Generalized Functions and Operations (an introduction)

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Introduction

Introductory lectures contain basic definitions and assertions of the distribution theory, structural properties, operations in $\mathcal{D}'$ (emphasizing differentiation) and regularization. They are devoted to students having some knowledge of functional analysis. Everything in these lectures is very well known for many years but since this sophisticated theory is usually used without understanding its foundations, I decided to present it to students of the Sokobanja summer school. Especially, the appendix can be useful to students of theoretical physics who are more interested in mathematical justification of basic constructions of distribution theory. I suggest the reading of the Appendix, first.

The main feature of various generalized function theories is to explain real models in cases when the classical analysis could not give satisfactory justifications of corresponding mathematical models. Especially, the problem of differentiation leads to the notion of a weak derivative and thus to the new concept of a function called generalized function. Results of Sobolev (around 1937.) were in this sense the most important although, at that time, Dirac's calculus already existed for around fifteen years. But the monograph "Theorie des distributions I, II" (1950/51) of L. Schwartz is considered as the most important contribution to the generalized function theory and L. Schwartz is considered as the inventor and the main contributor of distribution theory. He was the first who published a systematized distribution theory on the basis of the theory of locally convex spaces which was very much developed in the end of the forties. There are many approaches but the concept of Schwartz is now days generally accepted and in fact today it is a language of modern mathematics and physics.

1 Space of basic functions

We will always denote by Ω an open set in $\mathbb{R}^n$. Recall, the support of a continuous function $f : \Omega \to \mathbb{C}$ is the adherent set for the set $\{x \in \Omega; f(x) \neq 0\}$. It is denoted by $\text{supp}f$. If $x \in \text{supp}f$, then there does not exist a neighborhood of x where $f = 0$. Since $\overline{A \bigcup B} = \overline{A} \bigcup \overline{B}$ it follows $\text{supp}(f+g) \subset \text{supp}f \bigcup \text{supp}g$, where f and g are continuous.

$\mathcal{C}^m(\Omega), m \in \mathbb{N}_0$ or $m = \infty$, is the set of functions defined on Ω with all continuous derivatives up to order m. $\mathcal{C}_0^m(\Omega)$ is the subset of $\mathcal{C}^m(\Omega)$ consisting of functions with compact supports in Ω.

Clearly, $\mathcal{C}_0^m(\Omega) \subset \mathcal{C}_0^m(\mathbb{R}^n)$. With the usual addition and multiplication by scalars, $\mathcal{C}^m(\Omega)$ and $\mathcal{C}_0^m(\Omega)$ are vector spaces over $\mathbb{C}$. Elements of $\mathcal{C}^\infty(\Omega)$ are called smooth functions.

Example 1.1 The function

$$f(x) = \begin{cases} 0, & |x| \geq 1 \\ \exp(\frac{1}{|x|^2-1}), & |x| < 1 \end{cases} \quad (|x|^2 = x_1^2 + ... + x_n^2),$$

is an element of $\mathcal{C}_0^\infty(\mathbb{R}^n)$ and $\text{supp}f = Z(0,1)$, where $Z(0,1) \subset \mathbb{R}^n$ is the closed ball with the center at 0 and radius 1. (Open ball is denoted by $L(x_0, r)$.) Put $g(x) = c^{-1}f(x),\ x \in \mathbb{R}$, where $c = \int\limits_{\mathbb{R}^n} f(x)dx$. The function:

$$\omega_\varepsilon(x) = \epsilon^{-n} g(x\epsilon^{-1}), \varepsilon > 0, \quad x \in \mathbb{R}, \tag{1.1}$$

will be often used. It has the following properties:

$$\int_{\mathbb{R}^n} \omega_\varepsilon(x)dx = 1, \mathrm{supp}\omega_\varepsilon(x) = \{x; |x| \le \varepsilon\} \text{ and } \omega_\varepsilon(x) \ge 0.$$

With $\varepsilon = \dfrac{1}{n}, n \in \mathbb{N}$, we use the notation

$$\omega_{1/n}(x) = \delta_n(x), \quad x \in \mathbb{R}. \tag{1.2}$$

Denote by $\mathcal{C}_0^m(K), m \in \mathbb{N}_0$ or $m = \infty$, where K is a compact set in Ω ($K \subset\subset \Omega$), a subspace of $\mathcal{C}_0^m(\Omega)$ which elements are supported by K ($\mathrm{supp}\phi \subset K$). We define in $\mathcal{C}_0^\infty(K)$, a sequence of norms $\{p_{K,m};\ m \in \mathbb{N}_0\}$:

$$p_{K,m}(\phi) = \sum_{|j| \le m} \sup_{x \in K} |\phi^{(j)}(x)|.$$

With this norms, $\mathcal{C}_0^\infty(K)$ is a locally convex space; A basis of neighborhoods of zero is constituted by sets

$$V_{K,m,\nu} \equiv \{\psi \in \mathcal{C}_0^\infty(K); p_{K,m}(\psi) < \frac{1}{\nu}\}, \nu \in \mathbb{N}, m \in \mathbb{N}_0.$$

Vector space $\mathcal{C}_0^\infty(K)$ endowed with the quoted topology is the space of test functions $\mathcal{D}(K)$.

It follows that a sequence (ψ_k) in $\mathcal{D}(K)$ converges to $\psi \in \mathcal{D}(K)$ if and only if $(\psi_k^{(\alpha)})$ converges to $\psi^{(\alpha)}$ as $k \to \infty$, uniformly on K, for every $\alpha \in \mathbb{N}_0^n$.

Thorem 1.1 *a) If K is a compact subset of Ω, then there exists $\psi \in \mathcal{C}_0^\infty(\Omega)$ with the properties $0 \le \psi \le 1$ and $\psi(x) = 1$ in a neighborhood of K.*

b) If je $f \in \mathcal{C}^m(\Omega), m \in \mathbb{N}_0$ or $m = \infty$, then for every compact set $K \subset \Omega$ there exists an $f_K \in \mathcal{C}_0^m(\Omega)$ such that $f(x) = f_K(x)$ in a neighborhood of K.

$\mathcal{D}(K)$ is a projective limit of a sequence of normed spaces $(\mathcal{C}_0^\infty(K); p_{K,m})$, $m \in \mathbb{N}_0$ (see the appendix). We will explain this. The inclusion mappings

$$i_s^m, m \geq s, i_s^m : (\mathcal{C}_0^\infty(K), p_{K,m}) \to (\mathcal{C}_0^\infty(K), p_{K,s})$$

are continuous, since $p_{K,0}(\phi) \leq p_{K,1}(\phi) \leq ....$ The sequence of normed spaces with the mappings i_s^m makes a projective spectar. The topology in $\mathcal{D}(K)$ is the coarsest topology such that the mappings $i^m : \mathcal{D}(K) \to (\mathcal{C}_0^\infty(K), p_{K,m}),\ m \in \mathbb{N}_0$, are continuous.

We have that $\mathcal{D}(K)$ is a Fréchet space with the metric

$$d(f,g) = \sum_{m=0}^{\infty} 2^{-m} \frac{p_{K,m}(f-g)}{1 + p_{K,m}(f-g)}.$$

Since every Fréchet space is barreled, it follows that $\mathcal{D}(K)$ is also barreled (see the appendix).

If sets K_1 and K_2 are compact and $K_1 \subset K_2$, then $\mathcal{C}_0^\infty(K_1) \subset \mathcal{C}_0^\infty(K_2)$ and that the topology in $\mathcal{D}(K_1)$ is equal to the topology induced by $\mathcal{D}(K_2)$ on $\mathcal{D}(K_1)$. For an open set $\Omega \subset \mathbb{R}^n$ there exists a sequence (K_ν) of compact sets such that $K_\nu \subset K_{\nu+1}$ i $\bigcup_{\nu=1}^{\infty} K_\nu = \Omega$. Then the sequence of locally convex spaces $(\mathcal{D}(K_\nu))$ with respect to inclusion mappings $i_s^m : \mathcal{D}(K_m) \to \mathcal{D}(K_s), m \leq s$, constitutes a strict inductive spectar. Supply $\mathcal{C}_0^\infty(\Omega)$ by the topology of a strict inductive limit. This is the finest locally convex toplogy in $\mathcal{C}_0^\infty(\Omega)$ such that the identity mappings $i^m : \mathcal{D}(K_m) \to \mathcal{C}_0^\infty(\Omega)$ are continuous (see the appendix). The vector space $\mathcal{C}_0^\infty(\Omega)$ endowed with the above topology is the basic Schwartz space $\mathcal{D}(\Omega)$.

Note that in the definition of toplogy in $\mathcal{D}(\Omega)$ this topology does not depend on a sequence (K_ν) of compact sets with the properties quoted above.

If we take from every space $\mathcal{D}(K_\nu)$ a circled convex neighborhood of zero $U_\nu \in \mathcal{B}_\nu$, where $\mathcal{B}_\nu$ is a basis of neighborhoods of zero in $\mathcal{D}(K_\nu)$, $\nu \in \mathbb{N}$, then the convex hull (envelope) U of the set $V = \bigcup\limits_{\nu\in\mathbb{N}} U_\nu$:

$$U = \{\phi \in \mathcal{C}_0^\infty(\Omega); \phi = \sum_{j=1}^{n} \beta_j\psi_j, \psi_j \in U_{\nu_j}, \beta_j \geq 0, \beta_1 + ... + \beta_n = 1\}$$

is a circled convex neighborhood of zero in the topology of $\mathcal{D}(\Omega)$. In this way we can form the basis of neighborhoods in $\mathcal{D}(\Omega)$.

Thorem 1.2 *(i) A linear mapping T of the space $\mathcal{D}(\Omega)$ into a locally convex space E is continuous if and only if it is continuous on $\mathcal{D}(K)$ for every compact set $K \subset \Omega$.*

(ii) A set $A \subset \mathcal{D}(\Omega)$ is bounded if and only if there exists a compact set $K \subset \Omega$ such that $A \subset \mathcal{D}(K)$ and it is bounded in $\mathcal{D}(K)$.

(iii) A sequence $(\phi_\nu) \in \mathcal{D}(\Omega)^{\mathbb{N}}$ converges in $\mathcal{D}(\Omega)$ if and only if there exists a compact set K such that $(\phi_\nu) \in \mathcal{D}(K)^{\mathbb{N}}$ and that it converges in $\mathcal{D}(K)$

(iv) $\mathcal{D}(\Omega)$ is sequentially complete.

(Moreover, it is well known that a strict inductive limit of complete spaces is complete.)

Example 1.2 Let $f(x)$ be as in Example 1.1. Let $g_\nu(x) = \dfrac{1}{\nu}f(x),\ x \in \mathbb{R}^n,\ \nu \in \mathbb{N}$. All the members of the sequence have supports contained in the closed ball $Z(0.1)$; The sequence (g_ν) converges in $\mathcal{D}(Z(0,1))$ to zero and so it converges in $\mathcal{D}(\mathbb{R}^n)$. On the other hand the sequence $k_\nu(x) = \dfrac{1}{\nu}f(\dfrac{x}{\nu}),\ x \in$

$\mathbb{R}^n$, $\nu \in \mathbb{N}$, does not converge in $\mathcal{D}(\mathbb{R}^n)$, because supp$k_\nu = Z(0,\nu)$, and there does not exist a compact set K containing all sets supp k_ν.

Since the inductive limit of barreled spaces is barreled, we have:

Thorem 1.3 *A closed and bounded set in $\mathcal{D}(\Omega)$ is compact. In particular, $\mathcal{D}(\Omega)$ is a Montel space (see the appendix).*

Note that the space $\mathcal{D}(\Omega)$ is not metrisabile.

2 Space of distributions

A continuous linear functional on $\mathcal{D}(\Omega)$ is called distribution. The space of distributions is denoted by $\mathcal{D}'(\Omega)$. We denote distributions as functions: $f, g, ...$;

$$f : \phi \mapsto \langle f, \phi \rangle = (f, \overline{\phi}),$$

where $\overline{\phi}$ is the complex conjugation for ϕ.

