Symmetric products of surfaces; a unifying theme for topology and physics

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Abstract

This is a review paper about symmetric products of spaces $SP^n(X) := X^n/S_n$. We focus our attention on the case of 2-manifolds X and make a journey through selected topics of algebraic topology, algebraic geometry, mathematical physics, theoretical mechanics etc. where these objects play an important role, demonstrating along the way the fundamental unity of diverse fields of physics and mathematics.

1 Introduction

In recent years we have all witnessed a remarkable and extremely stimulating exchange of deep and sophisticated ideas between Geometry and Physics and in particular between quantum physics and topology. The student or a young scientist in one of these fields is often urged to master elements of the other field as quickly as possible, and to develop basic skills and intuition necessary for understanding the contemporary research in both disciplines. The topology and geometry of manifolds plays a central role in mathematics and likewise in physics.

The understanding of duality phenomena for manifolds, mastering the calculus of characteristic classes, as well as understanding the role of fundamental invariants like the signature or Euler characteristic are just examples of what is on the beginning of the growing list of prerequisites for a student in these areas.

A graduate student of mathematics alone is often in position to take many specialized courses covering different aspects of manifold theory and related areas before an unified picture emerges and she or he reaches the necessary level of maturity.

Obviously this state of affairs is somewhat unsatisfactory and clearly it can only get worse in the future. One of objectives of our review is to follow a single mathematical object, a remarkable $(2n)$-dimensional manifold $SP^n(M)$, the n-th *symmetric product* of a surface M, on a guided tour "transversely" through mathematics with occasional contacts with physics. We hope that the reader will find this trip amusing and the information interesting and complementary to the usual topology textbook presentations.

By definition, the n-th symmetric product of a space X is defined as $SP^n(X) := X^n/S_n$. In other words a point $D \in SP^n(X)$ is an unordered collection of n-points in X, often denoted by $D = x_1 + \ldots + x_n$ where the points $x_i \in X$ are not necessarily distinct. More generally a G-symmetric product of X is defined by $SP^n_G(X) = X^n/G$ where $G \subset S_n$ is a subgroup of the symmetric group on n-letters.

If M is a 2-dimensional manifold, a surface for short, then $SP^n(M)$ is also a manifold, Section 2. In each theory, relevant examples are essential for illustrating and understanding general theorems and as a guide for intuition. Symmetric powers of surfaces provide a list of interesting and nontrivial examples illustrating many phenomena of manifold theory.

These manifolds are interesting objects which make surprising appearance at crossroads of many disciplines of mathematics and mathematical physics.

If $X = M_g$ is a surface of genus g, say a nonsingular algebraic curve, then $SP^n(M_g)$, interpreted as the space of all effective divisors of order n, serves as the domain of the classical Jacobi map,[2], [25],

$$\mu : SP^n(M_g) \to \mathrm{Jac}(M_g).$$

In Algebraic Topology, the spaces $SP^\infty(X) := \mathrm{colim}_{n\in\mathbb{N}}\, SP^n(X)$ have, at least for connected CW-complexes, a remarkable homotopical decomposition

$$SP^\infty(X) \simeq \prod_{n=1}^{\infty} K(H_n(X,\mathbb{Z}), \ltimes),$$

due to Dold and Thom, [17],[44]. In particular the Eilenberg-MacLane space $K(\mathbb{Z}, \ltimes)$ has a natural "geometric realization" $SP^\infty(S^n)$. Much more recent are results which connect special divisor spaces with the functional spaces of holomorphic maps between surfaces and other complex manifolds, [47], [13],[36]. This is a rich theory which can be seen as a part of the evergreen topological theme of comparing functional spaces with various particle configuration spaces. It also appears that symmetric products play more and more important role in mathematical physics, say in matrix string theory, [5], [12], [14], [15], [21].

The case of closed, open, orientable or nonorientable surfaces is as already observed of special interest since their symmetric products are genuine manifolds. These manifolds were studied in [41], [42], [18], [4] and they undoubtedly appeared in many other papers in different contexts. The signature of $SP^n(M_g)$ was determined by Macdonald [41], the signature of $SP^n(M)$ for more general closed, even dimensional manifolds was calculated by Hirzebruch [32]. Their calculation is based on the evaluation of the L-polynomial and the celebrated Hirzebruch signature theorem. Zagier in [55] used the Atiyah-Singer G-signature theorem and obtained a formula for the signature of any G-symmetric product $SP^n_G(M)$.

The contemporary as well as the classical character of these objects is particularly well illustrated by Arnold in [4], where the homeomorphism $SP^n(RP^2) \cong RP^{2n}$, [18], is interpreted as a "quaternionic" analogue of $SP^n(\mathbb{CP}^{\nVdash}) \cong \mathbb{CP}^{\ltimes}$ and directly connected with some classical results of Maxwell about spherical functions.

In Section 2 we give a brief exposition of some of these results with pointers to relevant references. Sections 3 and 4 reflect the research interests of the authors and provide new examples of applications of symmetric products.

2 Symmetric products are everywhere!

2.1 Examples

Definition 2.1. *The n-th symmetric product or the n-th symmetric power of a space X is*

$$SP^n(X) = X^n/S_n$$

where S_n is the symmetric group in n letters.

Here are the first examples of symmetric products of familiar spaces.

(1) $SP^n([\mathbf{0},\mathbf{1}]) = \mathbf{\Delta^n}$ is an n-simplex.

Since $[\mathbf{0},\mathbf{1}]$ is totally ordered, $SP^n([0,1]) = \{x_1 + .. + x_n \mid 0 \leq x_1 \leq .. \leq x_n \leq 1\} = \Delta^n$. The same idea can be used to show that $SP^n(\mathbb{R})$ is a closed polyhedral cone in $\mathbb{R}^{\ltimes}$.

(2) $SP^n(\mathbb{C}) = \mathbb{C}^{\ltimes}$ and $SP^\infty(\mathbb{C}) = \mathbb{C}^\infty$.

Every element $z_1 + .. + z_n \in SP^n(\mathbb{C})$ can be identified with the monic complex polynomial $p(z) = (z - z_1)...(z - z_n)$ with zeros in z_i. Also, $SP^n(\mathbb{C}) \simeq \mathbb{SP}^{\ltimes}(*) \cong *$.

(3) $SP^n(S^1) \simeq S^1$, $n > 0$, and as a consequence $SP^\infty(S^1) \simeq S^1$.

