy groups of the noncommutative gauge group as

$$\pi_n\Big(\,\overline{U(\infty)}\,\Big) = \pi_n\Big(U(\infty)\Big) = \begin{cases} \mathbb{Z} & , \quad n \ \text{ odd} \\ 0 & , \quad n \ \text{ even} \end{cases} . \tag{3.14}$$

This enables one to classify objects in the noncommutative gauge theory which have a direct topological characterization, such as anomalies and solitonic configurations [40].

3.3. Integrability

The fact that noncommutative gauge theory in two dimensions is an exactly solvable model is most likely to be connected with the fact that it is an integrable system. We will now indicate how this may indeed be the case by showing that Yang-Mills theory on the projective module $\mathcal{E}$ naturally defines an infinite-dimensional Hamiltonian system [16]. On the affine space $\mathcal{C}(\mathcal{E})$, with tangent space identified as (2.21), we can define a natural symplectic structure by the two-form

$$\omega[a,b] = \ \mathrm{Tr}\, a \wedge b \ , \quad a,b \in \Omega^1(\mathcal{E}) \ . \tag{3.15}$$

This form is clearly non-degenerate. It is also independent of the point $\nabla = \partial + A \in \mathcal{C}$ at which it is evaluated, and hence it is closed: $\delta\omega/\delta A = 0$. The important feature of the symplectic form (3.15) is that it is gauge invariant. Using the transformation rule (3.9) and cyclicity of the trace, we easily find

$$\omega\left[\delta_\lambda a\,,\ \delta_\lambda b\right] = \omega[a,b] \ . \tag{3.16}$$

It follows that the gauge group $\mathcal{G}$ acts symplectically on the affine space $\mathcal{C}$. Since $\mathcal{C}$ is contractible, there exists a corresponding moment map $\mu : \mathcal{C} \to \mathrm{End}(\mathcal{E})^*$ which naturally generates a system of Hamiltonian functionals $H_\lambda : \mathcal{C} \to \mathbb{R}$, $\lambda \in \mathrm{End}(\mathcal{E})$ through

$$H_\lambda[A] = \operatorname{Tr} \mu[A]\lambda \ . \tag{3.17}$$

As we show below, the moment map in this case is nothing but the noncommutative field strength

$$\mu[A] = F_A + \phi \ , \tag{3.18}$$

and so the Yang-Mills action (3.1) is the *square* of the moment map. The action of this Hamiltonian system is thereby given by the square of its Hamiltonian, rather than by just the Hamiltonian itself. Since $\pi_2(U(\infty)) = 0$, the map $\lambda \mapsto H_\lambda$ is a Lie algebra homomorphism from $\mathrm{End}(\mathcal{E})$ into the Poisson algebra on $\mathcal{C}$ induced by the symplectic structure (3.15). Whether or not this is enough to construct the infinitely many conserved charges required for complete integrability of the noncommutative Yang-Mills system is not presently understood.

To prove (3.18), we note that near a point $\nabla = \partial + A \in \mathcal{C}$ we have

$$F_{A+t\,a} = F_A + t\,\left[\nabla \stackrel{\wedge}{,} a\right] + O\left(t^2\right) \tag{3.19}$$

where $a \in \Omega^1(\mathcal{E})$ and $t \to 0$. By using the definition (2.22), we may then compute the functional derivative

$$\begin{aligned} \frac{\delta}{\delta A} H_\lambda[a] &= \frac{\delta}{\delta A} \operatorname{Tr}\, (F_A + \phi)\,\lambda \\ &= \operatorname{Tr}\, \left[\nabla \stackrel{\wedge}{,} a\right]\lambda \\ &= -\operatorname{Tr}\,[\nabla, \lambda] \wedge a \ , \end{aligned} \tag{3.20}$$

where we have used the Leibnitz rule and the integration by parts property (2.6) for the connection ∇. From (3.6) and (3.15) it therefore follows that

$$\frac{\delta}{\delta A} H_\lambda[a] = -\omega\,[\delta_\lambda A\,,\, a] \ , \tag{3.21}$$

which are just the Hamilton equations of motion in the present case. The existence of these flows is equivalent to the $\mathcal{G}$-invariance (3.16) of the symplectic structure. Note that the background flux ϕ plays no role in this derivation and could in principle be simply dropped from the moment map (3.18), thereby indicating once again its irrelevance with respect to dynamics.