Example 2.1 Dirac distribution $\delta(\cdot - x_0) \in \mathcal{D}'(\Omega)$, $x_0 \in \Omega$, is defined by

$$(\delta(x - x_0), \overline{\phi}(x)) = \langle \delta(x - x_0), \phi(x) \rangle = \phi(x_0), \phi \in \mathcal{D}(\Omega).$$

The following two theorems simplify very much the work with distributions.

Thorem 2.1 *A linear functional $f : \mathcal{D}(\Omega) \to \mathbb{C}$ belongs to $\mathcal{D}'(\Omega)$ if and only if if for every sequence $(\phi_\nu) \in \mathcal{D}(\Omega)^{\mathbb{N}}$ converging to zero in $\mathcal{D}(\Omega)$ it follows that (f, ϕ_ν) converges to zero in the set of complex numbers.*

Thorem 2.2 *Necessary and sufficient condition that a linear functional $f : \mathcal{D}(\Omega) \to \mathbb{C}$ is a distribution is that for every compact set $K \subset \Omega$ there* *exist constants $C > 0$ and $m \in \mathbb{N}$ such that for every $\phi \in \mathcal{D}(K)$*

$$|< f, \phi >| \leq C p_{K,m}(\phi). \tag{2.1}$$

Previous theorems are consequences of the fact that $\mathcal{D}(\Omega)$ is a bornological space (i.e. that a seminorm bounded on a bounded set in $\mathcal{D}(\Omega)$, is continuous on $\mathcal{D}(\Omega)$; see the appendix).

Recall that a family $\{A_i, i \in I\}$ of subsets of Ω is locally finite if for every $x \in \Omega$ there exists a neighborhood of $x, W(x)$, such that $W(x)$ has a non void intersection with at most finite number of sets $A_i, i \in I$.

Thorem 2.3 *A linear functional on $\mathcal{D}(\Omega)$ is a distribution if and only if there exists a family $R = \{\rho_\alpha;\ \alpha \in \mathbb{N}_0^n\}$ of continuous functions on Ω, such that the family of supports $\{\mathrm{supp}\rho_\alpha;\ \alpha \in \mathbb{N}_0^n\}$ is locally finite and*

$$|< u, \phi >| \leq \sum_{\alpha \in \mathbb{N}_0^n} \sup_{x \in \mathbb{R}^n} |\rho_\alpha(x) \partial^\alpha \phi(x)|, \phi \in \mathcal{C}_0^\infty(\Omega).$$

Let $\mathcal{R}$ be a family of all families: $R = \{\rho_\alpha; \alpha \in \mathbb{N}_0^n\}$ in $\mathcal{C}^0(\Omega)$ such that the family $\{\sup \rho_\alpha; \alpha \in \mathbb{N}_0^n\}$ is locally finite. Theorem 2.3 implies that the topology of $\mathcal{D}(\Omega)$ is defined by the uncountable family of seminorms

$$P_R(\phi) := \sum_{\alpha \in \mathbb{N}_0^n} \sup_x |\rho_\alpha \partial^\alpha \phi|,$$

where $R \in \mathcal{R}$. (With this family $\mathcal{C}_0^\infty(\Omega)$ becomes a locally convex space with the dual $\mathcal{D}'(\Omega)$.)

Let $f \in \mathcal{D}'(\Omega)$. Assume that there exists $p \in \mathbf{N}_0$ such that (2.1) holds with $m = p$ and every compact set $K \subset \Omega$. Then the smallest p with this property is called the order of f.

Example 2.2 Let $j \in \mathbb{N}_0^n$ $\Omega \ni 0$. Then $\delta^{(j)}; \phi \mapsto (-1)^{|j|} \phi^{(j)}(0)$ is a distribution of order $|j|$.

Let, now T be defined by

$$T : \phi \to \sum_{j=1}^{\infty} \phi^{(j)}(j), \phi \in \mathcal{D}(\mathbb{R}).$$

This is a distribution of infinite order.

Different topologies in $\mathcal{D}'(\Omega)$. The weak topology is defined by the family of seminorms: $\|f\|_\phi = | < f, \phi > |, \phi \in \mathcal{D}(\Omega)$. The topology of compact convergence is defined by the family of seminorms $\|f\|_K = \text{supp}\{| < f, \phi > |; \phi \in K\}$, where K is a compact subset od $\mathcal{D}(\Omega)$. The strong topology is defined by the family of seminorms: $\|f\|_B = \sup\{\langle |f, \phi\rangle |; \phi \in B\}$, where B is a bounded set in $\mathcal{D}(\Omega)$. They induce Hausdorff topologies. They are denoted by τ_w, τ_c, τ_b, and we have: $\tau_w \leq \tau_c \leq \tau_b$.

Thorem 2.4 *Topologies τ_c i τ_b in $\mathcal{D}'(\Omega)$ are equal.*

Clearly, τ_c is strictly finer than τ_w.

Thorem 2.5 *a) $\mathcal{D}'(\Omega)$ is sequentially complete with respect to the weak topology.*

b) Let (f_n) be a Cauchy sequence in $(\mathcal{D}'(\Omega), \tau_w)$ (in the sense of weak topology). Then it is a convergent one in the sense of strong topology.

In particular the convergence of sequences in the weak and strong topologies in $\mathcal{D}'(\Omega)$ coincide.

Moreover, $\mathcal{D}(\Omega)$ is bornological and it follows that $\mathcal{D}'(\Omega)$ is complete with respect to the strong topology in $\mathcal{D}'(\Omega)$. Also, as $\mathcal{D}(\Omega)$ is a Montel space, we have that $\mathcal{D}(\Omega)$ is reflexive i.e. the set $\mathcal{D}''(\Omega) = (\mathcal{D}'(\Omega))'$, of continuous linear functionals on $\mathcal{D}'(\Omega)$ with respect to the strong topology in $\mathcal{D}'(\Omega)$, equals $\mathcal{D}(\Omega)$ (semi reflectivity) and that the identity mapping

$\mathcal{D}(\Omega) \to \mathcal{D}''(\Omega)$ is continuous with respect to the strong topology in $\mathcal{D}''(\Omega)$ (reflexivity).

It is easy to check that the sequence (δ_n) from Example 1.1 converges weakly to $\delta-$ distribution. Thus, $\delta_n \to \delta$ in the sense of strong topology.

Thorem 2.6 *Let $B' \subset \mathcal{D}'(\Omega)$. The following assertions are equivalent:*

(i) B' is bounded in the weak topology. (ii) B' is bounded in the strong topology. (iii) B' is uniformly continuous subset of $\mathcal{D}'(\Omega)$.

Regular distributions and Radon measures. It is easy to show that if $f \in L^1_{loc}(\Omega)$ then,

$$\phi \mapsto < f, \phi >= \int_{R^n} f(x)\phi(x)dx, \phi \in \mathcal{D}(\Omega),$$

defines an element of $\mathcal{D}'(\Omega)$. It is called regular distribution and it is denoted by $\tilde{f}$.

Thorem 2.7 *a) If a sequence of functions (f_n) iz $L^1_{loc}(\Omega)$ converges to 0 in $L^1_{loc}(\Omega)$, then this sequence determines a sequence of distributions (f_n) which converges to 0 in $\mathcal{D}'(\Omega)$.*

b) If $f, g \in L^1_{loc}(\Omega)$ and if for every $\phi \in \mathcal{D}(\Omega)$, $\langle \tilde{f}, \phi \rangle = \langle \tilde{g}, \phi \rangle$, then f equals g almost everywhere in Ω.

Thus, $L^1_{loc}(\Omega)$ is isomorphic to a subspace of $\mathcal{D}'$, called the space of regular distributions. So, $\mathcal{D}(\Omega)$ is a subspace of $\mathcal{D}'(\Omega)$. Moreover it is a dense subspace of $\mathcal{D}'(\Omega)$.

Note that Dirac δ-distribution given in Example 2.1 is not a regular distribution.

If $u \in L^1_{loc}(\mathbb{R}^n)$, we put

$$u_\varepsilon(x) = (u * \omega_\varepsilon)(x) := \varepsilon^{-n} \int\limits_{\mathbb{R}^n} u(y) g\Big(\frac{x-y}{\varepsilon}\Big) dy =$$

$$\int\limits_{\mathbb{R}^n} u(x - \varepsilon y) g(y) dy, x \in \mathbb{R}^n. \tag{2.2}$$

where g and ω_ε are smooth functions from Example 1.1. It is clear that (2.2) is a smooth function The operation $*$ is called the convolution.

If $u \in L^1_{loc}(\Omega)$, then (2.2) does not have sense, in general. In that case we consider $x \in \Omega_{-\varepsilon}$, where $\Omega_{-\varepsilon} = \{x \in \Omega;\ d(x, (\mathbb{R}^n \setminus \Omega)) > \varepsilon\}$, and by

$$\begin{aligned} u_\varepsilon(x) &= (u * \omega_\varepsilon)(x) = \varepsilon^{-n} \int\limits_{\Omega} u(y) g\Big(\tfrac{x-y}{\varepsilon}\Big) dy \\ &= \int\limits_{|y| \le 1} u(x - \varepsilon y) g(y) dy,\ x \in \Omega_{-\varepsilon} \end{aligned} \tag{2.3}$$

is defined a smooth function on $\Omega_{-\varepsilon}$.

In the first and second part of the next theorem the convolution is understood in the sense of (2.2) because we define u and $u_\nu,\ \nu \in \mathbb{N}$, to be equal zero out of Ω. The sequence $(u_{\nu,\nu})$ in the second part is in $\mathcal{C}^\infty(\mathbb{R}^n)$. In the fourth part convolution is understood in the sense of (2.3).

Thorem 2.8 *(i) If u is an integrable function vanishing out of a compact set $K \subset \Omega$ and if $\varepsilon \le d(K, (\mathbb{R}^n \setminus \Omega))$, then $u_\varepsilon \in \mathcal{C}_0^\infty(\Omega)$.*

(ii) If a sequence (u_ν) in $L^p(\Omega)$, $p \in [1, \infty)$, in $L^p(\Omega)$-norm converges to $u \in L^p(\Omega)$, then $u_{\nu,\nu} = u_\nu * \delta_\nu \in \mathcal{C}^\infty(\mathbb{R}^n)$ for every $\nu \in \mathbb{N}$ and $u_{\nu,\nu} \to u$ u $L^p(\Omega)$.

(iii) If (u_ν) is a sequence of continuous functions in $\mathbb{R}^n$ which converges almost uniformly to u, then $(u_{\nu,\nu})$ is a sequence in $\mathcal{C}^\infty(\mathbb{R}^n)$ which converges almost uniformly to u.

(iv) If (u_ν) is a sequence in $L^p_{loc}(\Omega)$, $p \in [1, \infty)$, which converges to $u \in L^p_{loc}(\Omega)$ (in the sense of $L^p_{loc}(\Omega)$ convergence), then $(u_{\nu,\nu}) \in \mathcal{C}^\infty(\Omega_{-1/\nu})$ and in the sense of $L^p_{loc}(\Omega)$ convergence converges to u.

Thus we have:

Thorem 2.9 *$\mathcal{C}_0^\infty(\Omega)$ is dense in: (i) $L^p(\Omega)$, $p \geq 1$; (ii) $\mathcal{C}_0^0(\Omega)$, (iii) $L^p_{loc}(\Omega)$, $p \geq 1$.*

Since $L^p(\Omega), \mathcal{C}^0(\Omega)$, and $L^p_{loc}(\Omega)$ are subspaces of $L^1_{loc}(\Omega)$, they can be identified with the corresponding subspaces of the space of regular distributions.

Now we introduce the space of **Radon measures**. Let (K_ν) be a sequence of compact subsets of Ω such that $K_\nu \subset K_{\nu+1}$ i $\bigcup\limits_\nu K_\nu = \Omega$. We define in $\mathcal{C}_0^0(K_\nu)$, $\nu \in \mathbb{N}$, the norms

$$P_{K_\nu}(\phi) = \sup_{x \in K_\nu} |\phi(x)|, \qquad \phi \in \mathcal{C}_0^0(K_\nu).$$

The sequence of normed spaces $(\mathcal{C}_0^0(K_\nu),\ P_{K_\nu})$, with respect to identity mappings $i^\nu_{\nu+1} : (\mathcal{C}_0^0(K_\nu), P_{K_\nu}) \to (\mathcal{C}_0^0(K_{\nu+1}), P_{K_{\nu+1}})$, constitutes a strict inductive spectar.

