KÄHLER AND CALABI-YAU SPACES [1] [2]

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Abstract

This is a survey article which deals with some basic notions and results in geometry and cohomology theory of a remarkable class of complex manifolds so called Kähler spaces and their subclass of Calabi-Yau spaces. Our approach to these spaces includes differential geometry methods and technics. We emphasize in some details manifolds of the complex dimension 3, because of their role in the string theory. To explain what is a Calabi-Yau space we start with the definition of a smooth manifold, its tangent space, vector fields, a connection and the corresponding curvatures. We introduce also de Rham cohomology groups.

Then we develop the corresponding theory over manifolds with complex structure and point out properties of these spaces making them distinguished for extra studying. Among all complex manifolds we pay the attention especially on Kähler spaces and consider them in some details. Finally we define Calabi-Yau spaces. We deduce formalism on some level being not very precise whenever is possible and explain some notions very roughly. For more details we notify the corresponding references. Examples, involved in some subsections,

[1] Article based on lectures delivered by the author at the Summer School in Modern Mathematical Physics, Sokobanja, Yugoslavia, 13-25, August, 2001

[2] Research partially supported by DFG and Ministry of Science of Serbia, project Geometry, Education and Visualization with Applications

should clarify the corresponding notions, hoping it makes this text more readable.

1 Introduction

As we know a possible background configuration for superstrings is the space $M^4 \times \mathcal{C}$, where M^4 is a four dimensional Minkowski space and $\mathcal{C}$ is a three dimensional Calabi-Yau space. Since a Minkowski space is well known we are interested in the notion of Calabi-Yau space, in the framework of the applications in the superstrings theory. The main purpose of this note is to explain the meaning of the following definition.

Definition 1 *Calabi-Yau space is a compact (three dimensional) complex manifold with a Ricci flat Kähler metric.*

More precisely, we shall consider Calabi-Yau spaces using the complex manifold theory. Of course, to understand these spaces in all details one should be familiar with the algebraic geometry. The full generality overcomes this note and hence we develop this theory as much as possible to omit that parts which need the knowledge of algebraic geometry. We point out also that some notions will be explained only on an intuitive level to succeed at least to advise how this theory is rich and fruitful for a research. More details one can find in [13], [10], [8], [12], [14], [17], [18], etc.

The paper is organized in five sections:

2. Real manifolds
3. Complex manifolds
4. Kähler metrics
5. Ricci flat Kähler merics
6. Geometry and topology of Kähler and Calabi-Yau spaces.

We clarify all notions with the suitable examples for a reader who is interesting in physics, especially in the string theory.

The section 2. considers differential geometry of smooth manifolds and the cohomology theory over these spaces. This is the reason to be interested also in the differential forms. The section 3. deals with complex manifolds. All notions introduced in the section 2. are adopted to the existence of holomorphic transition functions. We consider Kähler spaces in the section 4. These spaces and their metric, Laplacian, as well as topology are studied in terms of Kähler form. To introduce Calabi-Yau spaces we need the curvature for the Levi-Civita connection of Kähler metric. We study these objects in the section 5. To consider geometry and topology of Kähler and Calabi-Yau spaces in full generality it overcomes this paper. Hence we present only few results of mathematicians from our University in the Section 6.

2 Real manifolds

The main purpose of this section is to introduce the basic notions to develop differential geometry of smooth manifolds: a manifold, coordinate systems, tangent and cotangent spaces, differential forms, a connection and the corresponding curvatures. Using the differential forms we develop also the cohomology theory.

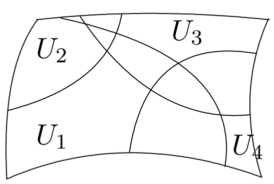

Figure 1: A manifold

2.1 Definition of a manifold

A manifold M is a space which looks like a Euclidean space $\mathbb{R}^n$ around each point. In other words, a manifold M is the union of open subsets U_i of $\mathbb{R}^n$ (see Fig. 1) , i.e.

$$M = \cup U_i, \qquad \forall U_i \subset \mathbb{R}^n.$$

Open subsets U_i in general case do not have to be open subsets of $\mathbb{R}^n$ but only homeomorphic to open subsets of $\mathbb{R}^n$.

One may illustrate this very important notion with some examples.

2.2 Examples

1. The Euclidean space $\mathbb{R}^n$; especially, for $n = 1$ $\mathbb{R} \equiv \mathbb{R}^1$ is a Euclidean line:

Figure 2: Euclidian line

2. *n-sphere* S^n *immersed in* $\mathbb{R}^{n+1}$ may be defined by $\sum_{i=1}^{n+1} x_i^2 = c^2$, where $c = \text{const}$ and $(x_1, \ldots, x_{n+1})$ are Desquartes coordinates. For $n = 0, 1, 2$ see Fig.3.