4. Classical Solutions

In this section we will come to the crux of our presentation, the exact solution of the classical noncommutative gauge theory. The technique we will employ is essentially an algebraic version of the Atiyah-Bott bundle splitting method [43] for obtaining the solutions of the Yang-Mills equations on ordinary Riemann surfaces. It relies heavily on the properties of Heisenberg modules over the noncommutative torus that we discussed in section 2. The exact classical solutions of noncommutative gauge theory on the torus were first constructed in [1, 2].

4.1. Equations of Motion

The classical field theory is defined by extremizing the action functional (3.1). We therefore seek the stationary points of noncommuative Yang-Mills theory which are determined by the equation

$$\frac{\delta}{\delta A}S[A]=0\ . \tag{4.1}$$

By using (2.22) this leads to the noncommutative Yang-Mills equations of motion

$$\Big[\nabla\,,\,[\nabla_1\,,\,\nabla_2]\Big]=0\ . \tag{4.2}$$

Under the operator-field correspondence, these equations may be cast into a more conventional form as

$$\partial_i F_A+A_i\star F_A-F_A\star A_i=0 \tag{4.3}$$

for $i=1,2$. We seek to characterize all such critical points of noncommutative gauge theory. This we will do within a particular homotopy class of the Grassmannian manifold of projectors of the algebra $\mathcal{A}$, characterized by a Heisenberg module $\mathcal{E}=\mathcal{E}_{p,q}$ of topological numbers $(p,q)\in\mathbb{Z}^2$ and dimension (2.16). As we will see, in this case there is a nice topological classification of the classical solutions on the space $\mathcal{C}$ of compatible connections of the Heisenberg module. In addition, by combining these results with those of section 3.3, it will follow immediately that the Yang-Mills action (3.1) is a gauge-equivariant Morse functional on $\mathcal{C}$.

4.2. BPS States

Recall from section 2.8 that a Heisenberg module $\mathcal{E}=\mathcal{E}_{p,q}$ is completely characterized by a connection $\nabla^{\rm c}\in\mathcal{C}$ of constant curvature (2.23,2.31). Such connections trivially satisfy the equations of motion (4.2), because their curvatures are proportional to the identity operator on $\mathcal{E}$ and are thereby central. What is striking about the constant curvature solutions is that they provide the absolute minimum value of the noncommutative Yang-Mills action (3.1) on the module $\mathcal{E}$ [1, 44],

$$S\left[\nabla^{\rm c}\right]=\inf_{\nabla\in\mathcal{C}}S[\nabla]\ . \tag{4.4}$$

They thereby constitute the stable classical vacuum states of the gauge theory. In an appropriate supersymmetric extension of the field theory, constant curvature connections correspond to $\frac{1}{2}$ BPS configurations [3, 45].

To prove this fact, we use the expansion (3.19) of the field strength to expand the action (3.1) about a constant curvature connection to get

$$S\left[\nabla^{\rm c}+t\,a\right]=S\left[\nabla^{\rm c}\right]+\frac{t^2}{2g^2}\ {\rm Tr}\ \left[\nabla^{\rm c}\ \stackrel{\wedge}{,}\ a\right]^2+O\left(t^4\right) \tag{4.5}$$

for $t\to0$, where we have used integration by parts (2.6) and the equations of motion (4.2) to eliminate the term of order t in (4.5). The order t^2 term is positive, and so we conclude that

$$S\left[\nabla^{\rm c}+a\right]\geq S\left[\nabla^{\rm c}\right]\qquad\forall a\in\Omega^1(\mathcal{E})\ . \tag{4.6}$$

To show that $\nabla^{\rm c}=\partial+A^{\rm c}$ is a *global* minimum, we can exploit the freedom of choice of the background flux ϕ, which we establish more rigorously in the next section, to identify it with the constant curvature of $\mathcal{E}_{p,q}$ by choosing

$$\phi=-F_{A^{\rm c}}=-\frac{2\pi\,q}{p-q\,\theta}\ . \tag{4.7}$$

Then $S[\nabla^{\rm c}]=0$, and since (3.1) is a positive functional, the connection $\nabla^{\rm c}$ corresponds to the ground state, as claimed. Henceforth we shall always assume that the natural boundary condition (4.7) has been chosen.