We denote by $\mathcal{K}_c(\Omega)$ the respective strict inductive limit. (One can easily show that the construction does not depend on a chosen sequence of compact sets). Then we have the definition:

Radon (complex) measure on Ω is a continuous linear functional on $\mathcal{K}_c(\Omega)$.

Thorem 2.10 *A necessary and sufficient condition for a linear functional $f : \mathcal{K}_c(\Omega) \to \mathbb{C}$ to be continuous is that for every compact set $K \subset \Omega$ there exists $C > 0$ such that $|\langle f, \phi \rangle| \leq C_{P_K}(\phi)$ for every $\phi \in \mathcal{C}_0^0(K)$.*

We will use the following simple assertion:

If $\mathcal{C}_0^\infty(\Omega)$ is dense in a locally convex space V and if the topology in $\mathcal{D}(\Omega)$ is finer than the topology which V induces on $\mathcal{C}_0^\infty(\Omega)$, then V', the dual of V, is algebraically isomorphic to a subspace of $\mathcal{D}'(\Omega)$ (and we make the corresponding identification).

Thus we have that $\mathcal{C}_0^0(\Omega)$ is dense in $\mathcal{K}_c(\Omega)$ and since the inclusion $\mathcal{D}(\Omega) \to \mathcal{K}_c(\Omega)$ is continuous, the space of Radon measures is isomorphic to a subspace of $\mathcal{D}'(\Omega)$. On the basis of Theorem 2.10 we have that Radon measures are distributions of zero order.

Example 2.3 Dirac δ- distribution is the Radon measure.

Let f be a complex valued function on Ω. Then by

$$\mu(\phi) := \sum_{x\in\Omega} f(x)\phi(x),\ \phi \in \mathcal{C}_0^0(\Omega),$$

with the assumption that these series converge unconditionally, is defined a measure called atomic measure; in short,

$$\mu(t) = \sum_{x\in\Omega} f(t)\delta(t-x),$$

where $\delta(t-x)$ is Dirac $\delta-$distribution concentrated at x.

The unconditional convergence means that for every $\varepsilon > 0$ there exists a finite set $J_0(\varepsilon) \subset \Omega$, such that for every finite set set $J \subset \Omega$ such that $J_0(\varepsilon) \subset J$, it follows $\left|\sum_{x\in J} f(x)\phi(x) - c\right| < \varepsilon$, where $c \in \mathbb{C}$ is the sum of this series.

Example 2.4 Lebesgue measure is defined by the Riemann integral:

$$\mu(\phi) = \int_Q \phi(t)dt, \qquad \phi \in \mathcal{C}_0^0(\Omega),$$

where Q is a ball containing the support of ϕ. One can show that the Lebesgue measure is the only one Radon measure on $\mathbb{R}^n$ (up to the product with a positive constant) such that for any $\phi \in \mathcal{C}_0^0(\mathbb{R}^n)$ and any $h \in \mathbb{R}^n$: $\mu(\phi(\cdot - h)) = \mu(\phi)$.

Example 2.5 If f is a continuous function on Ω and ν is a given Radon measure on Ω, then by

$$\mu(\phi) = \nu(f \cdot \phi) \tag{2.4}$$

is defined a Radon measure. We denote μ by $f \cdot \nu$. Moreover, one can show that μ is well defined by (2.4) if f is a locally integrable function with respect to ν. Then it is said that $\mu = f \cdot \nu$ has the density f or that μ has a "derivative" f with respect to ν.

Example 2.6 Lebesgue-Stieltjes measure is defined similarly as the Lebesgue measure (Example 2.4) but with the Lebesgue -Stieltjes integral instead of the Lebesgue integral (with a corresponding function of locally bounded variation).

Sheaf properties. Distributions f and g in $\mathcal{D}'(\Omega)$ are equal on an open set $\Omega_1 \subset \Omega$ if and only if their restrictions on Ω_1 (distributions applied on test functions supported by Ω_1) are equal on Ω_1. It is said that they are equal in a neighborhood of $x \in \Omega$ if they are equal in an open neighborhood of this point.

Thorem 2.11 *Let $f, g \in \mathcal{D}'(\Omega)$ such that they are equal in the neighborhood of every point of Ω. Then $f = g$ in Ω.*

The proof of this theorem is based on the following finite partition of unity:

Let $\Omega_1, ..., \Omega_k$ be open sets and let K be a compact set such that $K \subset \bigcup_{i=1}^k \Omega_i = \Omega$ and $K \cap \Omega_i \neq \emptyset$, $i = 1, ..., k$. Then there exist $\phi_i \in \mathcal{C}_0^\infty(\Omega_i)$, $i = 1, ..., k$, such that $\phi_i \geq 0$; $\sum_{i=1}^k \phi_i \leq 1$ and $\sum_{i=1}^k \phi_i(x) = 1$, in a neighborhood of K in Ω.

Now we can define the support of a distribution: The support of $f \in \mathcal{D}'(\Omega)$, supp$f$, is the set of points in Ω having a neighborhood where f is not equal to zero.

In other words, the support of $f \in \mathcal{D}'(\Omega)$ is the complement of the union of all open sets in Ω where $f = 0$. It is a closed set.

The next notion is of a great importance in the theory of partial differential equations.

The singular support of $f \in \mathcal{D}'(\Omega)$, sing supp$f$, is the set of points in Ω which do not have a neighborhood where f is a smooth function (i.e. f is defined as a regular distribution by a smooth function).

In other words the singular support is the complement of the union of all open sets where f is a smooth function.

Partition of unity. A family $\{U_\alpha;\ \alpha \in A\}$ of subsets of a topological space X is a cover of $W \subset X$ if $W \subset \bigcup_{\alpha \in A} U_\alpha$. It is an open cover if every U_α is an open set. Subfamily of $\{U_\alpha\}$ which also covers W is called a sub cover and $\{V_\beta;\ \beta \in B\}$ is called a refinement of $\{U_\alpha\}$ if it covers W and for every $\beta \in B$ there exists $\alpha \in A$ such that $V_\beta \subset U_\alpha$. Recall, a family $\{A_i;\ i \in I\}$ is locally finite in X if for every $x \in X$ there exists its neighborhood ω_x such that $\omega_x \cap A_i \neq \emptyset$ only for a finite number of indices $i \in I$.

We will assume that a topological space X is paracompat (every open cover of X has a locally finite sub cover).

Partition of unity on an open set $\Omega \subset X$ is a family $\{\phi_i;\ i \in I\}$ of continuous functions on Ω such that:

(a) The family of supports $(\text{supp}\phi_i;\ i \in I\}$ is locally finite.

(b) $\sum_{i \in I} \phi_i(x) = 1$ for every $x \in \Omega$ and $\phi_i(x) \geq 0$ for every $x \in \Omega$, $i \in I$.

(Note for every x the sum is finite.)

The partition of unity $\{\phi_i;\ i \in I\}$ corresponds to the cover $\{U_\alpha;\ \alpha \in A\}$ if for every $i \in I$ there exists α_i such that $\mathrm{supp}\phi_i \subset U_{\alpha_i}$.

Since the set $\mathbb{R}^n$ is paracompact we have:

Thorem 2.12 *Let $\Omega \subset \mathbb{R}^n$ and $\{U_\alpha;\ \alpha \in A\}$ be an open cover of Ω. Then there exist a countable partition of unity $\{\phi_i;\ i \in \mathbb{N}\}$ which corresponds to $\{U_\alpha\}$ such that functions ϕ are smooth and* $\mathrm{supp}\phi_i$ *are compact sets, for every $i \in \mathbb{N}$.*

Now one can prove the (second sheaf) property:

Thorem 2.13 *Let $\{\Omega_\alpha;\ \alpha \in A\}$ be a family of open set and let $f_\alpha \in \mathcal{D}'(\Omega_\alpha)$ $\alpha \in A$. Assume that for every $\alpha, \beta \in A$ such that $\Omega_\alpha \cap \Omega_\beta \neq \emptyset$ there holds: $f_\alpha = f_\beta$ on $\Omega_\alpha \cap \Omega_\beta$. Then, there exists one and only one distribution f on $\Omega = \bigcup\limits_\alpha \Omega_\alpha$ such that $f = f_\alpha$ on Ω_α.*

Without explanations, note that the sheaf of distributions is a fine sheaf but not flabby.

3 Operations in $\mathcal{D}'(\Omega)$

Operations on the whole $\mathcal{D}'(\Omega)$ are called regular operations while those operations defined only on appropriate distributions (proper subset of $\mathcal{D}'(\Omega)$) are called irregular ones.

Change of variables. Let

$$x = Ay + b, \qquad y \in \Omega_1, \tag{3.1}$$

where A is a regular linear transformation of Ω_1 onto Ω and $b \in \mathbb{R}^n$. (Ω and Ω_1 are open sets in $\mathbb{R}^n$). Let $f \in \mathcal{D}'(\Omega)$. Then $f(Ay+b) \in \mathcal{D}'(\Omega_1)$ is defined by

$$\langle f(Ay+b),\phi(y)\rangle := \langle f(x),\frac{\phi(A^{-1}(x-b))}{|detA|}\rangle,\ \phi\in\mathcal{D}(\Omega_1). \tag{3.2}$$

Since $\phi(y)\mapsto\phi(A^{-1}(x-b))$ is a mapping from $\mathcal{D}(\Omega_1)$ into $\mathcal{D}(\Omega)$, it follows that $f(Ay+b)\in\mathcal{D}'(\Omega_1)$.

If in (3.1) $A=I$ (I is the identity matrix), $b=-h=-(h_1,...,h_n)$, then (3.2) defines the translation: $\langle f(t-h),\phi(t)\rangle := \langle f(t),\phi(t+h)\rangle$.

If $A=-I,\ b=0$, (3.2) defines the transposition: $\langle f(-t),\ \phi(t)\rangle = \langle f(t),\phi(-t)\rangle$.

If $A=aI,\ a\neq 0,\ b=0$, (3.2) defines the homotety:

$$\langle f(at),\phi(t)\rangle := \langle f(t),\frac{1}{|a|^n}\phi(\frac{t}{a})\rangle.$$

It is said that $f\in\mathcal{D}'(\mathbb{R}^n)$ is homogeneeous of order λ if for every homotety $y=ax,\ a>0$: $f(ax)=a^\lambda f(x)$ i.e.

$$\langle f(x),\phi(\frac{x}{a})\rangle = a^{\lambda+n}\langle f(x),\phi(x)\rangle,\ \phi\in\mathcal{C}_0^\infty(\mathbb{R}^n),\ a>0.$$

This follows from

$$\langle f(ax),\phi(x)\rangle = \frac{1}{a^n}\langle f(x),\phi(\frac{x}{a})\rangle;\ \langle f(ax),\phi(x)\rangle = a^\lambda\langle f(x),\phi(x)\rangle.$$

For example, $\delta(x)$ is homogeneous of order $-n$.

If A is an orthogonal matrix ($A^{-1}=A^T$) and $b=0$, then (3.2) defines the rotation:

$$\langle f(At),\phi(t)\rangle := \langle f(t),\phi(A^Tt)\rangle \quad (|detA|=1).$$

For example, $\delta(x)$ is invariant with respect to the rotation around zero.