This result follows for example [44] from the fact that the map $\pi : SP^n(S^1) \to S^1$

$$e^{i\alpha_1} + .. + e^{i\alpha_n} \mapsto e^{i(\alpha_1 + .. + \alpha_n)}$$

is a fibration with a contractible fibre. Alternatively, one can use the homotopy equivalence $SP^n(S^1) \simeq SP^n(\mathbb{C} \setminus \{\nvdash\})$. Then from $SP^n(\mathbb{C}) \cong \mathbb{C}^{\ltimes}$ we deduce that $SP^n(S^1)$ is homotopic to the complement of a hyperplane.

$$SP^n(S^1) \simeq SP^n(\mathbb{C} \setminus \{\nvdash\}) \simeq \mathbb{C}^{\ltimes} \setminus \mathbb{H} \simeq \mathbb{S}^{\nVdash}.$$

Of course, the Dold-Thom theorem (Theorem 2.5) implies

$$SP^\infty(S^1) \simeq \prod_{n=1}^{\infty} K(H_n(S^1, \mathbb{Z}), \ltimes) \simeq \mathbb{K}(\mathbb{H}_{\nVdash}(\mathbb{S}^{\nVdash}, \mathbb{Z}), \nVdash) \simeq \mathbb{K}(\mathbb{Z}, \nVdash) = \mathbb{S}^{\nVdash}.$$

Note that $SP^2(S^1)$ is actually the Möbius band.

(4) $SP^n(S^2) = \mathbb{CP}^{\ltimes}$, $\ltimes > \nvdash$ and $SP^\infty(S^2) = \mathbb{CP}^\infty$.

First we identify $S^2 = \mathbb{C} \cup \{\infty\}$ and $\mathbb{CP}^{\ltimes} = \{\imath(\digamma) = \eth_{\ltimes}\digamma^{\ltimes} + .. + \eth_{\nvdash} \mid \eth_{\beth} \in \mathbb{C}\}/(\imath(\digamma) \sim \lambda\imath(\digamma), \lambda \neq \nvdash)$. The map

$$SP^n(S^2) \ni z_1 + .. + z_k + \infty + .. + \infty \mapsto (z + z_1)..(z + z_k) \in \mathbb{CP}^{\ltimes}$$

is well defined

$$\begin{aligned}
\lim_{z_1 \to \infty} (z + z_1)..(z + z_k) &= \lim_{z_1 \to \infty} (z^k + (z_1 + .. + z_k)z^{k-1} + .. + z_1..z_k) \\
&= \lim_{z_1 \to \infty} \frac{1}{z_1}(z^k + (z_1 + .. + z_k)z^{k-1} + .. + z_1..z_k) \\
&= z^k + (z_1 + .. + z_{k-1})z^{k-1} + .. + z_1..z_{k-1}
\end{aligned}$$

and easily checked to be a homeomorphism.

(5) $SP^2(S^n) = \mathrm{MapCone}(\Sigma^{\mathrm{n}}\mathbb{RP}^{\ltimes - \nVdash} \to \mathbb{S}^{\ltimes})$ [29].

2.2 Maxwell and Arnold

The symmetric power of a real projective plane is also a projective space, $SP^n(RP^2) \cong RP^{2n}$, [18]. Vladimir Arnold in [4] observed that this result is a direct consequence of the theorem on *multipole representation of spherical functions* of James Clerk Maxwell.

Recall that a *spherical function* of degree n on a unit sphere in $\mathbb{R}^{\nVDash}$ is the restriction to the sphere of a homogeneous harmonic polynomial of degree n.

Theorem 2.2. *The n-th derivative of the function $\frac{1}{r}$ along n constant (translation-invariant) vector fields $V_1, \ldots V_n$ in $\mathbb{R}^{\nVDash}$ coincides on the sphere with a spherical function of order n. Any nonzero spherical function ψ of degree n can be obtained by this construction from some n-tuple of nonzero vector fields. These n fields are uniquely defined by the function ψ (up to multiplication by nonzero constants and permutation of the n fields).*

In other words any spherical function is the restriction on the unit sphere of a function of the form

$$\mathcal{L}_{V_1} \ldots \mathcal{L}_{V_n}(\frac{1}{r})$$

where $\mathcal{L}_X(f)$ is the directional derivative (Lie derivative) of f in the direction of the vector field X. The following lemma can be proved by a simple inductive argument based on the formula

$$\frac{\partial}{\partial x}\frac{A}{r^b} = \frac{r^2(\partial A/\partial x) - bAx}{r^b + 2}.$$

Lemma 2.3. *The function $\mathcal{L}_{V_1} \ldots \mathcal{L}_{V_n}(\frac{1}{r})$ has the form P/r^{2n+1} where P is a homogeneous polynomial of degree n.*

Actually the polynomial P can be shown to be harmonic in $\mathbb{R}^{\nVDash}$, i.e. $\Delta(P) = 0$ where Δ is the Laplace operator $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$. This essentially follows from the well known connection between Δ and the spherical Laplacian $\tilde{\Delta}$,

$$\tilde{\Delta}F = r^2\Delta F - \Lambda F, \quad \Lambda := k^2 + k \tag{1}$$

where F is a homogeneous function on $\mathbb{R}^{\nVDash} \setminus \{\nvdash\}$ of degree k, [4]. Note that $\tilde{\Delta}$ is an operator defined of k-homogeneous functions on $\mathbb{R}^{\nVDash} \setminus \{\nvdash\}$, so perhaps a more appropriate notation would be $\tilde{\Delta}_k$. From the equation (1) one deduces many important properties of spherical functions and the associated harmonics in $\mathbb{R}^{\nVDash} \setminus \{\nvdash\}$. For example one immediately observes that a spherical function, defined as a restriction of a harmonic function, is an eigen function of the spherical Laplacian. Conversely, any eigen function ψ of the spherical Laplacian corresponding to the eigen value $\Lambda = n^2 + n$ can be extended in the ambient space $\mathbb{R}^{\nVDash} \setminus \{\nvdash\}$ to a homogeneous harmonic function in two ways, with the respective degrees n and $-n-1$.

Another important consequence is that there is bijective correspondence between the space SF_n of spherical functions of order n and the space HP_n of homogeneous, harmonic polynomials of order n. A detailed exposition of these and other beautiful facts about spherical functions can be found in [4] and [3].

Let $S^n(\mathbb{R}^{\nVDash})$ be the linear space of all homogeneous polynomials in $\mathbb{R}^{\nVDash}$ of degree n. An elementary fact is that $\dim(S^k(\mathbb{R}^{\nVDash})) = (\ltimes+\nvDash)(\ltimes+\nVdash)/\nvDash$. By an elementary inductive argument one shows that the linear map $\delta_n : S^n(\mathbb{R}^{\nVDash}) \to \mathbb{S}^{\ltimes-\nvDash}(\mathbb{R}^{\nVDash})$ defined as the restriction of the Laplacian Δ, is an epimorphism. It follows that the dimension of the space $HP_n = \mathrm{Ker}(\delta_n)$ is $(n+2)(n+1)/2 - n(n-1)/2 = 2n+1$ so the dimension of SF_n is also $2n+1$. Finally by Theorem 2.2, one observes that the symmetric power of RP^2 is homeomorphic to the projective space associated to SF_n, hence

$$SP^n(RP^2) \cong RP^{2n}.$$

2.3 Abel and Jacobi

Suppose that $M = M_g$ is a compact Riemann surface, i.e. a compact 2-manifold (of genus g) with a complex structure. Alternatively, M can be introduced as a nonsingular algebraic curve. Symmetric powers $SP^n(M)$ of a curve are ubiquitous in Algebraic Geometry, [20] [25] [28]. We cannot possibly do justice to most of these developments in this article. Keeping in mind our focus on the *topological manifold* $SP^n(M_g)$, we start with the following result [42] which is a generalization of the fact $SP^n(\mathbb{CP}^1) \cong \mathbb{CP}^n$.