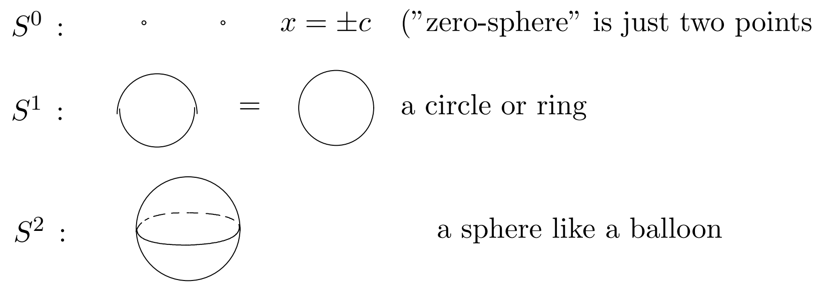

Figure 3: Spheres

3. *Projective spaces* $P^n(\mathbb{R})$.

   Let us explain this manifold giving its three equivalent models.

   So the projective line may be obtained from an affine line if one adds to that line one point, so called *infinite point.* If we do the same with all lines in an affine space $\mathcal{A}^n$, such that parallel affine lines have the common infinite point (see Fig. 4), one obtains the model $\mathcal{P}_a(\mathbb{R})$ of a projective space $P^n(\mathbb{R})$. The second model $\mathcal{P}_l(\mathbb{R})$ of a projective space $P^n(\mathbb{R})$ one may obtain assuming that points of $\mathcal{P}_l(\mathbb{R})$ are lines in $\mathbb{R}^{n+1}$ passing through the origin O.

   Finally, the third model $\mathcal{P}_s(\mathbb{R})$ is built from a sphere S^n with identified antipodal points.

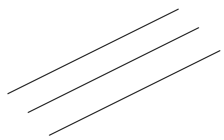

Figure 4: Parallel affine lines

4. *Group manifolds* are defined by the space of free parameters in the defining representation of a group. So we have:

   - $\mathbb{Z}_2 = \{(0,1), +_2\}$ is an additive group with the summation $+_2$ modulo 2; but $\mathbb{Z}_2$ is also the group generated by multiplication by (-1), and thus has elements ± 1.

     Consequently, $\mathbb{Z}_2$ may be identified with S^0, i.e. $\mathbb{Z}_2 = S^0$.

   - Let $U(1)$ be the group of multiplication by unimodular complex numbers with elements $e^{i\theta}$. Since $0 \leq \theta < 2\pi$ may parametrize a circle we have $U(1) = S^1$.

   - We know that

     $$SU(2) = \left\{ u = \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix}, a = x_1 + ix_2, b = x_3 + ix_4, \det u = |a|^2 + |b|^2 = \sum_{i=1}^{4} x_i^2 = 1 \right\}.$$

     As one can identify the parameter space of the group $SU(2)$ with the manifold of the 3-sphere S^3 it follows

$$SU(2) = S^3.$$

- One can verify also the following identification:

$$SO(3) = SU(2)/\mathbb{Z}_2 = P^3(\mathbb{R}),$$

where $SO(3)$ is the special orthogonal group of 3×3 type matrices.

So far, we gave only examples of manifolds. To see that all sets of points are not manifolds we illustrate it with some examples too. One-dimensional spaces which are not manifolds are given with the pictures (see Fig. 5)

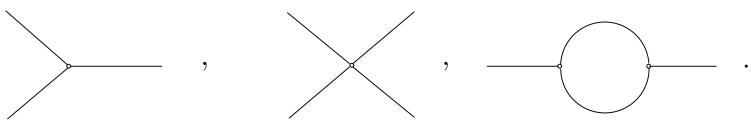

Figure 5: Spaces which are not manifolds

The condition that the space looks locally like $\mathbb{R}^1$ is not fulfilled at the junctions.

2.3 Boundary of a manifold

We do not introduce this notion in a formally correct way, as it overcomes the framework of this note. To understand this rather complicated notion on an intuitive level we explain it through some examples.

The boundary $\partial(\overline{AB})$ of a line segment $\overline{AB}$ consists of the two points A, B, and one may write $\partial(\overline{AB}) = \{A, B\}$. If D^2 is a disk, then its boundary ∂D^2 is S^1, i.e. $\partial D^2 = S^1$ (see Fig. 6 respectively).

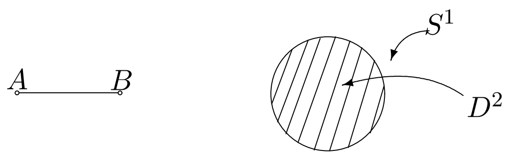

Figure 6: Boundaries of a segment and a disk

In general case if $\dim M = n$, and M has a boundary ∂M, then $\dim M = n - 1$ and $\partial\partial M = \emptyset$.