In general, let $y_1,...,y_n$, be smooth functions defined in $\Omega\subset\mathbb{R}^n$ such that

$$\alpha : x\mapsto y=(y_1(x),...,y_n(x)) \tag{3.3}$$

is the bijection of Ω onto $\Omega_1 \subset \mathbb{R}^n$ (then at every point $x \in \Omega$ the Jacobian

$$J = \frac{\partial(y_1, ..., y_n)}{\partial(x_1, ..., x_n)} := \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial y_n}{\partial x_1} & \cdots & \frac{\partial y_n}{\partial x_n} \end{vmatrix}$$

is different from zero). Mapping (3.3) has the smooth inverse $\alpha^{-1} : \Omega_1 \ni y \mapsto x \in \Omega$ and the mapping $\mathcal{D}(\Omega) \to \mathcal{D}(\Omega_1)$

$$\phi(x) \mapsto \phi(\alpha^{-1}(y))|J^{-1}(y)|, \quad y \in \Omega_1, \ x \in \Omega \tag{3.4}$$

is continuous. Thus, for $f \in \mathcal{D}'(\Omega_1)$,

$$\langle f(\alpha(x)), \phi(x)\rangle := \langle f(y), \phi(\alpha^{-1}(y))|J^{-1}(y)|\rangle, \ \phi \in \mathcal{D}(\Omega)$$

defines a mapping $\mathcal{D}'(\Omega_1) \to \mathcal{D}'(\Omega)$. It is adjoined to (3.4) : $f(y) \mapsto f(\alpha(x))$. It is called the pull back and denoted by α^* ($\alpha^* f(x) = f(\alpha(x))$). This mapping is continuous with respect to weak, respectively strong, topologies in $\mathcal{D}'(\Omega_1)$ and $\mathcal{D}'(\Omega)$.

Note, if a distribution $f(y)$ is defined by $f \in L^1_{loc}(\Omega_1)$, then $\alpha^* f$ is defined by the locally integrable function $f(\alpha(\cdot))$.

The change of variables can be defined as an irregular operation in cases when the above assumptions do not hold for α. Let us show this.

Let $\alpha : \Omega \to \Omega_1$ be a smooth mapping and $f \in \mathcal{D}'(\Omega_1)$. Let δ_n be as in (1.2) and $f_n = f * \delta_n$. If $\lim_{n\to\infty} f_n(\alpha(x))$ exists in $\mathcal{D}'(\Omega)$, then

$$f(\alpha(x)) := \lim_{n\to\infty} f_n(\alpha(x)) \quad \text{in } \mathcal{D}'(\Omega). \tag{3.5}$$

$$(\ \langle f(\alpha(x)), \phi\rangle = \lim_{n\to\infty} \langle f_n(\alpha(x)), \phi\rangle.\ )$$

Example 3.1 Using (3.5), one can show:

$$\delta(x^2 - a^2) = \frac{1}{2a}(\delta(x-a) + \delta(x+a)).$$

Value at the point. Let $f \in \mathcal{D}'(\mathbb{R}^n), x_0 \in \mathbb{R}^n$ and $y = \varepsilon x + x_0$ ($A = \varepsilon I,\ \varepsilon \in \mathbb{R},\ I$ is the identity matrix). If there is a constant C such that for

$$\lim_{\varepsilon \to 0} \langle f(x_0 + \varepsilon x), \phi(x) \rangle = \langle C, \phi \rangle,\ \phi \in \mathcal{C}_0^\infty(\mathbb{R}^n), \tag{3.6}$$

then it is said that$f$ has a value C at the point $x = x_0; \Lambda; f(x)_{x=x_0} = C$.

Note that (3.6) holds if and only if (3.6) holds for every sequence $\varepsilon_n \to 0,\ n \to \infty$.

The value at the point is an irregular operation. Delta distribution $\delta(x)$ does not have the value at $x_0 = 0$. Also the Heaviside function (distribution)

$$H(x) = \begin{cases} 1 & x > 0 \\ 0 & x \leq 0 \end{cases}$$

does not have a value at $x_0 = 0$. But if $\tilde{f}(x)$ is a regular distribution determined by a continuous function $f(x)$ such that $f(x_0) = C$, then the corresponding regular distribution$\tilde{f}(x)$ has the value C at $x = x_0$. This is a consequence of the Lebesgue theorem.

Let $f(x, y) \in \mathcal{D}'(\mathbb{R}^n \times \mathbb{R}^m),\ x_0 \in \mathbb{R}^n$. If for every $\phi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ and every $\psi \in \mathcal{C}_0^\infty(\mathbb{R}^m)$ there exists $g(y) \in \mathcal{D}'(\mathbb{R}^m)$ such that

$$\lim_{\varepsilon \to 0} \langle f(x_0 + \varepsilon x, y), \phi(x)\psi(y) \rangle = \int_{\mathbb{R}^n} \phi(x) dx \langle g(y),\ \psi(y) \rangle,$$

then it is said that $g(y)$ is the value of $f(x, y)$ over the manifold $\{x_0\} \times \mathbb{R}^m$, $f(x, y)|_{x=x_0} = g(y)$.

Multiplication of a distribution by a smooth function. Let $a \in \mathcal{C}^\infty(\Omega)$. Then the mapping:

$$\mathcal{D}(\Omega) \to \mathcal{D}(\Omega),\ \phi \mapsto a\,\phi,$$

is continuous and the adjoined mapping $\mathcal{D}'(\Omega) \to \mathcal{D}'(\Omega)$:

$$\langle a(x)f(x), \phi(x)\rangle := \langle f(x), a(x)\phi(x)\rangle, \quad \phi \in \mathcal{D}(\Omega),$$

is continuous with respect to weak, respectively strong, topologies in $\mathcal{D}'(\Omega)$.

If for example f is a measure, then we can assume weaker condition on a. We can assume that $a \in L^1_{loc}(\Omega)$ with respect to the measure f (cf. Example 3.3). Also, if f is a distribution of order m, then $a(x)f(x)$ is defined for $a \in \mathcal{C}^m(\Omega)$.

In general the product of two distributions is an irregular operation. It can be defined through the product of their regularized sequences or in case of "good position" of their wave fronts. This will not be explained in this short presentation.

Differentiation of distributions. Let $f \in \mathcal{D}'(\Omega)$. Then $\dfrac{\partial}{\partial x_i} f = f^{(e_i)}$ is defined by

$$\langle f^{(e_i)}(x), \phi(x)\rangle := -\langle f(x), \frac{\partial \phi(x)}{\partial x_i}\rangle, \; i = 1, ..., n. \tag{3.7}$$

Clearly, if $(\phi_\nu) \to 0$, then $\left(\dfrac{\partial \phi_\nu}{\partial x_i}\right) \to 0$, $\nu \to \infty$, in $\mathcal{D}(\Omega)$. This means that with (3.7) is defined a mapping $\mathcal{D}'(\Omega) \to \mathcal{D}'(\Omega)$ i.e., every distribution has all partial derivatives:

$$\langle \partial^\alpha f, \phi\rangle = (-1)^{|\alpha|}\langle f, \phi^{(\alpha)}\rangle$$

and the order of differentiation does not change the value $\partial^\alpha f$ in $\mathcal{D}'(\Omega)$.

Thorem 3.1 *Let $\alpha \in \mathbb{N}_0$. Then the differentiation ∂^α is a continuous and linear mapping $\mathcal{D}'(\Omega) \to \mathcal{D}'(\Omega)$ with respect to the weak (strong) topology in $\mathcal{D}'(\Omega)$.*

Example 3.2 Let us show $H'(x) = \delta$. For every $\phi \in \mathcal{D}(\mathbb{R})$, we have

$$\langle H'(x), \phi(x)\rangle = -\langle H(x), \phi'(x)\rangle = -\int_0^\infty \phi'(x)dx = \phi(0) = \langle \delta(x), \phi(x)\rangle.$$

Example 3.3 We have

$$\langle \delta^{(j)}(x-x_0), \phi(x)\rangle = (-1)^{|i|}\phi(x_0), \quad \phi(x) \in \mathcal{C}_0^\infty(\mathbb{R}).$$

Example 3.4 If we apply Theorem 3.1 on a (classically) convergent series, after differentiation we obtain a series which converges in $\mathcal{D}'(\Omega)$ although it does not converge in the classical sense.

Let us show that the trigonometric series $\sum_{\nu=-\infty}^{\infty} b_\nu e^{i\nu x}$, $\quad x \in \mathbb{R}$, converges in $\mathcal{D}'(\mathbb{R})$ if there exist $M > 0$ and $k \in \mathbb{N}_0$ such that

$$|b_\nu| \leq M|\nu|^k, \qquad \nu \neq 0. \tag{3.8}$$

We have

$$b_0 + \tilde{g}^{(p)} = \sum_{\nu=-\infty}^{\infty} b_\nu e^{i\nu x},\ x \in \mathbb{R},$$

where $p \in \mathbb{N}_0,\ p \geq k+2$, and $g(x)$ is a continuous periodic function defined by

$$g(x) = \sum_{\substack{\nu=-\infty \\ \nu\neq 0}}^{\infty} \frac{b_\nu e^{i\nu x}}{(i\nu)^p},\quad x \in \mathbb{R}^n. \tag{3.9}$$

($\tilde{g}$ denotes a regular distribution determined by g.) In fact from (3.8) it follows $\left|b_\nu/(i\nu)^p\right| \leq M/\nu^2,\ \nu \neq 0$. So, $g(x)$ is a continuous function and by differentiation (3.9) p-times, we have

$$b_0 + \tilde{g}^{(p)} = \sum_{\nu=-\infty}^{\infty} b_\nu e^{i\nu x}.$$

Example 3.5 One can easily prove the Leibnitz rule for $a(x)f(x),\ a \in \mathcal{C}^\infty(\Omega),\ f(x) \in \mathcal{D}'(\Omega)$:

$$\frac{\partial}{\partial x_j}(a(x)f(x)) = \frac{\partial}{\partial x_j}a(x)f(x) + a(x)\frac{\partial}{\partial x_j}f(x).$$

or for general $k \in \mathbb{N}^n$:

$$\partial^k(a(x)f(x)) = \sum_{|\alpha|\leq k} \binom{k}{\alpha} a^{(\alpha)}(x) f^{(k-\alpha)}(x). \tag{3.10}$$

We will generalize (3.10): Let $p(y),\ y \in \mathbb{R}^n$, be a polynomial of order $m \in \mathbb{N}$, $p(y) = \sum\limits_{|\alpha|\le m} c_\alpha y^\alpha$ ($y^\alpha = y_1^{\alpha_1}...y_n^{\alpha_n} \in \mathbb{R}$). Then we put $p(\partial) = \sum\limits_{|\alpha|\le m} c_\alpha \partial^\alpha$ and have (Hörmander's formula) for $f \in \mathcal{D}'(\mathbb{R}^n)$ and $a \in \mathcal{C}_0^\infty(\mathbb{R}^n)$:

$$p(\partial)(af) = \sum_{|\alpha|\le m} \frac{1}{\alpha!}\partial^\alpha a\, p^{(\alpha)}(\partial) f.$$

Relations of classical and distributional derivatives. Let a function $f(x),\ x \in \mathbb{R}$, have the continuous derivative in the open set $\mathbb{R} \setminus \bigcup_{j=1}^k \{x_i\}, \quad x_j \in \mathbb{R},\ j = 1, ..., k$. Assume that the jump of f at $x = x_j$ is $s_j = f(x_j + 0) - f(x_j - 0)$. Since f defines a regular distribution $\tilde{f}$ in $\mathcal{D}'(\mathbb{R})$, we have

$$\langle \frac{d}{dx}\tilde{f}, \phi\rangle = -\int_{-\infty}^{\infty} f(x)\phi'(x)dx = \sum_j \phi(x_j)s_j + \int_{-\infty}^{\infty} f'(x)\phi(x)dx,$$

where f' is a function equal to the ordinary derivative of $f(x)$ for $x \neq x_j,\ 1 \le j \le k$. This means

$$\frac{d}{dx}\tilde{f} = \tilde{f}' + \sum_j s_j \delta_{x_j}$$

where $\tilde{f}'$ is a distribution defined by f'.