Theorem 2.4. *Suppose that $n > 2g - 2$. Then there is a fibre bundle*

$$\mathbb{CP}^{n-g} \to SP^n(M_g) \to T^{2g} \tag{2}$$

where the fibre $\mathbb{CP}^{n-g}$ is a complex projective space of dimension $n - g$ and T^{2g} is a $2g$-dimensional torus.

We start an outline of the proof of this theorem with a brief exposition of the Abel–Jacobi map.

A standard fact [20] is that the complex vector space $\Omega(M_g)$ of *holomorphic* differential 1-forms is g-dimensional. Let $\omega_1, \dots, \omega_g$ be a basis of this space. Define the *subgroup of periods* $\mathrm{Per} = \mathrm{Per}(\omega_1, \dots, \omega_g)$ in $\mathbb{C}^g$ by the requirement that $v = (v_1, \dots, v_g) \in \mathrm{Per}$ if and only if for some $\alpha \in \pi_1(M_g)$

$$\text{for all } i = 1, \dots, g \quad v_i = \int_\alpha \omega_i\,.$$

Then Per is a discrete subgroup in $\mathbb{C}^g$ of maximal rank, hence $\mathrm{Jac}(M_g) := \mathbb{C}^g/\mathrm{Per}$ is a $(2g)$-dimensional torus called the *Jacobian* of the surface M_g.

Suppose that $b \in M_g$ is a base point. Given $x \in M_g$ and a path β connecting points b and x, let $u = (u_1, \dots, u_g)$ be a vector in $\mathbb{C}^g$ defined by

$$u_i := \int_\beta \omega_i\,.$$

The vector $u \in \mathbb{C}^{\mathfrak{d}}$ depends on β, however its image in $\mathrm{Jac}(M_g)$ depends only on the point x. This way arises the celebrated *Abel–Jacobi* map

$$\mu : M_g \to \mathrm{Jac}(M_g). \tag{3}$$

If $g = 1$, i.e. in the case of an *elliptic curve*, the map μ is an isomorphism. This is a famous turning point in mathematics, when the study of meromorphic functions on the curve was reduced to the study of meromorphic functions on a torus, i.e. the meromorphic functions in the complex plane $\mathbb{C}$ with 2 periods.

In the case of a general curve M_g, the map μ is far from being an isomorphism. Since $\mathrm{Jac}(M_g)$ is an abelian group, the Abel–Jacobi map μ can be extended to a symmetric power $SP^n(M_g)$ by a formula

$$\mu_n(D) := \mu(x_1) + \ldots + \mu(x_n)$$

where $D = x_1 + \ldots + x_n \in SP^n(M_g)$. If $n = g$ then $\mu_g : SP^g(M_g) \to T^{2g}$ is a "correct" replacement for the map μ.

An alternative description of the map μ_n is following. Let $\mathrm{Pic(M_g) := Div_0/Div_H}$ be the *Picard* group of M_g where Div_0 is the group of all divisors of degree 0 and Div_H is the group of principal divisors, i.e. the divisors of the form $D = (f)$ for some meromorphic function f. There is a map

$$\Phi : \mathrm{Div}_0 \to \mathrm{Jac}(M_g)$$

defined by $\Phi(D) = v = (v_1, \ldots, v_g)$ where

$$v_i := \int_c \omega_i \quad \text{for each} \quad i$$

and c is a 1-chain, i.e. a system of paths connecting points in D, such that $\partial(c) = D$. Abel's theorem [25] claims that the kernel of Φ is precisely the group Div_H of principal divisors, hence the induced map

$$\phi : \mathrm{Pic}(M_g) \to \mathrm{Jac}(M_g) \tag{4}$$

is a monomorphism. It turns out[1] that the map ϕ is an isomorphism. As a consequence, the Abel–Jacobi map $\mu = \mu_1$ (3), more precisely its higher dimensional

[1] Jacobi inversion problem, [25].

extension μ_n, has a twin map

$$\nu_n : SP^n(M_g) \to \mathrm{Pic}(M_g) \tag{5}$$

defined by $\nu_n(D) = D - nb$ (b is the base point in M_g), such that $\phi \circ \nu_n = \mu_n$.

As a consequence one can approach the proof of Theorem 2.4 via the map ν_n. In order to determine the inverse image $\nu_n^{-1}([D])$ for a given $[D] \in \mathrm{Pic}(M_g)$, one has to solve the Riemann–Roch problem i.e. to determine the dimension of the space of effective divisors with prescribed information about their zeros and poles. If $n > 2g - 2$ then, by the Riemann–Roch theorem, the dimension of the space of meromorphic functions f such that $E = D + (f)$ is an effective divisor of order n, is precisely $n - g + 1$. The representation of the divisor E in the form $D + (f)$ is not unique, namely if $D + (f) = D + (g)$ then $(f) = (g)$ and $g = cf$ for some nonzero constant $c \in \mathbb{C}$. It follows that $\nu_n^{-1}(D)$ is a complex projective space of dimension $n - g$ which finally explains the appearance of the fibre $\mathbb{CP}^{\ltimes-\eth}$ in the fibration (2).

2.4 The Poincaré polynomial of a symmetric product

In this section we compute the Betti numbers of general symmetric products. These results were originally obtained by I.G. Macdonald, [40].

Let $V = \oplus_{d\geq 0} V_d$ be a graded finite dimensional vector space. The associated Poincaré polynomial is defined by $P_t(V) = \sum_{d\geq 0} t^d \mathrm{dim} V_d$. It is easily shown that

$$P_t(V \oplus W) = P_t(V) + P_t(W),$$

$$P_t(V \otimes W) = P_t(V) P_t(W).$$

The symmetric algebra over the vector space V is defined by

$$S^*(V) = T^*(V)/v \otimes w - (-1)^{degv \cdot degw} w \otimes v.$$

It is naturally bigraded by

$$\mathrm{bdeg}\langle v_1, \ldots, v_n\rangle = (\sum_{j=1}^{n} \deg(v_j), n).$$