2.4 Coordinate systems

Let $M = \cup_{i\in A} U_i$, where U_i are open sets covering M, and $\phi_i : U_i \longrightarrow \mathbb{R}^n$ homeomorphisms, for all $i \in A$. Let $\Sigma = U_i \cap U_j \neq \emptyset$. Then

$$\phi_{ij} = \phi_j \circ \phi_i^{-1} : \phi_i(\Sigma) \longrightarrow \phi_j(\Sigma)$$

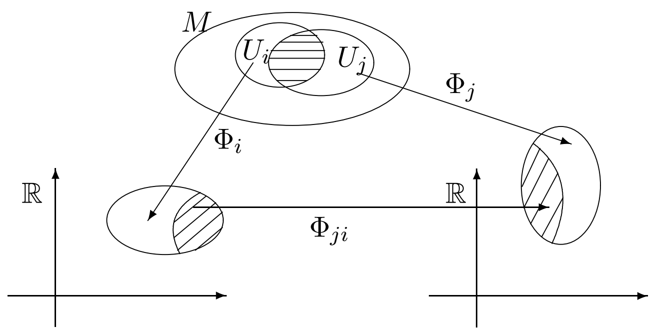

Figure 7: Transition functions

we call *the transition function* (see Fig. 7). If $p \in U_i$ it follows $\phi_i(p) = (x^1, \ldots, x^n)$ are *coordinates of the point* p with respect to *the coordinate system* ϕ_i which is defined on U_i. Having in mind properties of ϕ_{ij} we get some classes of manifolds. If all transition functions ϕ_{ji} have continuous partial derivatives of all orders we say M is C^∞ or *smooth manifold.* M is *a real analytic manifold* if all ϕ_{ij} are real analytic functions. Finally, M is *a complex manifold* if all ϕ_{ij} are holomorphic, i.e. complex valued functions with complex power series.

2.5 Examples

1. Let S^2 be a 2-*dimensional sphere.* One may show that S^2 can be covered with two open sets U_1 and U_2, such that $\emptyset \neq U_1 \cap U_2 = S^2 \setminus \{N, S\}$, $N \in U_1$, $S \in U_2$. We can define the corresponding coordinate systems ϕ_1, ϕ_2 (see Fig. 8) such that the transition function is the following one
$$\phi_{12}(x, y) = \left(\frac{x}{x^2 + y^2}, \frac{-y}{x^2 + y^2} \right).$$

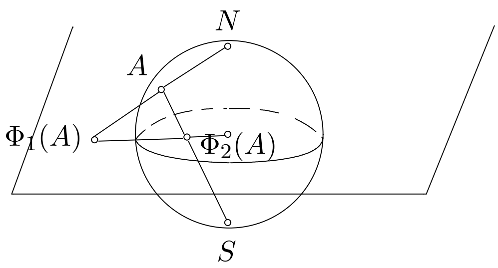

Figure 8: Coordinate systems of a sphere

Therefore S^2 is a smooth manifold.

If we put $z = x + iy$, consequently we have $\phi_{12}(z) = 1/z$ and hence S^2 is also the complex manifold.

2. *Lie groups in general.* Let A be a matrix. Then $\exp(A) = I + A + \ldots + \frac{1}{n!}A^n + \ldots$ converges to an invertible matrix. Assuming that a Lie group G is one of the followings: $GL(k, \mathbb{R})$, $GL(k, \mathbb{C})$, $U(k)$, $SU(k)$, $O(k)$, $SO(k)$, and $\mathfrak{g}$ its Lie algebra then $\exp : \mathfrak{g} \longrightarrow G$ is the diffeomorphism which defines a coordinate system near $I \in G$. A coordinate system near $g_0 \in G$ can be defined by mapping $\mathfrak{g} \longrightarrow g_0 \exp \mathfrak{g}$. The transition functions are thus given by left multiplication in the group.

2.6 Tangent space and cotangent space

Roughly speaking the tangent space T_pM of a smooth manifold M in a point p is the linear approximation of M in some neighbourhood U of p. To introduce T_pM more precisely, let $y = f(x)$ be a real map and $x = p+v$ be some point near p. Then expanding this function around p in a Taylor series it yields

$$f(x = p+v) = f(p) + v \left.\frac{df}{dx}\right|_{x=p} + \ldots$$

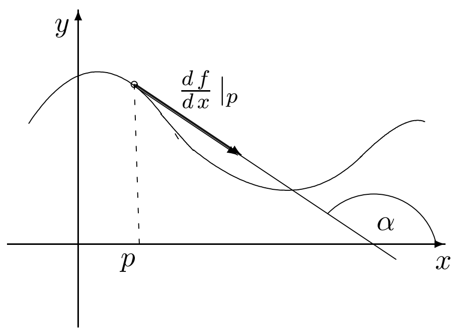

Figure 9: The tangent line of a curve

As we know $\left.\frac{df}{dx}\right|_p = \tan\alpha$, where α is the angle between x axis and *the tangent line of the curve* $y = f(x)$ *in the point* p (see Fig. 9). Considering

n-dimensional surface with coordinates x^i $(i = 1, \dots, n)$ the corresponding second term in Taylor series is the following one