Thorem 3.2 *If $f \in \mathcal{C}^p(\Omega)$, then the regular distribution $\tilde{f}$ satisfies:*

$$D^i\tilde{f} = (\widetilde{D^i f}) \quad \textit{for} \ \ |i| \le p \ \ (\Omega \subset \mathbb{R}^n,\ i \in \mathbb{N}^n).$$

Example 3.6 Let $f \in \mathcal{C}^1(\overline{\Omega})$ where $\overline{\Omega} = \Omega \cup \partial\Omega$ is a closed bounded set with the boundary $S = \partial\Omega$ consisting of a finitely many closed $(n-1)$-dimensional surfaces of $\mathcal{C}^1$-class which do not intersect each other. Put

$f = 0$ out of $\bar{\Omega}$. By partial integration we have

$$\langle \frac{\partial}{\partial x_j}\tilde{f}, \psi\rangle = -\int\limits_{\Omega} f(x)\frac{\partial}{\partial x_j}\psi(x)dx =$$

$$= \int\limits_{S} f(x)\psi(x)\cos(\nu, x_j)dS + \int\limits_{\Omega} \frac{\partial f}{\partial x_j}\psi(x)dx, \quad \psi \in \mathcal{C}_0^\infty(\mathbb{R}^n),$$

where ν is the outer normal on S (on $\mathcal{C}^1-$ parts of S) and (ν, x_j) - is the angle between the positive axes x_j and the normal ν and dS is a part of the surface. This implies

$$\frac{\partial}{\partial x_j}\tilde{f} = \Big(\tilde{\frac{\partial f}{\partial x_j}}\Big) + s \ \text{ on } \ \mathbb{R}, \tag{3.11}$$

where s is a distribution defined by $(s, \psi) = \int\limits_{S} f(x)\cos(\nu, x_j)\psi(x)dS$.

If $f \in \mathcal{C}^2(\Omega) \cap \mathcal{C}^1(\overline{\Omega})$ and $f = 0$ out of $\bar{\Omega}$ then (3.16) and $\frac{\partial}{\partial \nu} = \sum\limits_{j=1}^{n} \cos(x_j, \nu)\frac{\partial}{\partial x_j}$ imply Green's formula:

$$\langle \Delta\tilde{f}, \phi\rangle = \langle(\tilde{\Delta f}), \phi\rangle + \int\limits_{S} \frac{\partial}{\partial \nu}f(x)\phi dS - \int\limits_{S} f(x)\frac{\partial}{\partial \nu}\phi dS, \ \phi \in \mathcal{C}_0^\infty(\Omega), \tag{3.12}$$

where $\Delta = \sum\limits_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}$ is the Laplace operator.

Note that in (3.11) and (3.12) $\Big(\tilde{\frac{\partial f}{\partial x_j}}\Big)$ i $(\tilde{\Delta f})$ denotes regular distributions determined by locally integrable functions which are respectively equal to the ordinary partial derivatives $\frac{\partial}{\partial x_j}f$ i Δf where they exist in the classical sense.

Connections of differentiations in the classical and distributional sense is given in the next theorem.

Thorem 3.3 *Let u and f be continuous functions in Ω and $\frac{\partial}{\partial x_j}u = f$ in the sense of distributions: $\frac{\partial}{\partial x_j}\tilde{u} = \tilde{f}$, where $\tilde{u}$ and $\tilde{f}$ are corresponding regular distributions. Then $\frac{\partial}{\partial x_j}u = f$ in the classical sense.*

Example 3.7. This example illustrates the difference between the classical and the distributional derivatives. Let f be the characteristic function for the interval $[0,1]$. Then $u(x,t) = f(x-ct),\ x,t \in \mathbb{R}$, is a distributional (we call it the weak) solution to the differential equation

$$u_{tt} - c^2 u_{xx} = 0.$$

Since $f \in L^1(\mathbb{R})$, Theorem 2.10 implies that there exists a sequence $(f_n) \in \mathcal{C}_0^\infty(\mathbb{R})^{\mathbb{N}}$ converging in L^1 to f. Thus

$$u_n(x,t) := f_n(x-ct) \to u(x,t) \quad \text{as} \ \ n \to \infty, \ \ n \in \mathbb{N} \ \ (x,t \in \mathbb{R}),$$

in the sense of distributions. By previous theorem, $f_n(x-ct),\ n \in \mathbb{N}$, is the classical and weak solution to the given equation, and this implies that $u(x,t)$ is a weak (but not classical) solution of this equation.

Thorem 3.4 *Let $f \in L^p(\Omega),\ 1 \le p < \infty,\ \delta_n$ given in (1.2) and let $f^{(\alpha)} \in L^p(\Omega)$ in the sense of distributions.*

(i) If $y \in \Omega$ and $d(y, \mathbb{R}^n \setminus \Omega) > \frac{1}{k}$, then

$$(f^{(\alpha)} * \delta_k)(y) = (f * \delta_k)^{(\alpha)}(y).$$

(ii) If $\Omega_1 \subset \Omega$ and $(\Omega_1)_\varepsilon \subset \Omega$ for some $\varepsilon > 0$, then

$$\|(f * \delta_k)^{(\alpha)} - f^{(\alpha)}\|_{L^p(\Omega_1)} \to 0 \quad as \ \ k \to \infty.$$

Example 3.8. Let Ω be open so that $\bar{\Omega}$ is as in Example 3.6. Let it be divided with a smooth $(n-1)$-dimensional surface Γ of C^1-class into two domains Ω_1 and Ω_2 such that $\Omega = \Omega_1 \cup \Omega_2 \cup \Gamma$.

If $u \in \mathcal{C}(\Omega)$ and $u|_{\Omega_k} \in \mathcal{C}^1(\Omega_k),\ k = 1,2$, then u has the distributional partial derivatives (regular distributions) and

$$\frac{\partial}{\partial x_j}\tilde{u} = \sum_{k=1}^{2} \Big(\widetilde{\frac{\partial}{\partial x_j}(u|_{\Omega_k})}\Big). \tag{3.13}$$

Let $\phi \in \mathcal{C}_0^\infty(\Omega_k),\ k = 1, 2$. By Example 3.6

$$\int\limits_{\Omega_k} u(x)\frac{\partial}{\partial x_j}\phi(x)dx = -\int\limits_{\Omega_k} \Big(\frac{\partial}{\partial x_j}u(x)\Big)\phi(x)dx - \int\limits_{\Gamma} u(x)\phi(x)\cos(\nu, x_j)dr.$$

Since the direction of the normal ν on Γ is taken so that for $k = 1, 2$, it is directed out of Ω_k, we obtain

$$-\Big\langle\frac{\partial}{\partial x_j}u, \phi\Big\rangle = \Big\langle u, \frac{\partial}{\partial x_j}\phi\Big\rangle = \int\limits_{\Omega_1} u\frac{\partial}{\partial x_j}\phi\ dx + \int\limits_{\Omega_2} u\frac{\partial}{\partial x_j}\phi\ dx,$$

$$\langle\frac{\partial}{\partial x_j}u, \phi\rangle = \int\limits_{\Omega} v(x)\phi(x)dx,$$

where $v \in \mathcal{L}^\infty$ so that $v|_{\Omega_k} = (\frac{\partial}{\partial x_j}u)|_{\Omega_k})$, and this proves (3.13).

Structural theorem. One can easily prove that $f = C$ is the only distributional and classical solution to equation $f' = 0$ on $\Omega = (a, b)$. Also we have if $f \in \mathcal{D}'(\Omega)$, then for every $k \in \mathbb{N}_0$ there exists $g \in \mathcal{D}'(\Omega)$ such that $g^{(k)}(x) = f(x)$ and $g(x)$ is uniquely determined up to a polynomial of order $\leq k - 1$.

Now we will give a structural theorem for distributions which shows that $\mathcal{D}'(\Omega)$ is the smallest extension of $L^\infty_{loc}(\Omega)$ where every element has all the derivatives.

Thorem 3.5 *Let $f(x) \in \mathcal{D}'(\Omega)$ and $\Omega_1 \subset\subset \Omega \subset \mathbb{R}^n$.*

(i) There exist $g(x) \in L^\infty(\Omega_1)$ and $r \in \mathbb{N}$ such that in the sense of distributions

$$f(x) = \frac{\partial^{n\cdot r}}{\partial x_1^r ... \partial x_n^r} g(x).$$

($\Omega_1 \subset\subset \Omega$ means theta $\overline{\Omega}_1$ is a compact subset of Ω).

(ii) For every neighborhood Ω_2 of $\overline{\Omega}_1$ in Ω there exist a continuous function F in $\mathbf{R}^n$ and $s \in \mathbb{N}$ so thatsupp $F \subset \Omega_2$ such that

$$f = \frac{\partial^{n\cdot s}}{(\partial x_1)^s ... (\partial x_n)^s} F(x).$$

4 Regularization

Division of distributions by smooth functions. Let $a \in \mathcal{C}^\infty(\Omega)$ and $g \in \mathcal{D}'(\Omega)$. Consider

$$a(x)f(x) = g(x) \text{ in } \mathcal{D}'(\Omega). \tag{4.1}$$

If $a(x)$ is different from zero everywhere in Ω, this is a trivial problem since $1/a \in \mathcal{C}^\infty(\Omega)$ and $f = \frac{1}{a}g$.

Let $n = 1$ and let the zeros of a be isolated and of finite order. This means that for every zero point x_0 $(a(x_0) = 0)$ there exists $k \in \mathbb{N}$ such that $x \mapsto (x - x_0)^{-k}a(x_0)$ is finite in a neighborhood of x_0. So let $\{\alpha_k;\ k \in \mathbb{N}\}$ be isolated points of a; $\{U_i,\ i \in \mathbb{N}\}$ be a family of open sets which cover $\mathbb{R}$ such that every zero lies in at most one open set of this family. Denote by $\{\phi_i;\ i \in \mathbb{N}\}$ the partition of unity which corresponds to $\{U_i\}$. If a solution of (4.1) exists it is clear that

$$\sum_{i\in\mathbb{N}}(af, \phi_i\phi) = \sum_{i\in\mathbb{N}}(g, \phi_i\phi),\ \phi \in \mathcal{C}_0^\infty(\Omega),$$

i.e. that for every $i \in \mathbb{N}$ there exists a unique solution f_i in $\mathcal{D}'(U_i)$ to

$$a_i(x)f_i(x) = g_i(x) \quad \text{in} \quad \Omega,$$

where a_i and g_i are the corresponding restrictions of of a and g on U_i. Conversely, if we know the existence of solution to $a_if_i = g_i, i \in \mathbb{N}$, then using the above partition of unity we have that $\sum_{i\in\mathbb{N}} \phi_i g_i/a_i$ is a solution of (4.1).

Thus we will assume that a has only one zero of finite order. Moreover, assume that $x = 0 \in \Omega$ is a zero of order m: $a(x) = x^m b(x), x \in \Omega$, where $b \in \mathcal{C}^\infty(\Omega),\ b(x) \neq 0$ in Ω. After dividing by b this equation becomes

$$x^m f(x) = h(x),\ h \in \mathcal{D}'(\Omega). \tag{4.2}$$

Thorem 4.1 *Let* $m \in \mathbb{N}$. *(i) A necessary and sufficient condition that* $f_0(x)$ *satisfies* $x^m f_0(x) = 0$ *in* $\mathcal{D}'(\Omega)$ *is that*

$$f_0(x) = \sum_{\nu=0}^{m-1} C_\nu \delta^{(\nu)}(x),\ C_\nu \in \mathbb{C}. \tag{4.3}$$

(ii) Equation (4.2) has a solution in $\mathcal{D}'(\Omega)$.

Thus the solution of (4.2) if exists, is not unique: two solutions differ by a function of the form (4.3).

Let $H = \{\phi;$ there exists $\psi \in \mathcal{D}(\Omega), \phi = x^m\psi\}$. By

$$\langle f_1, \phi\rangle := \langle h, \psi\rangle,\ \ \phi = x^m\psi,\ \phi \in H,$$

is defined a continuous linear functional on H. By the Hahnn-Banach theorem (see the appendix) it follows that there exists a unique $\tilde{f} \in \mathcal{D}'(\Omega)$ such that $\tilde{f}_{|H} = f_1$. This $\tilde{f}$ satisfies (4.2).