We introduce the formal variable q by $S^*_q(V) = \sum_{n\geq 0} q^n S^n(V)$, where $S^n(V) = \{x \in S^*(V) \mid \text{bdeg}(x) = (\cdot, n)\}$ is the $n-$th symmetric power of V. Since $S^n(V \oplus W) = \oplus_{p+q=n} S^p(V) \otimes S^q(W)$, we have

$$S^*_q(V \oplus W) = S^*_q(V) S^*_q(W),$$

$$P_t(S^*_q(V \oplus W)) = P_t(S^*_q(V)) P_t(S^*_q(W)).$$

If $L = L_d$ is a 1-dimensional, graded vector space generated by a vector of degree d then,

$$S^*_q(L_d) = \begin{cases} 1 + qL + q^2 L^{\otimes 2} + \cdots & \text{, d even} \\ 1 + qL & \text{, d odd} \end{cases}, \quad \text{and}$$

$$P_t(S^*_q(L_d)) = \begin{cases} \frac{1}{1-qt^d} & \text{, d even} \\ 1 + qt^d & \text{, d odd} \end{cases}.$$

Since any graded vector space $V = \oplus_{d\geq 0} V_d$ can be decomposed into a sum of 1-dimensional, graded vector spaces, it follows that,

$$P_t(S^*_q(V)) = \frac{\prod_{d\, odd}(1 + qt^d)^{dim V_d}}{\prod_{d\, even}(1 - qt^d)^{dim V_d}}.$$

Now, if $V = H_*(X, \mathbb{Q})$ is the homology of a CW-complex X and β_d are its Betti numbers, then

$$H_*(SP^n(X), \mathbb{Q}) = (\mathbb{H}_*(\mathbb{X}, \mathbb{Q})^{\otimes \ltimes})^{\mathbb{S}_\ltimes} = \mathbb{S}^\ltimes(\mathbb{H}_*(\mathbb{X}, \mathbb{Q}) = \mathbb{S}^\ltimes(\mathbb{V})$$

and we get the Macdonald result [40]

$$\sum_{n\geq 0} q^n P_t(SP^n(X)) = \frac{\prod_{d\, odd}(1 + qt^d)^{\beta_d}}{\prod_{d\, even}(1 - qt^d)^{\beta_d}}.$$

In particular for $t = -1$ we get the generating function for Euler characteristics of symmetric powers of the space X

$$\sum_{n\geq 0} q^n \chi(SP^n(X)) = (1 - q)^{-\chi(X)}.$$

2.5 Dold-Thom theorems

Symmetric powers of spaces have been used in homotopy theory for the last fifty years. They for example appear in the study of iterated loop spaces (the symmetric products appear e.g. as fragments of model spaces for important spaces such as $\Omega^n\Sigma^n X$). Here we review some of the central results.

Let $(X,*)$ be a space with the base point $* \in X$. Assuming that $SP^0(X) = \{*\}$, for each $n \geq 0$ we define a natural inclusion

$$SP^n(X) \hookrightarrow SP^{n+1}(X), \quad x_1 + ... + x_n \mapsto x_1 + ... + x_n + *.$$

The colimit of the direct system of these inclusions is the so called infinite symmetric product $SP^\infty(X)$. The "we can always add" map $\mu : SP^n(X) \times SP^m(X) \to SP^{n+m}(X)$

$$(x_1 + ... + x_n, y_1 + ... + y_m) \mapsto x_1 + ... + x_n + y_1 + ... + y_m$$

induces associative multiplication on $SP^\infty(X)$ with the neutral element $* \in SP^0(X)$. Moreover, it can be proved that $SP^\infty(X)$ is the free commutative topological monoid generated by X with $*$ as the unit element. It may be natural to ask what can be said about $A^\infty(X)$, the free commutative topological group generated by X with $*$ as neutral and the topology of the quotient

$$A^\infty(X) = \coprod_{n,m\geq 1} SP^n(X) \times SP^m(P)/\sim$$

where the equivalence relation $\sim$ is defined by

$$(x_1 + .. + x_n, y_1 + .. + y_m) \sim (x_1 + .. + \hat{x}_i + .. + x_n, y_1 + .. + \hat{y}_j + .. + y_m)$$

if and only if $x_i = y_j$.

A nonzero element in $A^\infty(X)$ can be formally written as a difference $(x_1 + .. + x_n) - (y_1 + .. + y_m)$ where elements x_i, y_j are all different from the base point $*$.

Theorem 2.5. (Dold-Thom) *If $(X,*)$ is a connected CW-complex, then*

$$SP^\infty(X) \simeq \prod_{n=1}^{\infty} K(H_n(X,\mathbb{Z}), \ltimes)$$

where $K(G,n)$ is an Eilenberg-MacLane space, i. e. a CW-complex with the property that $\pi_n(K(G,n)) = G$ and $\pi_i(K(G,n)) = 0$ for each $i \neq n$.

Examples:

(1) $SP^{\infty}(S^n) \cong K(\mathbb{Z}, \ltimes)$.

(2) $SP^{\infty}(\mathbb{CP}^{\ltimes}) = \prod_{\daleth=\Bbbk}^{\ltimes} \mathbb{K}(\mathbb{Z}, \nvDash\daleth) = \prod_{\daleth=\Bbbk}^{\ltimes} \mathbb{SP}^{\infty}(\mathbb{S}^{\nvDash\daleth})$.

(3) $SP^{\infty}(S^n \cup_k e^{n+1}) = K(\mathbb{Z}/\daleth, \ltimes)$.

There are different proofs of the theorem of Dold and Thom, see [17],[44]. One possibility is to establish first the following relative of Theorem 2.5.

Theorem 2.6. *If $(X, *)$ is a connected CW-complex, then*

$$A^{\infty}(X) \simeq \prod_{n=1}^{\infty} K(\tilde{H}_n(X, \mathbb{Z}), \ltimes).$$

The proof of this theorem given in [44] is based on the following facts:

(i) $X \simeq Y \Longrightarrow A^{\infty}(X) \simeq A^{\infty}(Y)$
(a homotopy $H : X \times I \to Y$ yields a homotopy $A^{\infty}(H) : A^{\infty}(X) \times I \to A^{\infty}(Y)$).

(ii) $A^{\infty}(S^0) \cong \mathbb{Z}$.

(iii) $A^{\infty}(S^n) = K(\mathbb{Z}, \ltimes)$, (a cofibration sequence $S^n \hookrightarrow D^{n+1} \to S^{n+1}$ produces a fibration $A^{\infty}(D^{n+1}) \to A^{\infty}(S^{n+1})$ with $A^{\infty}(S^n)$ as a fibre, so an induction on the dimension can be applied).

(iv) $X \mapsto \pi_i(A^{\infty}(X))$ induces a reduced homology theory with integral coefficients (it satisfies Eilenberg-Steenrod axioms: (i), $A^{\infty}(*) \simeq *$, (iii)).

(v) The uniqueness of the homology theory satisfying the Eilenberg-Steenrod axioms implies that $\pi_i(A^{\infty}(X)) \cong \tilde{H}_i(X, \mathbb{Z})$.

Theorem 2.7. (Dold-Thom) *If $(X, *)$ is a connected CW-complex then the inclusion $SP^{\infty}(X) \hookrightarrow A^{\infty}(X)$*

$$x_1 + ... + x_n \mapsto x_1 + ... + x_n$$

is a homotopy equivalence.