$$\sum_{i=1}^{n} v^i \left. \frac{\partial f(x)}{\partial x^i} \right|_{x=p} := v^i \frac{\partial f(x)}{\partial x^i}, \tag{1}$$

where, from now on, we use *Einstein convention of summation* when we have repeated index. One may show $v^i \frac{\partial}{\partial x^i} \mid_{x=p}$ has an intrinsic meaning not only on a surface but also on an arbitrary smooth manifold M. Hence $v^i \frac{\partial}{\partial x^i}$ with smoothly varying coefficients $v^i(x)$ is called *a vector field*. Furthermore, the operators $\left\{\frac{\partial}{\partial x^i}\right\}$ at $x = p$ define a basis for the tangent space of M at the point p. One can prove also that T_pM is the vector space spanned by the tangents at p to all curves passing through p in the manifold M, and its dimension is n at each point $p \in M$.

Since we obtain the tangent space using the second term (1) in Taylor series it says for instance that at a point p a manifold M can be approximated by the vector space up to the first order, not merely up to the zero-th order. Further theory reveals that T_pM may be some of special vector spaces (Euclidean, Hermitian etc.) which depends on the structure of M.

The tangent space occurs naturally in classical mechanics. Let $L(q^i(t), \dot{q}^i(t))$ be a Lagrangian. Then t-derivatives can be defined using the implicit function rule

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \dot{q}^i \frac{\partial}{\partial q^i}. \tag{2}$$

The second term in the previous equation has the structure of a vector field. *Velocity space* in Lagrangian classical mechanics corresponds exactly to the tangent space of *the configuration space*: if M has coordinates $\{q^i\}$, then $T_q(M)$ has coordinates $\{\dot{q}^i\}$. Comparison (2) with (1) shows that the operators $\left\{\frac{\partial}{\partial q^i}\right\}$ form a basis for T_qM.

Since T_pM has a structure of a vector space, having in mind our knowledge of linear algebra we are ready to be interested in its dual space T_p^*M, $p \in M$, so called *the cotangent space* of a manifold at $p \in M$.

To understand better T_p^*M we recall some facts from the corresponding theory. Let E_i be a basis of V. Then the basis e^j of V^* is given by their inner product

$$< E_i, e^j >= \delta_i^j.$$

Especially, if $E_i = \frac{\partial}{\partial x^i}$ is a coordinate basis of T_pM then $e^i = dx^i$ is given by the differential line elements. We consider *the covector field* $U = u_i dx^i$.

A motive to study a cotangent space T_p^*M one may find also in classical mechanics as cotangent space corresponds to momentum space. As the basis vectors are given by the differential line elements dq^i, the covector fields are the followings $p_i dx^i$, assuming we identify $p_i = \partial L(q^i, \dot{q}^i)/\partial \dot{q}^i$. Using the construction of tensor product of vector spaces one may introduce *a tensor field* $\omega_{(l)}^{(k)}$ over M as

$$\omega_{(l)}^{(k)} = \omega_{j_1 \ldots j_l}^{i_1 \ldots i_k} \frac{\partial}{\partial x^{i_1}} \otimes \ldots \otimes \frac{\partial}{\partial x^{i_k}} \otimes dx^{j_1} \otimes dx^{j_l}.$$

Consequently, this is an element of

$$\underbrace{T_pM \otimes \ldots \otimes T_pM}_{k} \underbrace{T_p^*M \otimes \ldots \otimes T_p^*M}_{l}.$$

2.7 Differential forms

A tensor product includes neither symmetrization nor antisymmetrization of indices. Anyhow, it is shown the totally antisymmetric covariant tensor fields (elements of $T_p^*M \otimes \ldots \otimes T_p^*M$) are powerful tool for many practical calculations not only in differential geometry but also in cohomology theory, integration over a manifold, global analysis, etc. We call these tensor fields *differential forms.* We omit their consideration in full generality and pay the attention on important details expressed in terms of basis elements of $T_p^* M$, dual to coordinate basis of $T_p M$.

Let dx, dy be two basis elements of $T_p^* M$. *Cartan's wedge product* or *exterior product* $\wedge$ of dx and dy is defined by

$$dx \wedge dy = \frac{1}{2}(dx \otimes dy - dy \otimes dx).$$

One may check directly

$$dx \wedge dy = -dy \wedge dx,$$

$$dx \wedge dx = 0.$$

The differential line elements dx, dy are differential 1-forms or 1-forms and its wedge product defines 2-form $dx \wedge dy$. Using the linearity of tensor product $\otimes$ one may compute Cartan's wedge product of arbitrary two 1-forms $\alpha = \alpha_i dx^i$ and $\beta = \beta_j dx^j$, such that we have

$$\alpha \wedge \beta = \alpha_i \beta_j dx^i \wedge dx^j = -\beta \wedge \alpha.$$

Consequently, the wedge product is a rule for constructing 2-forms out of pairs of 1-forms.