Any other solution is of the form $\tilde{f}(x) + \sum_{\nu=0}^{m-1} C_\nu\delta^{(\nu)}$. If we consider (4.1) only on $\mathbb{R} \setminus \{0\}$, then the solution is unique:

$$f(x) = 1/x^m,\ x \in \mathbb{R} \setminus \{0\}.$$

This function does not have a locally integrable extension on $\mathbb{R}$. Note that there exists a distribution $\tilde{f}$ such that $f(x) = \tilde{f}(x),\ \ x \in \mathbb{R} \setminus \{0\} : \langle f, \phi\rangle = \langle \tilde{f}, \phi\rangle$ for every $\phi \in \mathcal{C}_0^\infty,\ \ \phi(x) = 0$ in a neighborhood of zero.

Thus we come to the definition of regularization:

Let $f \in L^1_{loc} \setminus \{0\}$. A distribution $\tilde{f} \in \mathcal{D}'(\Omega \setminus \{0\})$, which satisfies

$$\langle \tilde{f}(x), \phi(x)\rangle = \int_{\mathbb{R}^n} f(x)\phi(x)dx,\ \phi \in \mathcal{C}_0^\infty(\Omega \setminus \{0\})$$

($\phi(x) = 0$ in a neighborhood of x_0) is called the regularization of a function f (pseudo function).

A regularization of $f \in L^1_{loc}(\Omega \backslash \{x_0\})$ is a distribution in $\mathcal{D}'(\Omega)$ such that its restriction on $\Omega \backslash \{x_0\}$ equals regular distribution on $\Omega \backslash \{x_0\}$ determined by f, if it exists. Theorem 4.1 implies that the regularization is not unique, if exists.

By partition of unity, we reduce the problem of regularization for a function with discrete set of singularities to one point singularity.

Regularization is not always possible:

Example 4.1 Let $h \in L^1_{loc}(\mathbb{R}\backslash\{0\})$ such that $h(x) \geq H(|x|),\ x \in \mathbb{R}\backslash\{0\}$ where $H(|x|)$ increases faster than any power of $\dfrac{1}{|x|}$ as $|x| \to 0$. Then there does not exist the regularization of h. For example $h(x) = e^{\frac{1}{x}} \in \mathcal{D}'((0,\infty))$ does not have an extension on $\mathbf{R}$.

Thorem 4.2 *If $f \in L^1_{loc}(\Omega \setminus \{0\})$ and for some $m \in \mathbb{N}$, $f|x|^m \in L^1_{loc}(\Omega)$, then f has a regularization in Ω :*

$$\langle \tilde{f}(x), \phi(x) \rangle := \int\limits_{\Omega} f(x) \Big[\phi(x) - (\phi(0) + \sum_{i=1}^{m-1} \frac{1}{i!} d^i \phi(0)) H(\varepsilon - |x|) \Big] dx,$$

$\phi \in \mathcal{C}_0^\infty(\Omega)$, where

$$d^i\phi(0) = \Big(x_1 \frac{\partial}{\partial x_1} + ... + x_n \frac{\partial}{\partial x_n} \Big)^i \phi(x)|_{x=0},$$

$$H(\varepsilon - |x|) = \begin{cases} 1 & za\ \ |x| < \varepsilon \\ 0 & za\ \ |x| \geq \varepsilon, \end{cases}$$

ε is chosen so that $Z(0, \varepsilon) \subset \Omega$.

Regularizations by analytic continuation. Let Λ be an open set in $\mathbb{R}^n$ and $t \mapsto h_t(x)$ be a mapping $\Lambda \to \mathcal{D}'(\Omega)$ (a family of distributions). For every $\phi \in \mathcal{C}_0^\infty(\Omega)$, by $t \mapsto \langle h_t(x), \phi(x) \rangle$ is defined a function $\Lambda \to \mathbb{C}$.

The mapping $\Lambda \to \mathcal{D}'(\Omega),\ t \mapsto h_t(x)$ has a limit as $t \mapsto t_0$ ($t_0 \in \overline{\Lambda},\ \ t \in \Lambda$) if there exists $g \in \mathcal{D}'(\Omega)$ such that $h_t \to g$ in the sense of strong topology in $\mathcal{D}'(\Omega)$. Since the strong and the weak sequential convergences are equivalent, this is equivalent with

$$\lim_{t\to t_0}\langle h_t(x),\phi(x)\rangle = \langle g,\phi\rangle,\ \ \phi\in\mathcal{C}_0^\infty(\Omega).$$

Then we say that h_t has a limit g at t_0.

A family h_t is continuous at the point $t_0\in\Lambda$ if $g=h(t_0)$. (Again this is equivalent with:

$$\langle h_t,\phi\rangle\to\langle h_{t_0},\phi\rangle,\ \ \phi\in\mathcal{C}_0^\infty(\Omega),\ \ \text{as}\ \ t\to t_0\ \ \text{in}\ \ \Lambda.$$

In a similar way we introduce the differentiation of $t\mapsto h_t$ and, as above, we have that

"$\partial_t^k h_t$ exists in $\mathcal{D}'(\Omega)$ if and only if for every $\phi\in\mathcal{D}(\Omega)$ there exist the $k-$th derivative of the function $\Lambda\ni t\mapsto\langle h_t,\phi\rangle\in\mathbb{C}$."

Thorem 4.3 *a) Let $f(x)\in\mathcal{D}'(\mathbb{R}^n)$, then $\{f(x-h);\ h\in\mathbb{R}^n\}$ is a continuous family of distributions and*

$$f^{(e_j)}(x)=\lim_{r\to 0}\frac{f(x+re_j)-f(x)}{r},$$

in the sense of convergence in $\mathcal{D}'(\mathbb{R}^n)$.

b) If a family h_t has a partial derivative ∂_{t_j} at t_0, then for every $m\in\mathbb{N}_0^n$:

$$\frac{\partial}{\partial t_j}\Big|_{t=t_0}(\partial_x^m h_t)=\partial_x^m\Big(\frac{\partial}{\partial t_j}\Big|_{t=t_0}h_t\Big).$$

In the case that Λ is an open set in $\mathbb{C}$ and that for every $\phi\in\mathcal{C}_0^\infty(\Omega)$

$$\lambda\mapsto\langle h_\lambda(x),\phi(x)\rangle$$

is an analytic function in Λ, then we say that $h_\lambda(x)$ is an anlytic family or an analytic distribution (with respect to λ.)

Thorem 4.4 *(i) A family* $\{h_\lambda(x);\ \lambda \in \Lambda\} \subset \mathbb{C}$ *is analytic if and only if for every* $\lambda_0 \in \Lambda$

$$\lim_{\lambda\to\lambda_0} \frac{h_\lambda - h_{\lambda_0}}{\lambda - \lambda_0} \ \textit{exists in}\ \mathcal{D}'(\Omega).$$

(ii) For every $m \in \mathbb{N}^n$ *and* $k \in \mathbb{N}$

$$\frac{d^k}{d\lambda^k}(\partial_x^m h_\lambda) = \partial_x^m\Big(\frac{d^k}{d\lambda^k} h_\lambda\Big), \quad \lambda \in \Lambda.$$

(iii) For every $\lambda_0 \in \Lambda$ *and* λ *in an open neighbourhood of* λ_0 *there holds*

$$h_\lambda = \sum_{k=0}^{\infty} \frac{(\lambda-\lambda_0)^k}{k!}\Big(\frac{d^k}{d\lambda^k} h_\lambda\Big)\Big|_{\lambda=\lambda_0} \ \textit{(Taylor expansion).}$$

Let $f_\lambda \in \mathcal{D}'(\Omega),\ \ \lambda \in \Lambda$, be an analytic distribution and let for every $\phi \in \mathcal{C}_0^\infty(\Omega)$ the analytic function $\langle f_\lambda, \phi\rangle$ have an analytic continuation on an open set $\Lambda_1 \supset \Lambda$. Then:

Thorem 4.5 *For every* $\lambda_1 \in \Lambda_1, \quad f_{\lambda_1} \in \mathcal{D}'(\Omega)$, *where* $\langle f_{\lambda_1}, \phi\rangle$ *is the analytic continuation of* $\lambda \mapsto \langle f_\lambda, \phi\rangle$ *in the point* $\lambda_1 = \lambda$ *(for every* $\phi \in \mathcal{C}_0^\infty(\Omega)$*).*

Now we will explain the regularization through the analytic continuation by examples.

Example 4.2. Let x_+^λ be defined on $\mathbb{R}$ by

$$x_+^\lambda = \begin{cases} x^\lambda & x > 0 \\ 0 & x \le 0. \end{cases}$$

For $Re\lambda > -1,\ x_+^\lambda \in L^1_{loc}(\mathbb{R})$ defines a regular distribution

$$\langle x_+^\lambda, \phi\rangle = \int_0^\infty x^\lambda \phi(x)dx. \tag{4.4}$$

By Thorem 4.4:

$$\frac{d}{d\lambda}\langle x_+^\lambda, \phi(x)\rangle = \langle \frac{d}{d\lambda} x_+^\lambda, \phi\rangle = \int_0^\infty x^\lambda \ell n x \phi(x) dx.$$

Thus, $x_+^\lambda$is an analytic distribution on $\{\lambda \in \mathbb{C};\ Re\lambda > -1\}$. If $\langle x_+^\lambda, \phi\rangle$ in (4.4) is written in the form

$$(4.5) \qquad \int_0^1 x^\lambda(\phi(x) - \phi(0))dx + \int_1^\infty x^\lambda \phi(x)dx + \frac{\phi(0)}{\lambda+1},$$

(since $\frac{\phi(0)}{\lambda+1} = \int_0^1 \phi(0)x^\lambda dx$) it follows that the first member is an analytic function for $Re\lambda > -2$, that the second member is an analytic function for every $\lambda \in \mathbb{C}$ and that the third member is an analytic function for $\lambda \neq -1$. This means that $x_+^\lambda,\ Re\lambda > -1$ is by (4.5) analytically continued to $Re\lambda > -2,\ \lambda \neq -1$. Continuing this procedure, we get x_+^λ is analytically continued by

$$\langle x_+^\lambda, \phi\rangle = \int_0^\infty x^\lambda \phi(x)dx = \int_0^1 x^\lambda[\phi(x) - \phi(0) - ...$$

$$-... - \frac{x^{n-1}}{(n-1)!}\phi^{(n-1)}(0)]dx + \int_1^\infty x^\lambda \phi(x)dx$$

$$+\sum_{k=1}^n \frac{\phi^{(k-1)}(0)}{(k-1)!(\lambda+k)}$$

into the domain $Re\lambda > -(n+1),\ \lambda \neq -1, -2, ..., -n$, where we use

$$\sum_{k=1}^n \frac{\phi^{(k-1)}(0)}{(k-1)!(\lambda+k)} = \int_0^1 x^\lambda(\phi(0) + x\phi'(0) + ... + \frac{x^{n-1}}{(n-1)!}\phi^{(n-1)}(0))dx.$$

We can show

$$(x_+^\lambda)' = \lambda x_+^{\lambda-1} \qquad \lambda \neq -1, -2, ...$$

For $Re\lambda > 0$ it is clear and from the unicity of the analytic continuation, it follows for $\lambda \neq -1, -2, ...$

Note that if $-\lambda \in \mathbb{N}$ then the regularization is made by the use of Taylor expansion.