By using the comparison theorem for spectral sequences, one can prove Theorem 2.7 for spheres. Then adding cell after cell, with the use of the 5-lemma, the theorem is established for every connected CW-complex.

As a consequence one concludes that both $X \mapsto SP^n(X)$ and $X \mapsto SP^\infty(X)$ are *homotopy contravariant functors*.

Corollary 2.8. $SP^\infty(X \vee Y) \simeq SP^\infty(X) \times SP^\infty(Y)$.

This is a direct consequence of the Dold-Thom theorem (Theorem 2.5) and the following facts $\tilde{H}_n(X \vee Y, \mathbb{Z}) \cong \tilde{\mathbb{H}}_{\ltimes}(\mathbb{X}, \mathbb{Z}) \oplus \tilde{\mathbb{H}}_{\ltimes}(\mathbb{Y}, \mathbb{Z})$, $K(G \times H, n) = K(G, n) \times K(H, n)$.

2.6 Steenrod and Milgram; homology of symmetric products

The inclusion map $i : SP^n(X) \hookrightarrow SP^{n+1}(X)$ is very useful in computations with symmetric products. For example this map induces a long exact sequence in homology (with any coefficient group)

$$.. \to H_*(SP^n(X)) \xrightarrow{i_*} H_*(SP^{n+1}(X)) \to H_*(SP^{n+1}(X), SP^n(X)) \to .. \qquad (6)$$

Consequently it is essential to understand the associated map i_*. A solution of this problem was announced by Norman Steenrod in [49] but this proof has never been published. Albrecht Dold proved in [16] the following theorem.

Theorem 2.9. *Let $(X, *)$ be a connected CW-complex, G an arbitrary coefficient group and $i_n : SP^n(X) \hookrightarrow SP^{n+1}(X)$, $j_n : SP^n(X) \hookrightarrow SP^\infty(X)$ the natural inclusions. Then $(i_n)_*$ is an inclusion onto a direct summand, i. e. there is a splitting exact sequence*

$$0 \to H_n(SP^n(X), G) \xrightarrow{(i_n)_*} H_n(SP^{n+1}(X), G).$$

In light of Theorem 2.9, the long exact sequence (6) becomes

$$.. \xrightarrow{0} H_*(SP^n(X)) \xrightarrow{i_*} H_*(SP^{n+1}(X)) \xrightarrow{onto} H_*(SP^{n+1}(X), SP^n(X)) \xrightarrow{0} ..$$

and implies

$$
\begin{aligned}
H_*(SP^n(X),G) &= \bigoplus_{i=1}^{n} H_*(SP^i(X),SP^{i-1}(X),G),\\
H_*(SP^\infty(X),G) &= \bigoplus_{i=1}^{\infty} H_*(SP^i(X),SP^{i-1}(X),G).
\end{aligned}
$$

Here we used the commutativity of the following diagram

$$
\begin{array}{ccc}
H_n(SP^n(X),G) & \xrightarrow{(i_n)_*} & H_n(SP^{n+1}(X),G)\\
{\scriptstyle (j_n)_*}\searrow & & \swarrow{\scriptstyle (j_{n+1})_*}\\
& H_n(SP^\infty(X),G) &
\end{array}
$$

This suggests that there should exist a natural filtration

$$H_1 \subseteq H_2 \subseteq .. \subseteq H_n \subseteq ..H_*(SP^\infty(X),G)$$

of $H_*(SP^\infty(X),G)$ such that $H_n \diagup H_{n-1} \cong H_*(SP^n(X),SP^{n-1}(X),G)$. Since the homology is compactly supported, for each $\alpha \in H_m(SP^\infty(X),G)$ we define

$$n_\alpha = \min\{r \mid (\exists \alpha_r \in H_m(SP^r(X),G))\, i_r(\alpha_r) = \alpha\}.$$

Thus, there is a filtration

$$H_n = \{\alpha \in H_*(SP^\infty(X),G) \mid n_\alpha \leq n\}$$

and it is obvious that $H_n \subseteq H_{n+1}$ and

$$H_n \diagup H_{n-1} = \{\alpha \in H_*(SP^\infty(X),G) \mid n_\alpha = n\} \cong H_*(SP^n(X),SP^{n-1}(X)).$$

Also, every filtration element H_n is additionally filtered with the groups

$$F_{n,m} = H_m(SP^\infty(X),G) \cap H_n.$$

Now the "we can always add" map $\mu : SP^n(X) \times SP^m(X) \to SP^{n+m}(X)$ induces a Pontriagin product on filtration elements

$$\mu : F_{n,m} \otimes F_{i,j} \to F_{n+i,m+j}.$$

Hence, for "untwisted" coefficients, say for a field $\mathbb{K}$,

$$
\begin{aligned}
H_*(SP^\infty(X),\mathbb{K}) &= \bigoplus_{i=1}^{\infty} H_*(SP^i(X),SP^{i-1}(X),\mathbb{K})\\
&= \bigoplus_{i=1}^{\infty}\bigoplus_{j=1}^{\infty} H_j(SP^i(X),SP^{i-1}(X),\mathbb{K}) = \bigoplus_{\beth=\nvDash}^{\infty}\bigoplus_{\gimel=\nvDash}^{\infty} \mathbb{F}_{\beth,\gimel}\diagup\mathbb{F}_{\beth-\nvDash,\gimel}
\end{aligned}
$$

is a bigraded, commutative associative algebra with a neutral element.

James Milgram in [43] gave an another idea for calculating the homology of symmetric product $SP^\infty(X)$. The first step is a homology decomposition of the space X in a wedge $\bigvee_{i\in I} M_i$ of Moore spaces. So instead of $H_*(SP^\infty(X))$ we calculate $H_*(SP^\infty(\bigvee_{i\in I} M_i))$. Knowing that $SP^\infty(X\vee Y)\simeq SP^\infty(X)\times SP^\infty(Y)$ it can be proved that there is bigraded algebra isomorphism

$$H_*(SP^\infty(X\vee Y),\mathbb{K})\cong \mathbb{H}_*(\mathbb{SP}^\infty(\mathbb{X}),\mathbb{K})\otimes \mathbb{H}_*(\mathbb{SP}^\infty(\mathbb{Y}),\mathbb{K}).$$

Hence,

$$H_*(SP^\infty(X),\mathbb{K})\cong \mathbb{H}_*(\mathbb{SP}^\infty(\bigvee_{\beth\in\mathbb{I}}\mathbb{M}_\beth),\mathbb{K})\cong \bigotimes_{\beth\in\mathbb{I}}\mathbb{H}_*(\mathbb{SP}^\infty(\mathbb{M}_\beth),\mathbb{K}).$$

So it remains to determine $H_*(SP^\infty(M))$ for a Moore space M. Recall that M is a Moore space of type (G,n) if M is CW-complex with one 0-cell and other cells only in dimensions n and $n+1$, such that $H_n(M)=G$ and $\tilde{H}_i(M)=0$ for $i\neq n$. According to Dold-Thom theorem

$$H_*(SP^\infty(X),\mathbb{K})\cong \mathbb{H}_*(\mathbb{K}(\mathbb{G},\ltimes),\mathbb{K}).$$

Finally, $H_*(K(G,n),\mathbb{K})$ is determined from the spectral sequence of the fibre space

$$\begin{array}{ccc} K(H,n) & \longrightarrow & K(G,n) \\ & & \downarrow \\ & & K(K,n) \end{array}$$

where $0\to H\to G\to K\to 0$ is exact sequence of abelian groups.