Let $\Lambda^p(x)$ be the set of p-forms at a point x, and $C^\infty(\Lambda^p)$ the space of smooth p-forms whose elements are anti-symmetric tensors $f_{ij\ldots}(x)$ having p-indices contracted with the wedge products of p-differentials. This is a vector space of dimension $n!/p!(n-p)!$. To clarify rather complicated explanation in terms of indices and contractions we give explicitly elements of $C^\infty(\Lambda^p)$ and corresponding dimensions for $p = 0, 1, 2, 3, \ldots, n-1, n$:

$$
\begin{array}{ll}
C^\infty(\Lambda^0) = \{f(x)\} & \dim = 1 \\
C^\infty(\Lambda^1) = \{f(x)dx^i\} & \dim = n \\
C^\infty(\Lambda^2) = \{f(x)_{ij}dx^i \wedge dx^j\} & \dim = \frac{n(n-1)}{2!} \\
C^\infty(\Lambda^3) = \{f(x)_{ijk}dx^i \wedge dx^j \wedge dx^k\} & \dim = \frac{n(n-1)(n-2)}{3!} \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \ldots\ldots\ldots \\
C^\infty(\Lambda^{n-1}) = \{f_{i_1\ldots i_{n-1}}dx^{i_1} \wedge \ldots dx^{i_n}\} & \dim = n \\
C^\infty(\Lambda^n) = \{f_{i_1\ldots i_n}dx^{i_1} \wedge dx^{i_n}\} & \dim = 1.
\end{array}
$$

Several important properties of p-forms and the wedge products are recognizable at once. First of all the wedge product of p- and q-form is $(p+q)$-form and hence

$$\Lambda^p(x) \wedge \Lambda^q(x) = \Lambda^{p+q}(x).$$

Because of antisymmetry properties we have

$$\Lambda^p = 0, \qquad p \geq n.$$

Consequently, the wedge product and linear operations among differential forms give the resulting form which always belong to the original set of spaces denoted by

$$\Lambda^* = \Lambda^0 \oplus \Lambda^1 \oplus \ldots \oplus \Lambda^n.$$

This is a graded algebra called *Cartan's exterior algebra of differential forms.*

Since arbitrary p-form depends on smooth functions $f_{ij\ldots}$ we may introduce some maps between these spaces $C^\infty(\Lambda^p)$ $(p = 0, 1, \ldots, n)$ in terms of the differentiation. So we have

$$
\begin{array}{ll}
d : C^\infty(\Lambda^0) \longrightarrow C^\infty(\Lambda^1); & d(f(x)) = \frac{\partial f}{\partial x^i}dx^i, \\
d : C^\infty(\Lambda^1) \longrightarrow C^\infty(\Lambda^2); & d(f_i(x)dx^i) = \frac{\partial f_i}{\partial x^j}dx^j \wedge dx^i, \\
d : C^\infty(\Lambda^2) \longrightarrow C^\infty(\Lambda^3); & d(f_{ij}(x)dx^i \wedge dx^j) = \frac{\partial f_{ij}}{\partial x^k}dx^k \wedge dx^i \wedge dx^j.
\end{array}
$$

One may reveals simple for $p = 0, 1 \ldots, n$ it makes sense the following map

$$d \, : \, C^{\infty}(\Lambda^p) \longrightarrow C^{\infty}(\Lambda^{p+1}),$$

such that

$$d^2 = 0.$$

This map is called *the exterior derivative.* Pay the attention for $p = 0, 1, 2$ it is given exactly by the foregoing relations and henceforth one may generalize a rule of exterior derivation of an arbitrary $p-$form.
The exterior derivation is connected with the wedge product of $p-$ and $q-$forms respectively by the equation

$$d(\alpha_p \wedge \beta_q) = d\alpha_p \wedge \beta_q + (-1)^p \alpha_p \wedge d\beta_q,$$

which seems like "generalized Leibnitz rule".

2.8 Example

One may check that possible $p-$forms α_p in 2-dimensional space M are these ones:

$$\begin{aligned} \alpha_0 &= f(x, y), \\ \alpha_1 &= u(x, y)dx + v(x, y)dy, \\ \alpha_2 &= \phi(x, y)dx \wedge dy, \end{aligned}$$

and their exterior derivatives

$$\begin{aligned} d\alpha_0 &= \tfrac{\partial f}{\partial x}dx + \tfrac{\partial f}{\partial y}dy, \\ d\alpha_1 &= \left(\tfrac{\partial v}{\partial x} - \tfrac{\partial u}{\partial y}\right) dx \wedge dy, \\ d\alpha_2 &= 0. \end{aligned}$$

Among all $p-$forms we have two remarkable classes, important for example in the cohomology theory. A $p-$ form ω is *closed* if $d\omega = 0$. A $p-$form ω is *exact* if $\omega = d\alpha$ for some globally defined $(p-1)$-form α. It is simple to see that every exact form $\omega (= d\alpha)$ is also closed one, as $d\omega = d^2\alpha = 0$. There is an interesting phenomenon for $M = \mathbb{R}^n$. Every closed form defined on $\mathbb{R}^n$ is also exact. If M is an arbitrary manifold this is true only locally, but not globally in general.