Using the regularization of x_+^λ we can formulate the regularization for x_-^λ, $-\lambda \notin \mathbb{N}$, where

$$x_-^\lambda = \begin{cases} (-x)^\lambda & x < 0 \\ 0 & x \geq 0. \end{cases}$$

For $Re\lambda > -(n+1)$, $\lambda \neq -1, -2, ..., -n$:

$$\langle x_-^\lambda, \phi\rangle = \int\limits_0^1 x^\lambda[\phi(-x) - \phi(0) - ... - (-1)^{n-1}\frac{x^{n-1}}{(n-1)!}\phi^{(n-1)}(0)]dx$$

$$+\int\limits_1^\infty x^\lambda \phi(-x)dx + \sum_{k=1}^{n} \frac{(-1)^{k-1}\phi^{(k-1)}(0)}{(k-1)!(\lambda+k)}.$$

Example 4.3 Let $|x|^\lambda = x_+^\lambda + x_-^\lambda$ i $|x|^\lambda sgnx = x_+^\lambda - x_-^\lambda, Re\lambda > -1.$ Then

$$\langle |x|^\lambda, \phi(x)\rangle = \int\limits_0^1 x^\lambda[\phi(x) + \phi(-x) - 2(\phi(0) + \frac{x^2}{2!}\phi''(0) + ...$$

(4.6)
$$+\frac{x^{2m-2}}{(2m-2)!}\phi^{(2m-2)}(0))]dx + \int\limits_1^\infty x^\lambda[\phi(x) + \phi(-x)]dx$$
$$+2\sum_{k=0}^{m-1} \frac{\phi^{(2k)}(0)}{(2k)!(\lambda+2k+1)}$$

for $Re\lambda > -(2m+1)$ i $\lambda \neq -1, -3, ..., -(2m-1)$.

$$\langle |x|^\lambda sgnx, \phi\rangle = \int_0^1 x^\lambda[\phi(x) - \phi(-x) - 2(x\phi'(0)$$

$$(4.7) \qquad + \frac{x^3}{3!}\phi^{(3)}(0) + ... + \frac{x^{2m-1}}{(2m-1)!}\phi^{(2m-1)}(0))]dx$$

$$+ \int_1^\infty x^\lambda[\phi(x) - \phi(-x)]dx + 2\sum_{k=0}^{m-1} \frac{\phi^{(2k)}(0)}{(2k+1)!(\lambda + 2k + 2)}$$

for $Re\lambda > -(2m+2),\ \lambda \neq -2, -4, ..., -2m.$

Since even negative numbers are in the domain of analytic continuation for $|x|^\lambda$ and $|x|^{-2k} = x^{-2k}$, from (4.6) we get regularizations for $x^{-2k},\ k \in \mathbb{N}$. Similarly (4.7) implies regularizations for $x^{-(2k+1)},\ \ k \in \mathbb{N}$.

One can see that $x^{-(2k+1)}$ or x^{-2k} solutions to (4.2) if m is even or odd.

Example 4.4 Let $x \in \mathbb{R}^n$ i $f(x) = |x|^\lambda = r^\lambda$. If $Re\lambda > -n$ then f defines a regular distribution

$$\langle r^\lambda, \phi\rangle = \int_{\mathbb{R}^n} t^\lambda \phi(x)dx.$$

By differentiation with respect to λ for $\lambda > -n$, we can enter into the integral. It follows that $|x|^\lambda$, $Re\lambda > -n$, defines an analytic distribution. Now we will make the analytic continuation. Put $x = r\ \dot{x}$ where $\dot{x}$ is a variable on the unite sphere S^{n-1}. We have

$$\begin{aligned} \int_{\mathbb{R}^n} |x|^\lambda \phi(x)dx \ &= \int_0^\infty \int_{S^{n-1}} r^\lambda \phi(r\ \dot{x}) r^{n-1} dr d\ \dot{x} \\ &= \int_0^\infty r^{\lambda+n-1} \int_{S^{n-1}} \phi(r\ \dot{x})d\ \dot{x}\ . \end{aligned}$$

The integral $\dfrac{1}{|S^{n-1}|}\displaystyle\int_{S^{n-1}} \phi(r\ \dot{x})d\ \dot{x}$, where $|S^{n-1}|$ is the surface area of the sphere S^{n-1}, called the average (mean value) of ϕ on the sphere with the radius r, is denoted by $S_\phi(r)$. Thus

$$\int\limits_{\mathbb{R}^n} |x|^\lambda \phi(x)dx = |S^{n-1}| \int\limits_0^\infty r^{\lambda+n-1} S_\phi(r)dr.$$

Since the support of ϕ is contained in a sphere of a bounded radius, the function $r \mapsto S_\phi(r)$ is bounded. By definition, $S_\phi(r)$ is a smooth function of r if $r > 0$. Taylor expansion of $\phi(x)$ gives

$$S_\phi(r) = \frac{1}{|D^{n-1}|} \int\limits_{S^{n-1}} \Big[\phi(0) + \sum_{j=1}^n \frac{\partial \phi(0)}{\partial x_j} x_j$$

$$+\frac{1}{2} \sum_{i,j=1}^n \frac{\partial^2 \phi(0)}{\partial x_i \partial x_j} x_i x_j + ... + \frac{1}{k!} d^k \phi(\xi)\Big] dx,$$

where $x_j = rw_j$ and w_j is a product of corresponding "sin" and "cos" functions, $1 \leq j \leq n$. Integration of addends containing an even number of factors, say $2m$, x_j gives the addend of the form $a_m r^{2m}$. By integration of addends containing an odd number of factors x_j, one gets the value zero.

So we obtain $S_\phi(r) = \phi(0) + a_1 r^2 + a_2 r^4 + ... + a_k r^{2k} + o(r^{2k})$, and this implies that S_ϕ has all the derivatives at zero. Thus $S_\phi(r) \in \mathcal{C}_0^\infty(\mathbb{R})$ and for $Re\lambda > -n$,

$$\langle |x|^\lambda, \phi\rangle = \langle |S^{n-1}|\, r_+^{\lambda+n-1},\ \ S_\phi(r)\rangle, \tag{4.8}$$

where the right hand side $|S^{n-1}| r_+^{\lambda+n-1}$ is applied to the test function in $\mathcal{C}_0^\infty(\mathbb{R})$. By Example 4.2 it follows that (4.8) can analytically be continued in the complex plane without points $\lambda = -n, -n-1, ...$.

5 Appendix

General notions of locally convex spaces

Denote by K a field of real numbers $\mathbb{R}$ or complex numbers $\mathbb{C}$. A vector space L over K endowed with a topology τ is called a vector topological space, in short VTS or VT space, if the mappings $L \times L \to L: \ (x, y) \mapsto x+y$ and $K \times L \to L: \ (\lambda, x) \mapsto \lambda x$ are continuous.

It is said that $A \subset L$ is an absorbing set if for every finite set $B \subset L$ there exists $\lambda_B \in K$ such that $B \subset \lambda A$ for $|\lambda| > |\lambda_B|$. $A \subset L$ is circled if $\lambda A \subset A$ for every $|\lambda| \leq 1$. The circled envelope of a set $A \subset L$ is the intersection of all circled sets containing A.

A set $B \subset L$, L is VTS, is bounded if for every neighborhood of zero U there exists $\lambda \in K$ such that $B \subset \lambda U$. A compact set of a VT space is a bounded set. If f is a continuous mapping of a VT space E into VT space F and if A is a bounded subset of E, then $f(A)$ is a bounded set in F.

Hausdorff VT space is metrizable if and only if it has a denumerable basis of neighborhoods of zero.

A set $A \subset L$ on a field K, is convex if $\{\lambda x + (1-\lambda)y; \ 0 \leq \lambda \leq 1\} \subset A$ for every $x, y \in A$. The convex hull of an arbitrary set A is the intersection of all convex sets in L containing A.

VTS (L, τ) is locally convex space, in short LCS or LC space, if it has a basis of neighborhoods of zero consisting of convex sets. A necessary and sufficient condition for a VT space to be an LC space is that its topology is defined by a family of seminorms.

If U is circled and convex neighborhood of zero in VT space (L, τ), then $p_U(x) = \inf\{\lambda > 0; \ x \in \lambda U\}$ defines a seminorm on L. Let $\{U_\alpha; \ \alpha \in A\}$ be a basis of neighborhoods of zero in LC space (L, τ) which are circled and convex. Then $\{p_\alpha; \ \alpha \in A\}$ is the corresponding family of seminorms. Conversely, if $\{p_\alpha; \ \alpha \in A\}$ is a family of seminorms in a vector space L, then this family induces in L a structure of an LCS. The basis of neighborhoods of zero consists of sets of the form $\{x;\ p_\alpha(x) < \varepsilon\},\ \varepsilon > 0,\ \alpha \in A$. Denote by (L_α, p_α) a LCS L endowed with the topology on L induced by the seminorm p_α. A topology τ of an LC space (L, τ) is the projective topology in L with respect to the identity mappings $i_\alpha : L \to (L_\alpha, p_\alpha)$.

Family of seminorms $\{p_\alpha; \alpha \in A\}$ on a vector space L separates points if for every $x \neq 0$ there exists $\alpha \in A$ such that $p_\alpha(x) \neq 0$. In that case the corresponding topology is Hausdorff. The converse also holds. If (L, τ) is a Hausdorff LCS then a family of seminorms $\{p_\alpha;\ \alpha \in A\}$ defining the topology τ, separates points.

Complete and metrizable LCS is called Fréchet space in short F-space, and a complete normed space is called Banach space, in short B-space.

Hahn-Banach theorem. Let L be a vector space over K, p be a seminorm on L and M a subspace of L. If f is a linear functional on M such that $|f(x)| \leq p(x)$ for every $x \in M$, then there exists a linear functional F on L such that $F|_M = f$ where $F|_M = f$ is a restriction of F on M, and $|F(x)| \leq p(x)$ for every $x \in L$.

Projective topology on LCS. Let L i $L_\alpha,\ \alpha \in A$, be vector spaces on $K;\ f_\alpha,\ \alpha \in A$, be linear mappings $f_\alpha : L \to L_\alpha$ and let $\tau_\alpha,\ \alpha \in A$, be locally convex topologies on L_α. Let τ be the corresponding projective topology on L. A neighborhood of an arbitrary point $x \in L$ is the intersection of a finite number of sets $f_\alpha^{-1}(U_\alpha)$, where U_α is a convex neighborhood of the point $f_\alpha(x)$ in L_α. Topology τ is invariant with respect to translations and it makes L to be a LC space (L, τ) . It is Hausdorff if and only if for every $x \neq 0$ there exists in L_α a neighborhood of zero U_α such that $f_\alpha(x) \notin U_\alpha$.

A mapping $f : E \to L$, where (E, τ_0) is a VTS and (L, τ) is LC space with the topology τ described above is continuous if and only if for every $\alpha \in A$ the mapping $f_\alpha \circ f : E \to L_\alpha$ continuous.

A subspace of an LC space with respect to the induced topology is an LCS. A product of LC spaces is an LCS.

Projective limit. Let $\mathcal{A} = \{(L_\alpha, \tau_\alpha);\ \alpha \in A\}$ be a family of LC spaces and A a partially ordered and directed set. Suppose that for every pair $(\alpha, \beta) \in A^2$, so that $\alpha \leq \beta$, is defined a continuous linear mapping $\pi_\alpha^\beta : L_\beta \to L_\alpha$ with the properties: (i) π_α^α is the identity mapping; (ii) if $\alpha \leq \beta \leq \gamma$, then $\pi_\alpha^\gamma = \pi_\alpha^\beta \circ \pi_\beta^\gamma$. In that case $\mathcal{A}$ is called a projective spectar with respect to the family $\{\pi_\alpha^\beta;\ (\alpha, \beta) \in A^2,\ \alpha \leq \beta\}$. In the case $A = \mathbb{N}$, a projective spectar is called a projective sequence. The set of elements $x = (x_\alpha) \in \prod\limits_{\alpha \in A} L_\alpha$ such that $x_\alpha = \pi_\alpha^\beta(x_\beta)$ for every $(\alpha, \beta) \in A^2,\ \alpha \leq \beta$, is, denoted by L. It is a subspace of $\prod\limits_\alpha L_\alpha$. If $p_\alpha,\ \alpha \in A$, denotes the projection of L on L_α then the space (L, τ), where τ is a projective topology in L with respect to the family of projections $\{p_\alpha;\ \alpha \in A\}$, is called the projective topological limit of spaces $\{(L_\alpha, \tau_\alpha)\}$ with respect to the mappings $\{\pi_\alpha^\beta\}$ and we write

$$\lim_{\overleftarrow{\alpha}} (L_\alpha, \tau_\alpha) = (L, \tau).$$

If (L_α, τ_α) are Hausdorff spaces, then it holds for (L, τ); and if (L_α, τ_α) are complete, then it also holds for (L, τ). Especially, the projective limit of a sequence of F-spaces is an F-space.