For example if X is compact Riemann surface of genus g there is a homology isomorphism $H_*(X,\mathbb{Z})\cong \mathbb{H}_*(\bigvee_{\nVdash}^{\nvDash\eth}\mathbb{S}^{\nVdash}\vee\mathbb{S}^{\nvDash})$ and so

$$H_*(SP^\infty(X),\mathbb{Z})\cong \ngeq(\nVdash)\otimes ..\otimes \ngeq(\nvDash\eth)\otimes \lneqq[\mho]$$

where $\deg e_i=1,\ e_i\in F_{1,1},\ \deg f=2,\ f\in F_{1,2}$ and $\Gamma[f]$ is the divided power algebra.

Question: What can one say about the group $H_*(SP^\infty(X),\mathbb{Z})$ in the case of a simply connected 4-manifold X?

2.7 Divisor spaces

Graeme Segal studied in [47] the topology of rational functions of the form $f = p(z)/q(z)$ where p and q are monic polynomials of degree n which do not have a common root. The space F_n of such polynomials can be identified with a subspace of the space M_n of all self maps of the Riemann sphere $S^2 = \mathbb{C} \cup \{\infty\}$ of degree n which take ∞ to 1. The first Segal's result is that the inclusion $F_n \to M_n$ is a homotopy equivalence up to dimension n.

This and other results of Segal served as a motivation for J. Milgram [44] and other mathematicians to study the so called *divisor spaces* or *particle space* as approximations for important, infinite dimensional functional spaces $\mathrm{Map}(U, V)$ and their subspaces. For a 2-dimensional complex manifold X one defines the divisor space

$$\mathrm{Div_k(X) = \{(\Sigma_{i=1}^k x_i, \Sigma_{i=1}^k y_i) \in SP^k(X) \times SP^k(X) \mid \{x_1, .., x_k\} \cap \{y_1, .., y_k\} = \emptyset\}.}$$

Kallel [36] introduces even more general spaces

$$\mathrm{Div_{k_1,..k_n}(X) = \{(\Sigma_{i=1}^{k_j} x_{ij})_{j=1}^n \in \prod_{j=1}^n SP^{k_j}(X) \mid \bigcap_{j=1}^n \{x_{1j}, .., x_{k_j j}\} = \emptyset\}.}$$

By building appropriate model and application of spectral sequence he proved the following result

$$H_*(\mathrm{Div_{k_1,..k_n}(X \backslash \{x\})\mathbb{K},}) \cong \mathrm{Tor}_{\mathrm{2nk-*,k}}^{\mathrm{H_*(SP^\infty(X))}}(\mathbb{K}, \mathbb{H}_*(\mathbb{SP}^\infty(\mathbb{X}); \mathbb{K})^{\otimes \Bbbk})$$

for the Riemann surface X of genus g and a coefficient field $\mathbb{K}$.

2.8 Dupont and Lusztig

Suppose that X is a compact, closed *unorientable* 2-manifold such that

$$\dim(H_1(X, \mathbb{Q})) = \eth\,. \tag{7}$$

The following theorem was proved by J.L. Dupont and G. Lusztig in [18].

Theorem 2.10. *For $n \geq g$, the symmetric product $SP^n(X)$ is diffeomorphic to a $(2n-g)$-dimensional real projective bundle over the g-dimensional torus T^g.*

The proof of Dupont and Lusztig follows the idea of the proof of Theorem 2.4. The starting point is an observation that for a given unorientable X, satisfying the condition (7), there exists a Riemann surface Y together with a fixed point free, antiholomorphic involution $T : Y \to Y$ such that $X \cong Y/(\mathbb{Z}/2)$ with the $\mathbb{Z}/2$-action determined by T. Then the genus of Y is g. Let $\mathcal{J}_{2n}$ be the set of all isomorphism classes of holomorphic line bundles on Y with the Chern class equal to $2n$. Then $\mathcal{J}_{2n}$ is a free homogeneous space of the complex torus (identified as $\mathcal{J}_0$), of complex dimension g.

Let L_y be the holomorphic line bundle associated to the divisor y for some $y \in Y$. Following [41], the map

$$(y_1, \ldots, y_{2n}) \mapsto L_{y_1} \otimes \ldots \otimes L_{y_{2n}}$$

defines for $n \geq g$ a holomorphic projective bundle over $\mathcal{J}_{2n}$,

$$\mathbb{CP}^{2n-g} \longrightarrow SP^{2n}(Y) \xrightarrow{\ \Phi\ } \mathcal{J}_{2n} \tag{8}$$

The involution T acts antiholomorphically on $SP^{2n}(Y)$ by the formula

$$T(y_1, \ldots, y_{2n}) = (Ty_1, \ldots, Ty_{2n})$$

and on $\mathcal{J}_{2n}$ by the formula $T(L) = T^*(\bar{L})$. It is easily verified that the map Φ in (8) is equivariant with respect to this action. Hence it takes one fibre of this map into another in an antiholomorphic way preserving their projective structures.

One deduces from here that the fixed point set A of $T : SP^{2n}(Y) \to SP^{2n}(Y)$ is a real projective bundle over a union of components, denoted by $\mathcal{J}'_n$, of the fixed point set B of $T : \mathcal{J}_{2n} \to \mathcal{J}_{2n}$. Using the fact that $T : Y \to Y$ is fix-point free, one sees that A can be identified with $SP^n(Y/T) = SP^n(X)$, hence in particular it is connected. The space B is a free homogeneous space of the subgroup of $\mathcal{J}_0$ fixed by T, hence a union of g-dimensional tori. It follows that $\mathcal{J}'_n$ is a real g-dimensional torus and the theorem is proved.

2.9 New invariants of 3–manifolds

In this section we briefly outline the role of symmetric powers of Riemann surfaces in a recent progress [46] in constructing invariants of 3-manifolds via Floer homology. Floer [19] originally defined his groups for a symplectic manifold (M, ω)

and a pair Σ_1 and Σ_2 of its Lagrangian submanifolds. P. Ozsváth and Z. Szabó show in [46] how a similar theory, producing new invariants of 3-manifolds, can be developed with the symmetric power $SP^g(M_g)$ in the role of the symplectic manifold M.