4.3. Unstable Vacua

Constant curvature connections can also be used to construct *all* solutions to the classical equations of motion of noncommutative gauge theory. Most of these stationary points will be local maxima or saddle-points, and hence will be unstable. Nevertheless, they are still perfectly good field configurations and play an important role in the quantum dynamics of the gauge theory, as we will see in the second part of these lecture notes. Their general structure can be deduced as follows.

For any connection $\nabla=\partial+A\in\mathcal{C}(\mathcal{E})$, the adjoint action of the field strength F_A on the graded differential algebra (2.20) is generated through the self-adjoint linear operator $\Xi_\nabla:\Omega(\mathcal{E})\to\Omega(\mathcal{E})$ defined by

$$\Xi_\nabla(\alpha)=\left[F_A\,,\,\alpha\right] \tag{4.8}$$

for $\alpha \in \Omega(\mathcal{E})$. Let us consider this operator for ∇ near a critical point, i.e. a solution of the equations of motion. We denote such a generic critical point by $\nabla^{\rm cl} = \partial + A^{\rm cl}$. Then the Yang-Mills equations (4.2) can be written as

$$\left[F_{A^{\rm cl}}\,,\,\nabla_i^{\rm cl}\right] = 0 \ , \quad i = 1, 2 \ . \tag{4.9}$$

This implies that the field strength definition $F_{A^{\rm cl}} = [\nabla_1^{\rm cl}, \nabla_2^{\rm cl}]$ for an on-shell gauge field corresponds (up to an irrelevant shift by the constant curvature ϕ) to a Heisenberg Lie algebra in $\mathrm{End}(\mathcal{E})$, with generators $\nabla_1^{\rm cl}$, $\nabla_2^{\rm cl}$ and $F_{A^{\rm cl}}$, and with central element the field strength. From this fact it follows that the eigenvalues $c_k \in \mathcal{A}$ of the operator (4.8) are central elements near $\nabla = \nabla^{\rm cl}$, i.e. they are proportional to the unit $\mathbb{1}$ of $\mathcal{A}$ in the algebraic setting, or equivalently constant functions on $\mathbf{T}^2$ under the operator-field correspondence.

In particular, this means that in the neighbourhood of any critical point there is an eigenspace decomposition $\Omega(\mathcal{E}) = \bigoplus_k \Omega_k$, where the operator Ξ_∇ acts on each Ω_k as multiplication by a fixed scalar $c_k \in \mathbb{R}$. We can interpret the eigenspaces $\Omega_k = \Omega(\mathcal{E}_{p_k,q_k})$ as the differential algebras of submodules $\mathcal{E}_{p_k,q_k} \subset \mathcal{E}_{p,q}$. Thus near a classical solution the module $\mathcal{E}_{p,q}$ may be regarded as admitting a natural direct sum decomposition into projective submodules,

$$\mathcal{E}_{p,q} = \bigoplus\nolimits_k \mathcal{E}_{p_k,q_k} \ . \tag{4.10}$$

We stress that (4.10) does *not* mean that the given Heisenberg module of the noncommutative gauge theory is reducible. It simply reflects the behaviour of connections on $\mathcal{E}_{p,q}$ near a stationary point of the noncommutative Yang-Mills action. Given a collection of connections $\nabla_{(k)}$ on $\mathcal{E}_{p_k,q_k}$, the action functional (3.1) possesses the additivity property

$$S\left[\bigoplus\nolimits_k \nabla_{(k)}\right] = \sum\nolimits_k S\left[\nabla_{(k)}\right] \tag{4.11}$$

with respect to the decomposition (4.10).

From (4.9) it also follows that each critical connection preserves the submodules, $\nabla_i^{\rm cl} : \mathcal{E}_{p_k,q_k} \to \mathcal{E}_{p_k,q_k}$, and hence we may define a connection $\nabla_{(k)}^{\rm c} = \nabla^{\rm cl}|_{\mathcal{E}_{p_k,q_k}}$ of constant curvature $F_{A^{\rm cl}}|_{\mathcal{E}_{p_k,q_k}}$ on each $\mathcal{E}_{p_k,q_k}$. From the results of the previous subsection, we know that each term $S[\nabla_{(k)}]$ on the right-hand side of (4.11) is minimized by $\nabla_{(k)}^{\rm c}$. It thus follows from (4.11) that the noncommutative Yang-Mills action has a critical point

$$\nabla^{\rm cl} = \bigoplus\nolimits_k \nabla_{(k)}^{\rm c} \tag{4.12}$$

on the module splitting (4.10).