We are ready to introduce the p^{th} cohomology group, $H^p(M)$, one of the important topological notions for our subsequent studies.

Definition 2 *The p^{th} cohomology group $H^p(M)$ is the set of all closed $p-$forms where two forms ω and ω' are considered equivalent if $\omega - \omega'$ is exact.*

$H^p(M)$ are real cohomology groups with the structure of real vector spaces even its name points out only the group structure.

The dimension of $H^p(M)$ is also significant topological notion, named the p^{th} *Betti number*, which we denote $b_p := \dim H^p(M)$.

It is simple to check that $H^0(M)$ is the space of constant functions. To find b_p or $H^p(M)$ in general case it might be a very complicated problem. We compute them here in the simplest case just to clarify the previous definitions.

2.9 Examples

- $b_0(M) = 1$ for a connected manifold M, and generally b_0 is equal to the number of connected pieces of the manifold.

- $\dim H^p(\mathbb{R}^n) = 0$, for $p > 0$ and $\dim H^0(\mathbb{R}^n) = 1$.

- $H^n(M, \mathbb{R}) = \mathbb{R}$, as all forms ω_n, elements of $H^n(M, \mathbb{R})$, differ from a volume element of M for a constant factor.

  We recall once again H^0 consists of the constant functions and H^n consists of the constant multiplies of the volume element.

- Let S^n be an n-dimensional sphere. Then

$$\dim H^p(S^n, \mathbb{R}) = 0, \quad \text{for } 0 < p < n,$$
$$\dim H^0(S^n, \mathbb{R}) = \dim H^n(S^n, \mathbb{R}) = 1.$$

- Let $T^2 = S^1 \times S^1$ be a two-dimensional torus. We denote by $\theta_1, \theta_2,\ 0 \leq \theta_i < 2\pi$ coordinates on each of the two circles making up the torus. It is simple to see $d\theta_i$ are closed but not exact forms, since θ_i are defined only modulo 2π and are not global coordinates. Consequently

$$\dim H^1(T^2, \mathbb{R}) = 2.$$

$H^2(T^2; \mathbb{R})$ is generated by $d\theta_1 \wedge d\theta_2$ and furthermore using Küneth formula we get

$$H^2(T^2 = S^1 \times S^1; \mathbb{R}) = H^1(S^1, \mathbb{R}) \otimes H^1(S^1, \mathbb{R}),$$

and hence

$$\dim H^2(T^2, \mathbb{R}) = 1.$$

We correspond *the Euler number* $\aleph(M)$ to M via alternating sum of b_p. More precisely we have

$$\aleph(M) = \sum_{p=0}^{n} (-1)^p b_p.$$

The theory, represented so far here in terms of differential forms, has not a metric character. For this one, that follows, we need also a metric on M. Let

$$ds^2 = g_{\mu\nu}(x) dx^\mu dx^\nu, \quad g_{\mu\nu} = g_{\nu\mu}$$

be *the first fundamental form* of M or a *Riemannian metric*. Let ω and τ be two arbitrary $p-$forms. Then

$$\langle \omega | \tau \rangle := \int_M \omega^{\mu_1 \ldots \mu_p} \tau_{\mu_1 \ldots \mu_p} \sqrt{g} dx^1 \wedge \ldots \wedge dx^n$$

is their inner product, where

$$g = \det(g_{\mu\nu}).$$

To introduce *the Hodge operator* $* : H^p(M) \longrightarrow H^{n-p}(M)$ one may rewrite the inner product of ω and τ in the following way

$$\langle\omega|\tau\rangle := \int_M \omega \wedge *\tau.$$

If $\omega_p \in C^\infty(\Lambda^p M)$ then

$$**\,\omega_p = (-1)^{p(n-p)}\omega_p.$$

We use the Hodge operator $*$ to introduce the operator δ which is adjoint of exterior derivative. One may check by direct computations that

$$\delta := (-1)^{np+n+1} * d*$$

is an operator

$$\delta : C^\infty(\Lambda^p) \longrightarrow C^\infty(\Lambda^{p-1})$$

with the properties:

$$\begin{aligned} \langle\omega_p|d\tau_{p-1}\rangle &= \langle\delta\omega_p|\tau_{p-1}\rangle, \\ \delta\delta\omega_p &= 0. \end{aligned}$$