Recall that a Heegaard splitting of a 3-manifold U is a decomposition of the form $U = U_0 \cup_\Sigma U_1$, where U_0 and U_1 are two handlebodies glued together by an orientation preserving diffeomorphism $\phi : \partial(U_0) \to \partial(U_1)$ of their boundaries. The common boundary Σ is assumed to be a Riemann surface M_g of genus g. The isotopy class of the diffeomorphism ϕ, and the associated Heegaard splitting, are determined by two collections $\{\alpha_1, \ldots, \alpha_g\}$ and $\{\beta_1, \ldots, \beta_g\}$ of simple, closed curves in $\Sigma = M_g$. The symmetric power $SP^g(\Sigma)$ is a complex manifold, and if $\langle \cdot, \cdot \rangle$ is an associated Hermitian metric and J its complex structure, then the 2-form ω defined by $\omega(X,Y) := \langle X, JY \rangle$, (if closed) turns $SP^g(\Sigma)$ into a symplectic manifold. Actually Ozsváth and Szabó show how Floer homology groups, reflecting the properties of the input 3-manifold U, can be defined with the torii $T_1 = \alpha_1 \times \ldots \times \alpha_g$ and $T_2 = \beta_1 \times \ldots \times \beta_g$ in the role of Lagrangian submanifolds Σ_1 and Σ_2. These are in general only totally real submanifolds of $SP^g(\Sigma)$ and in order to define Floer homology groups they need an additional hypothesis that U carries a structure of a Spin^c manifold.

This is just a beginning of a beautiful and interesting theory and the reader is referred to [46] and subsequent publications for details.

3 Symmetric powers of open surfaces

3.1 Signature of symmetric products of punctured surfaces

Definition 3.1. *Given a Riemann surface of genus g, the associated* open *or* punctured *surface $M_{g,k}$ is defined by*

$$M_{g,k} = M_g \setminus \{\alpha_1, \ldots, \alpha_k\}$$

where $\{\alpha_1, \ldots, \alpha_k\}$ is a collection of k distinct points in M_g.

One of the main results of [7] is the proof of the existence of punctured surfaces $M = M_{g,k}$ and $N = M_{g',k'}$ such that the associated symmetric products $SP^{2n}(M)$

and $SP^{2n}(N)$ are not homeomorphic although M and N have the same homotopy type. Actually it was shown that this is the case for the punctured surfaces $M_{g,k}$ and $M_{g',k'}$ which satisfy the conditions:

- $2g + k = 2g' + k'$,
- $g \neq g'$ and $\max\{g, g'\} \geq n$.

The key ingredient in the proof that the associated symmetric products $SP^{2n}(M_{g,k})$ and $SP^{2n}(M_{g',k'})$ are not homeomorphic is the computation of the signature of these (open!) manifolds.

Theorem 3.2. ([7])

$$\mathrm{Sign}(SP^{2n}(M_{g,k})) = (-1)^n \binom{g}{n} . \tag{9}$$

3.2 A connection with $(m+k, m)$-groups

An $(m+k, m)$-grupoid or a vector valued grupoid (G, f) is simply a map $f : G^{m+k} \to G^m$. The analogs of commutativity, associativity and other algebraic laws can be formulated for these objects and the corresponding algebraic structures are called $(m+k, m)$-semigroups, $(m+k, m)$-groups etc.

The theory of vector valued algebraic structures was developed in the eighties by G. Čupona, D. Dimovski, K. Trenčevski and their collaborators, see [51], [52] and the references in [51]. Perhaps a motivation for the study of these objects, aside from the intrinsic algebraic interest, can be found in a growing interest in vector valued structures following the development of the *theory of operads*, [39].

Our point of departure is an observation[2] that if (M, f) is a topological, commutative $(m+k, m)$-group, then the symmetric product $SP^m(M)$ admits the structure of a commutative Lie group.

Here are relevant excerpts from [51]:

- **Theorem 6.1.** ([52] *Theorem 3.5* & *Prop. 3.2 Chap. III) If (M, f) is locally euclidean, topological, commutative $(m+k, m)$-group for $m \geq 2$, then*

[2]We are indebted to Prof. Kostadin Trenčevski for the information that the symmetric products of surfaces are relevant for the theory of $(m+k, m)$-groups.

$dim(M) = 2$, M is oriented manifold not homeomorphic to the sphere S^2 and

$$SP^m(M) \cong \mathbf{R}^u \times (S^1)^v.$$

- **Conjecture.** *Each connected, locally euclidean, topological, commutative $(m+k, m)$-group is isomorphic to an affine com$(m+k, m)$-group.*

It is clear that the results from Section 3.1 are relevant for this conjecture. In other words the signature computation (Theorem 3.2) rule out many open surfaces M as possible ground spaces for a structure of a locally euclidean $(m+k, m)$-group. A more detailed analysis will be published in a subsequent publication.

4 Genera of symmetric powers

Let X be a closed, oriented manifold with orientation preserving action of a finite group G. Rational cohomology of the orbit space X/G is naturally isomorphic to $G-$invariant part of rational cohomology of X. The equivariant Euler characteristic is defined as

$$\chi(g, X) = \sum_{j\geq 0}(-1)^j \operatorname{tr} g^*|_{H^j(X)}.$$

It is an easy fact from representation theory that

$$\chi(X/G) = \frac{1}{|G|}\sum_{g\in G}\chi(g, X).$$

The same formula holds for signatures of manifolds [55]

$$\operatorname{sign}(X/G) = \frac{1}{|G|}\sum_{g\in G}\operatorname{sign}(g, X).$$

Let $\varphi : \Omega^* \otimes \mathbb{Q} \longrightarrow \mathbb{R}$ be an arbitrary Hirzebruch genus, i.e. a homomorphism from an appropriate bordism ring to a ring R (usually integers or complex numbers). It means that φ behaves well under disjoint sums and products of manifolds equipped with some additional structures (orientation, almost complex or spin etc.).

Let $\mathcal{E} : \Gamma(E_0) \xrightarrow{D_0} \Gamma(E_1) \xrightarrow{D_1} \cdots \xrightarrow{D_{q-1}} \Gamma(E_q)$ be an elliptic complex of differential operators on X whose index $ind\{\mathcal{E}\} = \sum_{|=\prime}^{\mathrm{II}}(-\infty)^{|}\dim_{\mathbb{C}}\mathcal{H}^{|}(\mathcal{E})$ is equal to the genus $\varphi(X)$. Classical complexes and operators, such as the de Rham complex for oriented manifolds, the Dolbeault complex for complex manifolds and the Dirac operator for $Spin^c$ manifolds, are all elliptic. By the Atiyah-Singer Index Theorem [34], the indices of these complexes can be calculated topologically by the formula

$$ind(\mathcal{E}) = \Big((\sum_{\rangle=\prime}^{\mathrm{II}} (-\infty)^{\rangle} \rfloor \langle(\mathcal{E}_{\rangle})) \prod_{|=\infty}^{\backslash} (\frac{\S_|}{\infty - \rceil^{-\S_|}} \cdot \frac{\infty}{\infty - \rceil^{\S_|}}) \Big) [\mathcal{X}],$$