We have thereby deduced that *every* classical solution of noncommutative gauge theory is of the form (4.12) and is characterized by a submodule decomposition (4.10) of the original Heisenberg module. Generically, there could be many different types of splittings (4.10), but there are two important constraints which must be met:

- *Dimension additivity:* Since (4.10) is a direct sum decomposition, we have $\dim \mathcal{E}_{p,q} = \sum_k \dim \mathcal{E}_{p_k,q_k}$, or equivalently $p - q\,\theta = \sum_k (p_k - q_k\theta)$.
- *K-theory charge conservation:* Demanding in addition that the Chern number be additive, $q = \sum_k q_k$, implies $(p,q) = \sum_k (p_k, q_k)$.

These conservation laws are very natural in the string theory setting, in which the positive cone of the K-theory group (2.14) represents the full lattice of D2–D0 brane charges. Note that locally any classical solution $\nabla^{\rm cl}$ may be regarded as a particular (non-BPS) combination of BPS field configurations.

4.4. Partitions

The conclusions just arrived at in the previous subsection show that, for all θ, any solution of the classical equations of motion for Yang-Mills theory on the projective module $\mathcal{E}_{p,q}$ is completely characterized by a *partition* [2, 16], which we define as a collection of integers $(\boldsymbol{p},\boldsymbol{q}) = \{(p_k, q_k)\}$ satisfying the constraints

$$\begin{aligned} p_k - q_k\theta &> 0 \ , \\ \sum\nolimits_k (p_k - q_k\theta) &= p - q\,\theta \ , \\ \sum\nolimits_k q_k &= q \ . \end{aligned} \tag{4.13}$$

In addition, to avoid overcounting, it is sometimes useful to impose a further ordering constraint $p_k - q_k\theta \leq p_{k+1} - q_{k+1}\theta \ \ \forall k$, and regard any two partitions as the same if they coincide after rearranging their components according to this ordering. We may then characterize the components of a partition by integers $\nu_a > 0$ which are defined as the number of partition components that have the $a^{\rm th}$ least dimension $p_a - q_a\theta$. The integer

$$|\boldsymbol{\nu}| = \sum\nolimits_a \nu_a \tag{4.14}$$

is then the total number of components in the given partition. Note that if θ is an irrational number, then the constraint on the magnetic charges in (4.13) is automatically ensured by the dimension additivity. This is not the case, however, for rational-valued noncommutativity parameters. In particular, in the commutative case $\theta = 0$ this last constraint distinguishes between "physical" Yang-Mills theory, which would take into account the sum over all topological charges $q \in \mathbb{Z}$ (i.e. all isomorphism classes of principal $U(p)$ bundles over $\mathbf{T}^2$), and Yang-Mills theory defined on a particular projective module. In the noncommutative setting, only gauge theory on a fixed projective module appears amenable to an unambiguous definition, reflecting the loss of structure group in this case.

With this at hand, we can finally evaluate the noncommutative Yang-Mills action on a classical solution, with corresponding partition $(\boldsymbol{p},\boldsymbol{q})$. By using the additivity property (4.11) of the action, the formula (2.31) for the constant curvature of a Heisenberg module, and the Murray-von Neumann dimension formula $\mathrm{Tr}\,\mathbb{1}|_{\mathcal{E}_{p_k,q_k}} = p_k - q_k\theta$, we easily arrive at

$$\begin{aligned} S(\boldsymbol{p},\boldsymbol{q}) &= S\left[\bigoplus\nolimits_k \nabla^{\rm c}_{(k)}\right] \\ &= \frac{2\pi^2}{g^2}\sum\nolimits_k\,(p_k-q_k\theta)\left(\frac{q_k}{p_k-q_k\theta}-\frac{q}{p-q\,\theta}\right)^2 \ . \end{aligned} \tag{4.15}$$