We consider a special case of a projective spectar. Let $((L_n, \tau_n))$ be a sequence of LC spaces such that L_{n+1} is a subspace od L_n and that the topology τ_{n+1} u L_{n+1} is finer that the one induced by τ_n on L_{n+1} for every $n \in \mathbb{N}$. With respect to identity mappings $i_m^n : L_n \to L_m,\ n \geq m$, this sequence is a projective sequence. Then the projective limit space L equals $\bigcap\limits_{n \in \mathbb{N}} L_n$, and the topology of the projective limit in L has a basis of neighborhoods of zero constituted by finite intersections of neighborhoods of zero in (L_n, τ_n).

Inductive topology of LC spaces. Let $\{L_\alpha, \tau_\alpha);\ \alpha \in A\}$ be a family of LC spaces over the field K and $\{g_\alpha;\ \alpha \in A\}$ be a family of linear mappings from $L_\alpha,\ \alpha \in A$, into a vector space L, so that L is a linear hull of $\bigcup\limits_{\alpha \in A} g_\alpha(L_\alpha)$ (a set of all linear combinations of elements in the union). The finest locally convex topology in L, denoted by τ, such that the mappings $g_\alpha,\ \alpha \in A$ are continuous is called inductive topology in L. Since the trivial topology in L, is an LC topology, it follows that the family $\mathcal{F}$ of all LC topologies in L for which the mappings $g_\alpha,\ \alpha \in A$, are continuous, is not empty.

A basis of neighborhoods of zero in τ consists of a set of circled absorbing convex sets $U \subset L$ such that $g_\alpha^{-1}(U)$ is a neighborhood of zero in (L_α, τ_α) for all $\alpha \in A$.

Denote by $\Gamma(M)$ a convex circled envelope of a set M. $\{\Gamma(\bigcup\limits_{\alpha \in A} g_\alpha(U_\alpha));\ U_\alpha \in \mathcal{B}_\alpha, \alpha \in A\}$, where $\mathcal{B}_\alpha$ is a basis of neighborhoods of zero in L_α, makes a basis in L. If spaces $\{(L_\alpha, \tau_\alpha);\ \alpha \in A\}$ are Hausdorff, it does not follow that (L, τ) is Hausdorff.

A linear mapping f of the LC space (L, τ) described above, into LCS (E, τ_0) is continuous if and only if $f \circ g_\alpha : (L_\alpha \tau_\alpha) \to (E, \tau_0)$ is continuous for every α.

The factor topology in L/M, where M is a subspace of LCS L, is an LCS. Direct topological sum of a family $\{(L_\alpha, \tau_\alpha);\ \alpha \in A\}$ of LCS, $\sum\limits_{\alpha \in A} (L_\alpha, \tau_\alpha)$, is an LCS.

Inductive limit. Let A be a partially ordered and directed set and let a family of LC spaces $\mathcal{F} = \{(L_\alpha, \tau_\alpha);\ \alpha \in A\}$ satisfies that for every pair $(\alpha, \beta) \in A^2$ and $\alpha \leq \beta$ there exists a continuous linear mapping $\kappa_\beta^\alpha : L_\alpha \to L_\beta$ with the properties: (i) κ_α^α is the identity mapping; (ii) if $\alpha \leq \beta \leq \gamma$, then $\kappa_\gamma^\alpha = \kappa_\gamma^\beta \circ \kappa_\beta^\alpha$.

We say that a family $\mathcal{F}$ with the family of mappings $\{\kappa_\beta^\alpha;\ (\alpha, \beta) \in$

$A^2,\ \alpha \leq \beta\}$ makes an inductive spectar which, in case $A = \mathbb{N}$, is called the inductive sequence. The disjoint union $\bigsqcup\limits_{\alpha \in A} L_\alpha$ is endowed with the relation of equivalence: $x_\alpha \sim x_\beta$ $(x_\alpha \in L_\alpha, x_\beta \in L_\beta)$ if there exists γ such that $\gamma \geq \alpha,\ \gamma \geq \beta$ and $\kappa^\alpha_\gamma(x_\alpha) = \kappa^\beta_\gamma(x_\beta)$. We define the addition of equivalence classes in a usual way (by the sum of representatives) as well as the scalar multiplication with elements in K. With this operations the set of equivalence classes $\tilde{x}$ is a vector space L called the algebraic inductive limit of a family $\mathcal{F}$ with respect to the family $\{\kappa^\alpha_\beta\}$. Let $\alpha \in A$ and $g_\alpha : (L_\alpha, \tau_\alpha) \to L,\ \alpha \in A$, map every element in L_α into the corresponding class of equivalence in L to which this element belongs. The space L endowed with the inductive topology τ with respect to mappings $g_\alpha : (L_\alpha, \tau_\alpha) \to L,\ \alpha \in A$, is called the inductive topological limit or in short, the inductive limit of the family $\{(L_\alpha, \tau_\alpha)\}$ with respect to $\{\kappa^\alpha_\beta\}$; we write $(L, \tau) = \varinjlim\limits_{\alpha}(L_\alpha, \tau_\alpha)$ where we often omit the signs for the topologies.

Let u be a mapping from $\sum_{\alpha \in A}(L_\alpha, \tau_\alpha)$ into L defined by $u(\bigoplus\limits_{k=1}^{n} x_{\alpha_k}) = \sum\limits_{k=1}^{n} [x_{\alpha_k}]$, where $[x_{\alpha_k}]$ are the classes of equivalence determined by $x_{\alpha_k},\ k = 1, ..., n$. Spaces $\varinjlim\limits_{\alpha}(L_\alpha, \tau_\alpha)$ and $\sum\limits_{\alpha \in A}(L_\alpha, \tau_\alpha)/ker u$, where $ker\ u$ is the kernel of the mapping u, are topologically isomorphic.

Strict inductive limit. Let $((L_n, \tau_n))$ be a sequence of LC spaces such that L_n is a subspace of L_{n+1} and the topology τ_n in L_n equals the topology which (L_{n+1}, τ_{n+1}) induces on $L_n,\ n \in \mathbb{N}$. This sequence with respect to inclusions $i^n_{n+1} : L_n \to L_{n+1}$ is called the strict inductive sequence and $(L, \tau) = \varinjlim\limits_{\alpha}(L_n, \tau_n)$ is called the strict inductive limit of the sequence $((L_n, \tau_n))$. If $(L_n, \tau_n),\ n \in \mathbb{N}$, are Hausdorff spaces then (L, τ) is Hausdorff space and the topology which τ induces on $L_n,\ n \in \mathbb{N}$, equals the topology τ_n. If (L_n, τ_n) are complete spaces then the same holds for (L, τ). If $L_n$is closed in (L_{n+1}, τ_{n+1}) for every $n \in \mathbb{N}$, then the set $B \subset L$ is bounded in u (L,τ) if and only if there exists $n_0 \in \mathbb{N}$ such that $B \subset L_{n_0}$ and B is bounded in (L_{n_0}, τ_{n_0}).

Barreled and bornological spaces. A convex, circled, absorbing and closed set is called a barreled set. An LC space (L,τ) is barreled if the set of all barrels in L makes a basis of neighborhoods of zero in τ. An F-space is a barreled space. The strict inductive limit of F-spaces or B-spaces is a barreled space.

An LC space (L,τ) is a bornological space if every circled convex subset $A \subset L$ which absorbs all bounded subsets in L, is a neighborhood of zero in τ. Any metric space is a bornological space as well as the strict inductive limit of F-spaces or B-spaces.

Montel space is an LC space which is Hausdorff, barreled and every closed and bounded set in it is compact.

Let (L,τ) be a VT space and $\sigma = \{B_\alpha;\ \alpha \in A\}$ be a family of bounded subsets in L. It is said that σ is a total family of bounded sets if the linear hull $\bigcup\limits_{\alpha \in A} B_\alpha$ equals L. Let (E,τ_1) be a VT space and (F,τ_2) an LC space. We denote by $\mathcal{L}(E,F)$ a set of continuous linear mappings from (E,τ_1) into (E,τ_2). Let σ be a total family of bounded sets in E, S an arbitrary element in σ and V an arbitrary neighborhood of zero in (E,τ). Subsets in $\mathcal{L}(E,F)$ of the form $M(S,V) = \{f;\ f(S) \subset V\}$ make a basis of neighborhoods of zero in the vector space $\mathcal{L}(E,F)$ so that $\mathcal{L}(E,F)$ is an LCS. If σ is a family of all finite subsets in E, then the respective topology in $\mathcal{L}(E,F)$ is called the weak topology and it is denoted by τ_w. If σ is a family of all compact subsets in E, then the respective topology in $\mathcal{L}(E,F)$ is called the topology of compact convergence; it is denoted by τ_c. If σ is a family of all bounded subsets of E, then the respective topology in $\mathcal{L}(E,F)$ is called the strong topology and it is denoted by τ_b. We have the following relation between topologies in $\mathcal{L}(E,F)$: $\tau_w \leq \tau_c \leq \tau_b$. The cases when $F = K$ ($= \mathbb{R}$ or $\mathbb{C}$) are of special interest. For example, for (E,τ) the families of bounded sets in $\mathcal{L}(E,\mathbb{C})$ with the topologies τ_s, τ_k and τ_b are equal.

The space of continuous linear mappings $(E,\tau) \to \mathbb{C}$ is denoted by E'. The weak topology in E is defined by the family of seminorms $\{P_f;\ f \in E'\},\ P_f(x) = |f(x)|,\ \ x \in E$.

The finest LC topology in E so that E' is the space of continuous linear functionals, is called Mackey topology. Topology in E defined by the family of seminorms $\{p_B;\ B \in \mathcal{B}\},\ p_B(x) = \sup\limits_{f\in B} |f(x)|,\ x \in E$, where $\mathcal{B}$ is a family of bounded sets in E', is called the strong topology in E.

0.26. Transposed mapping. Let E and F be vector spaces over K and E^* and F^* be their algebraic duals (algebraic dual of E is the set of linear functionals on E). If u is a linear mapping from E into F, then the transposed (adjoint) mapping for u, $u^* : F^* \to E^*$ is defined as follows: If $y^* \in F^*$, then $u^*(y^*) = y^* \circ u \in E^*$. With the notation $x^*(x) = \langle x^*, x\rangle,\ x^* \in E,\ x \in E$, we have $\langle y^*, u(x)\rangle = \langle u^*(y^*), x\rangle$.

Let (E,τ_E) i (F,τ_F) be VT spaces over K, E' and F' their respective spaces of continuous linear functionals (duals) and let u be a continuous mapping from (E,τ_E) into (F,τ_F). If the spaces E' and F' are endowed with the weak topologies, respectively with topologies of compact convergence, respectively with strong topologies, then the mapping $u' : F' \to E'$, where u' is the restriction of u^* on F', is continuous.

F.Riesz theorem. Let $(H,(,))$ be a Hilbert space on $\mathbb{C}$ and H' be its dual space. There exists a bijective mapping $\kappa : (H', \|\ \|') \to (H, \|\ \|)$ (where $\|f\|' = \sup |f(\phi)|;\ \|\phi\| \leq 1\})$ is the dual norm, with the properties: If $f \in H'$ and x is an arbitrary element in H, then $f(x) = (x, \kappa(f))$ and $\|f\|' = \|\kappa f\|;\ \kappa(f_1 + f_2) = \kappa(f_1) + \kappa(f_2);\ \kappa(\alpha f) = \overline{\alpha}\kappa(f)$. (mapping κ is antylinear). It follows that κ is an isometry and that H' is also a Hilbert space.

Reflexive space. Let (E, τ) be an LC space and E' be the dual of E. The space of continuous linear functionals on (E', τ_b), denoted by E'', is called the second dual of E. Clearly, $E \subset E''$. Denote by j the mapping $j : E \longrightarrow E''$ so that if $x \in E$ then $j(x)$ is the functional on (E', τ_b) defined by $E' \ni x' \mapsto \langle x', x \rangle \in K$. This mapping is injective. It is said that (E, τ) semi reflexive if the mapping j is sirjective and if j is the homeomorphism of (E, τ) ont (E'', τ_b) then it is called that $E, \tau)$ is a reflexive space.

Finite-dimensional Hausdorff spaces as well as Hilbert spaces are reflexive. Montel space is also reflexive.