where x_i are the Chern roots of X and $ch(E_i)$ are the Chern characters of bundles E_i. The equivariant genus for manifolds with finite group action compatible with operators in the given complex $\mathcal{E}$ on X is defined by

$$\varphi(g, X) = \sum_{j=0}^{q} (-1)^j \mathrm{tr}\, g^*|_{H^j(\mathcal{E})}.$$

Motivated by the formulas for the Euler characteristic and the signature, we define the *orbit* $\varphi-$genus of the orbit space as the averaging sum of equivariant genera

$$\varphi(X/G) = \frac{1}{|G|} \sum_{g \in G} \varphi(g, X). \tag{10}$$

The cyclic index $\phi(\omega_r, M^r)$ plays a special role in calculations of genera of symmetric powers defined by (10). Namely,

$$\varphi(SP^n(M)) = \frac{1}{n!} \sum_{\sigma \in S_n} \varphi(\sigma, M^n) = \sum_{\alpha_1 + 2\alpha_2 + \cdots n\alpha_n = n} \frac{\prod_{r=1}^{n} \varphi(\omega_r, M^r)^{\alpha_r}}{1^{\alpha_1} \cdots n^{\alpha_n} \alpha_1! \cdots \alpha_n!},$$

where the last equality holds because of the cycle decomposition of permutations. We use the generating function of cyclic index to obtain the corresponding generating functions for genera of symmetric powers

$$\sum_{n=1}^{\infty} \varphi(SP^n(X)) t^n = exp\Big(\sum_{n=1}^{\infty} \varphi(\omega_r, X^r) \frac{t^r}{r} \Big).$$

Let us calculate the *orbit* χ_y-characteristic and the elliptic genus of symmetric powers $SP^n(S_g)$ of complex curves S_g.

Recall that χ_y-characteristic is the index of the Dolbeault complex associated to the complex manifolds X which following [34] can be computed by the formula

$$\chi_y(X) = (\prod_{j=1}^{n} \frac{x_j(1 + ye^{-x_j})}{1 - e^{-x_j}})[X].$$

For a complex curve $X = S_g$ of genus g it follows from the equivariant Atiyah-Singer Index Theorem that

$$\chi_y(\omega_r, S_g^r) = x \cdot \Big(\frac{1+ye^{-x}}{1-e^{-x}} \cdot \frac{1+ye^{i\lambda_1 - x}}{1-e^{i\lambda_1 - x}} \cdots \frac{1+ye^{i\lambda_{r-1} - x}}{1-e^{i\lambda_{r-1} - x}}\Big)[S_g].$$

We need to find the coefficient of x in the Taylor expansion of above product, and it turns out to be

$$\chi_y(\omega_r, S_g^r) = (1-g)(1+(-y)^r).$$

This gives the following [27]

$$\sum_{n=1}^{\infty} \chi_y(SP^n(S_g))t^n = exp\Big((1-g)\sum_{r=1}^{\infty}(1+(-y)^r)\frac{t^r}{r}\Big) = \big((1-t)(1+yt)\big)^{g-1}.$$

In the case of the projective line ($g = 0$), we know that $SP^n(\mathbb{CP}^{\nVdash}) = \mathbb{CP}^{\ltimes}$, and our result agrees with the derivative of logarithm for χ_y-characteristic, which is

$$g'(t) = \sum_{n=1}^{\infty} \chi_y(\mathbb{CP}^{\ltimes})\approx^{\ltimes} = \frac{\nVdash}{(\nVdash - \approx)(\nVdash + \curvearrowright\approx)}.$$

For $y = -1, 0, 1$ we obtain the generating functions for the Euler characteristics, Todd genera and signatures respectively.

The *elliptic genus* $Ell(X)$ is introduced by Witten as the equivariant genus of the natural circle action on the free loop space $\mathcal{LX}$ of a closed, oriented manifold X. Its logarithm is given by the elliptic integral

$$g(y) = \int_0^y (1-2\delta t^2 + \varepsilon t^4)^{-1/2}dt.$$

The characteristic power series of the elliptic genus is $Q_{Ell}(x) = \frac{x}{f(x)}$, where $f(x)$ is the solution of the differential equation $(f')^2 = 1-2\delta t^2 + \varepsilon f^4$. Using its product expansion we have the formula

$$Ell(X) = \varepsilon^{n/2}\Big(\prod_{i=1}^{2n} x_i \frac{1+e^{-x_i}}{1-e^{-x_i}} \prod_{k=1}^{\infty} \frac{1+q^k e^{-x_i}}{1-q^k e^{-x_i}} \cdot \frac{1+q^k e^{x_i}}{1-q^k e^{x_i}}\Big)[X].$$

Hence, for complex curve of genus g, by the equivariant Atiyah-Singer Index Theorem, we have

$$Ell(\omega_r, S_g^r) = \varepsilon^{1/4}\Big(x\prod_{j=0}^{r-1} \frac{1+e^{i\lambda_j - x}}{1-e^{i\lambda_j - x}} \prod_{k=1}^{\infty} \frac{1+q^k e^{-i\lambda_j + x}}{1-q^k e^{-i\lambda_j + x}} \frac{1+q^k e^{i\lambda_j - x}}{1-q^k e^{i\lambda_j - x}}\Big)[S_g].$$

So we are able to determine the needed coefficient of x in above product as

$$Ell(\omega_r, S_g^r) = \begin{cases} 0 & \text{r odd} \\ (2-2g)\varepsilon^{1/4} & \text{r even} \end{cases}$$

which gives

$$\sum_{n=1}^{\infty} Ell(SP^n(S_g))t^n = \frac{1}{(1-t^2)^{(1-g)\varepsilon^{\frac{1}{4}}}}.$$

Note that in the case $g = 0$ it is different from the logarithm of the elliptic genus.

The above formula was proved in greater generality in [56], [9]. Here we describe another approach to orbifold genera, motivated by String theory. There is a definition of orbifold Euler characteristic for manifolds with group actions [33]

$$\chi(X, G) = \sum_{\{g\}} \chi(X^g/C(g)).$$

Generalizing this formula for an arbitrary genus, we define the corresponding orbifold genera

$$\varphi(X, G) = \sum_{\{g\}} \frac{1}{|C(g)|} \sum_{h \in C(g)} \varphi(h, X^g)$$

where $\{g\}$ is the conjugacy class and $C(g)$ is the centralizer of an element $g \in G$.

In the case of orbifold elliptic genera of symmetric powers, for a manifold X with elliptic genus $Ell(X) = \sum_{m,l} c(m,l)y^l q^m$, we have the formula due to Dijkgraaf, Moore, Verlinde, Verlinde [15]

$$\sum_{n \geq 0} Ell(X^n, S_n)t^n = \prod_{i=1}^{\infty} \prod_{l,m} \frac{1}{(1-t^i y^l q^m)^{c(m,l)}}.$$

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