From the classical action one can in fact determine a number of important facts about the critical point set of noncommutative gauge theory. First of all, it is straightforward to see that, up to gauge equivalence, the action (4.15) vanishes *only* for the trivial partition $(\boldsymbol{p},\boldsymbol{q})=\{(p,q)\}$ corresponding to the Heisenberg module $\mathcal{E}_{p,q}$ itself. This is equivalent to the statement that the gauge theory is globally minimized by its constant curvature connection $\nabla^{\rm c}$, as we showed in section 4.2. Secondly, for any fixed finite action solution of the noncommutative Yang-Mills equations of motion, it is straightforward to show from (4.15) that each partition contains finitely many components [16]. This shows that the number (4.14) is indeed finite and permits one to pick out a minimum dimension submodule $\mathcal{E}_{p_1,q_1}$ as explained above. Finally, one can show that the set of values of the noncommutative Yang-Mills action on the critical point set (the set of all partitions) is discrete [2], so that the action indeed defines a bonafide Morse functional on $\mathcal{C}(\mathcal{E}_{p,q})$.

5. Morita Duality

In noncommutative geometry an important role is played by the notion of "Morita equivalence" of C^*-algebras. Two algebras are Morita equivalent if they have the same representation theory. The one-to-one correspondence between their projective modules means in particular that they have the same K-theory. In section 2.8 we in fact encountered a very important case of this duality for noncommutative tori. Namely, the two algebras $\mathcal{A}_\theta$ and $\mathcal{A}_{\theta'}$, with noncommutativity parameters related through (2.34), are both represented on the Heisenberg module $\mathcal{E}=\mathcal{E}_{p,q}$ and commute with each other in the representation. They are therefore Morita equivalent and in this context $\mathcal{E}$ is refered to as a Morita equivalence bimodule between $\mathcal{A}_\theta$ and $\mathcal{A}_{\theta'}$.

Given that noncommutative gauge theories are constructed from the representations of the algebra, this equivalence has the potential of supplying a novel duality relation between field theories defined on different noncommutative spaces. In fact, this is one of the reasons for the excitement about noncommutative Yang-Mills theory. The Morita equivalence between two noncommutative tori turns out to be equivalent to the fact that the underlying tori $\mathbf{T}^2$ are related by open string T-duality [45, 46]. The noncommutative Yang-Mills action is invariant under the equivalence [44], provided we use an extended notion of Morita equivalence known as gauge Morita equivalence. Under the extended equivalence, not only do we obtain a one-to-one correspondence between projective modules over different noncommutative tori associated with different topological numbers, but we also augment this with transformations of connections between the modules. The conclusion is then that noncommutative Yang-Mills theory is *invariant* under T-duality [4, 47]. This is in marked contrast to the situation in ordinary Yang-Mills theory which is not duality invariant. T-duality invariance is one of the remarkable features of noncommutative field theories that follow from their origins in string theory, yet can be defined and analysed within the framework of quantum field theory. In this final section we will present the Morita transformation rules and describe various consequences of them.

The full open string T-duality group on $\mathbf{T}^2$ is $SO(2,2,\mathbb{Z})$ and it acts on the K-theory ring $\mathrm{K}_0(\mathcal{A})\oplus\mathrm{K}_1(\mathcal{A})$ in a spinor representation, in precisely the same way in which it acts on the Ramond-Ramond charges of D-branes. In generic spacetime dimension d this group is $SO(d,d,\mathbb{Z})$. The special feature of two dimensions is that there is an isomorphism

$$SO(2,2,\mathbb{Z}) = SL(2,\mathbb{Z})\times SL(2,\mathbb{Z}) \ . \tag{5.1}$$

One of these $SL(2,\mathbb{Z})$ factors is just the geometrical automorphism group of the torus $\mathbf{T}^2$ which acts by discrete Möbius transformations of its Teichmüller modulus τ. It is the same symmetry group that is present in the commutative case and so will not be discussed any further. The other $SL(2,\mathbb{Z})$ acts on the Kähler modulus of $\mathbf{T}^2$ and has no counterpart in ordinary geometry. It acts by discrete Möbius transformations of the noncommutativity parameter θ. In fact, a well-known theorem of noncommutative geometry [23] asserts that two noncommutative tori $\mathcal{A}=\mathcal{A}_\theta$ and $\mathcal{A}'=\mathcal{A}_{\theta'}$ are Morita equivalent if and only if

$$\theta' = \frac{m\,\theta+n}{r\,\theta+s} \ , \quad \begin{pmatrix} m & n \\ r & s\end{pmatrix} \in SL(2,\mathbb{Z}) \ . \tag{5.2}$$

This coincides with the transformation rule for the quantity $G+B$ under T-duality [4], where G is the open string metric and B the antisymmetric tensor field.