One may use these operators d and δ to introduce *the Laplacian*

$$\Delta = (d+\delta)^2 := d\delta + \delta d,$$

which acts on $p-$forms ($0 \leq p \leq n$) on M. We compute

$$\begin{aligned} \langle\omega_p|\Delta\omega_p\rangle &= \langle\omega_p|d\delta\omega_p\rangle + \langle\omega_p|\delta d\omega_p\rangle \\ &= \langle\delta\omega_p|\delta\omega_p\rangle + \langle d\omega_p|d\omega_p\rangle \geq 0 \end{aligned}$$

to see Δ is *a positive operator.*

A remarkable class of $p-$forms are *harmonic* whose image under the action of Δ is zero, i.e.

$$\Delta\omega_p = 0.$$

Since Δ is positive the condition ω_p being harmonic is equivalent to these ones $d\omega_p = 0$ and $\delta\omega_p = 0$:

$$\Delta\omega_p = 0 \Longleftrightarrow d\omega_p = 0,\ \ \delta\omega_p = 0.$$

Among differential forms exact, co-exact and harmonic forms are sufficient to express an arbitrary form in their terms. More precisely we have the statement in the well-known Hodge theorem (1952):

Theorem 2.1 *If M is a compact manifold without boundary, any $p-$form ω_p can be uniquely decomposed as a sum of exact, co-exact and harmonic forms*

$$\omega_p = d\alpha_{p-1} + \delta\beta_{p+1} + \gamma_p,$$

where γ_p is a harmonic $p-$form.

Hodge theorem implies there is a unique harmonic representative for each equivalence class in $H^p(M)$.

How one can deal with these operators we illustrate on $M = \mathbb{R}^2$ with the standard Euclidean metric:

Basis of Λ^*: $(1,\, dx,\, dy,\, dx \wedge dy)$,
Hodge $*$: $*(1,\, dx,\, dy,\, dx \wedge dy) = (dx \wedge dy,\, dy,\, -dx,\, 1)$,
$\delta f(x,y) = 0$,
$\delta(udx + vdy) = -(\partial_x u + \partial_y v)$,
$\delta\phi dx \wedge dy = -\partial_x\phi dy + \partial_y\phi dx$,
$\Delta f = -(\partial_x^2 f + \partial_y^2 f)$.

2.10 Connections on a manifold

Some analysis of the previous consideration tell us that we do not connect neither vectors from tangent spaces T_mM and T_nM in any two points $m, n \in M$ nor differential forms, built over T^*_mM, T^*_nM. Anyhow we know that any vector $v \in T_p\mathbb{R}^2$ may be corresponds to other $v' \in T_q\mathbb{R}^n$ under parallel displacement

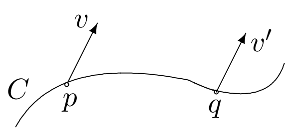

Figure 10: Parallel displacement of a vector along curve

along any curve C (see Fig. 10). To generalize this idea on any manifold we need to introduce a notion of a connection. We present here the Cartan approach via forms and later on we give relations to classical tensor calculus approach.

Let $e^a = a^a{}_\mu dx^\mu$ and $E_a = E_a{}^\mu \frac{\partial}{\partial x^\mu}$ be any orthonormal basis of T^*_xM and T_xM respectively. *The affine spin connection one-form* $\omega^a{}_b$ is given by the relation

$$de^a + \omega^a{}_b \wedge e^b \equiv T^a \equiv \tfrac{1}{2} T^a{}_{bc} e^b \wedge e^c, \tag{3}$$

where T^u is called *the torsion* $2-$*form* of M. *The curvature* $2-$*form* $R^a{}_b$ is defined by

$$R^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b = \frac{1}{2} R^a{}_{bcd} e^c \wedge e^d. \tag{4}$$

The equations (3) and (4) are called *the Cartan's structure equations.*

We differentiate (4) to obtain *the Bianchi identities*

$$dR^a{}_b + \omega^a{}_c \wedge R^c{}_b - R^a{}_c \wedge \omega^c{}_b = 0.$$

Let $V^a{}_b$ be a differential form of degree p. Then its covariant derivative is defined as

$$DV^a{}_b = dV^a{}_b + \omega^a{}_c \wedge V^c{}_b - (-1)^p V^a{}_c \wedge \omega^c{}_b.$$

The Levi-Civita spin connection is an affine spin connection which satisfies two conditions:

$$\begin{aligned} &\text{metricity}: \quad \omega_{ab} = -\omega_{ba}, \\ &\text{no torsion}: \quad T^a = de^a + \omega^a{}_b \wedge e^b = 0. \end{aligned}$$

For a given metric $(g_{\alpha\beta})$ on M there exists the unique Levi-Civita spin connection.
Using the classical tensor calculus these two conditions are expressed as

$$\begin{aligned} &\text{metricity}: \quad g_{\mu\nu;\alpha} = \partial_\alpha g_{\mu\nu} - \Gamma^\lambda_{\alpha\mu} g_{\lambda\nu} - \Gamma^\lambda_{\alpha\nu} g_{\mu\lambda} = 0, \\ &\text{no torsion}: \quad T^\mu{}_{\alpha\beta} = \tfrac{1}{2}(\Gamma^\mu_{\alpha\beta} - \Gamma^\mu_{\beta\alpha}) = 0, \end{aligned}$$

where *the Cristoffel symbols* $\Gamma^\mu_{\alpha\beta}$ are determined by the relations

$$\Gamma^\mu_{\alpha\beta} = \frac{1}{2} g^{\mu\nu} (\partial_\alpha g_{\nu\beta} + \partial_\beta g_{\nu\alpha} - \partial_\nu g_{\alpha\beta}).$$