The same transformation sends a projective module $\mathcal{E}$ of topological numbers (p,q) over the algebra $\mathcal{A}$ to a module $\mathcal{E}'$ of topological numbers

(p', q') over $\mathcal{A}'$, in such a way that the pair of K-theory charges transforms as an $SL(2, \mathbb{Z})$ doublet

$$\begin{pmatrix} p' \\ q' \end{pmatrix} = \begin{pmatrix} m & n \\ r & s \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} \ . \tag{5.3}$$

From (5.2) and (5.3) it is straightforward to show that the Murray-von Neumann dimensions (2.16) of the modules are related by

$$\dim \mathcal{E}' = \frac{\dim \mathcal{E}}{|r\,\theta + s|} \ , \tag{5.4}$$

and hence that the canonical trace on $\mathcal{E}$ is rescaled as

$$\mathrm{Tr}\,' = |r\,\theta + s| \ \mathrm{Tr} \ . \tag{5.5}$$

In particular, the Morita equivalence bimodule for the Heisenberg modules $\mathcal{E}_{1,0}$ and $\mathcal{E}_{p,q}$ can be constructed by quantizing open strings in a background B-field where the strings stretch from a single D2-brane to a cluster of p coincident D2-branes carrying q units of D0-brane charge [4].

Under the gauge Morita equivalence the field strengths of connections on the modules $\mathcal{E}$ and $\mathcal{E}'$ are related through [45]

$$F'_{A'} = (r\,\theta + s)^2\, F_A + 2\pi\, r\,(r\,\theta + s) \ . \tag{5.6}$$

The invariance of the noncommutative Yang-Mills action (3.1) is then guaranteed by the transformation properties

$$\begin{aligned} g'^{\,2} &= |r\,\theta + s|^3\, g^2 \ , \\ \phi' &= (r\,\theta + s)^2\, \phi - 2\pi\, r\,(r\,\theta + s) \ . \end{aligned} \tag{5.7}$$

This mapping exhibits explicitly the physical role [4, 44, 45, 47, 48] of the background flux ϕ, as it is required to absorb the inhomogeneous term in the transformation (5.6). In particular, it can shifted in and out of the action in a variety of different ways with no effect on the field theory, thus showing precisely why it had no real bearing on our previous analyses.

What is particularly important for us is the way that the solution spaces of two dual gauge theories map into one another [16]. The Morita transformation acts diagonally on the critical point sets of noncommutative gauge theories, providing a one-to-one correspondence between their constant curvature connections [45], and hence their classical solutions. This implies, in particular, that the basic structure of partitions between dual gauge theories is the same. It is straightforward to check that the classical action (4.15) is invariant under the transformations (5.2), (5.3) and (5.7). Moreover, in the supersymmetric gauge theory the entire spectrum of BPS states is invariant under the Morita duality [47].

Finally, let us note that an ordinary $\theta = 0$ gauge theory is mapped to a noncommutative gauge theory with rational-valued noncommutativity parameter $\theta' = n/s$ and Yang-Mills coupling constant $g'^{\,2} = s^3\, g^2$ under the Morita map (5.2) and (5.7). In this way it is possible to embed ordinary Yang-Mills theory into the wider class of noncommutative gauge theories, and this embedding has been used to suggest that noncommutative Yang-Mills theory can be used to impose very stringent constraints on ordinary large N gauge theories on tori [49]. In particular, an ordinary $U(p)$ gauge theory with 't Hooft flux is Morita equivalent to a rational $\theta' = n/s$ noncommutative gauge theory with single-valued gauge fields [44]. In this way the non-Abelian colour degrees of freedom of commutative Yang-Mills theory may be traded for spacetime noncommutativity.

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