The components of the corresponding *Riemann curvature* are as

$$R^\alpha{}_{\beta\mu\nu} = \partial_\mu \Gamma^\alpha_{\nu\beta} - \partial_\nu \Gamma^\alpha_{\mu\beta} + \Gamma^\alpha_{\mu\gamma} \Gamma^\gamma_{\nu\beta} - \Gamma^\alpha_{\nu\gamma} \Gamma^\gamma_{\mu\beta};$$

its Ricci tensor has the components

$$\rho_{\mu\nu} = R_{\mu\alpha\nu\beta} g^{\alpha\beta};$$

and finally, we have *the scalar curvature*:

$$\tau = \rho_{\mu\nu} g^{\mu\nu}.$$

The Riemann curvature gives the complete information about "the shape of a manifold M". But, since it has $n^2(n^2-1)/12$ independent components, it might be rather inconvenient to use all of them. For many investigations it is enough to use the Ricci tensor and sometimes only the scalar curvature. Further theory reveals that curvature is precisely the second order effect which essentially measures deviation from the Euclidean case.

2.11 The holonomy group

We finish this short review about Riemannian geometry with holonomy group. A holonomy group of the space M^n in the case when it differs from $SO(n)$ carries a considerable amount of information about the differential geometry and topology of the space. For example, knowing the holonomy group of a symmetric space, one can effectively calculate its cohomology algebra (which in this case is isomorphic to the algebra of parallel differential forms). A holonomy group heavily depends on the parallel transport with respect to any connection ∇.

Let ∇ be a given connection, e a basis at $T_p(M)$, γ a curve passing through a point $p \in \gamma$. We denote by e' the image of e, obtained by parallel transport along γ with respect to ∇ (see Fig. 11).

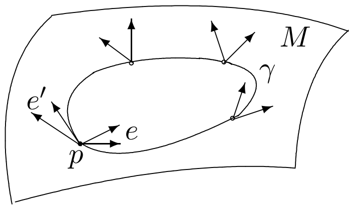

Figure 11: Parallel transport of a basis

Consequently we may write $e' = ge$, where $g \in GL(n, \mathbb{R})$. If we make the same procedure along all closed curves γ, then all $g \in GL(n, \mathbb{R})$ build the subgroup $G \subset GL(n, \mathbb{R})$. This subgroup is independent of the original point p and is called *the holonomy group of the connection*. We have that $G \subset SO(n, \mathbb{R})$ if the connection preserves a metric (and M is orientable). We refer [4], [1], [2], [16] for more details.

3 Complex manifolds

We have introduced already complex manifolds in the section 2. requiring the transition functions being holomorphic. In this section we use another approach to define a complex manifold. This approach will be used later just to announce how many geometries on the same manifold is possible to have and how are reach some of these geometries with corresponding properties. To develop the theory of complex manifolds we will have in mind all what we have introduced in the previous section (coordinate systems, tangent and cotangent spaces, differential forms, cohomologies, Hodge theorem) to see how is it possible to adopt these objects to the holomorphic transition functions. Of course, the existence of holomorphic transition functions implies various consequences and sometimes a specific approach to these previously mentioned objects.

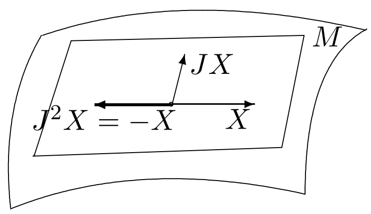

Figure 12: A complex structure

A complex manifold is a real manifold endowed with a complex structure. *A complex structure* on a real manifold M is an endomorphism J, $J^2 = -id$ satisfying an integrability condition $\nabla J = 0$ (see Fig. 12). As a consequence of this integrability condition one can prove that we may introduce local complex coordinates z^j on M such that the transition functions between different coordinate patches are holomorphic. For a complex manifold we have the corresponding real and complex dimensions:

$$\dim_{\mathbb{R}} M = 2n, \quad \dim_{\mathbb{C}} M = n.$$

From here it follows that only evendimensional real manifolds may admit a complex structure. But, it is interesting all these manifolds do not admit anyhow a complex structure. This fact makes this theory more attractive.

We do not consider now the criterions of its existence. We mention only examples of manifolds which admit and which do not admit any complex structure. So S^6 does not admit a complex structure, while $S^2 = P^1(\mathbb{C})$ does admit as well as $S^p \times S^q$, for p, q odd numbers. Consequently $S^3 \times S^3$, $S^2 \times S^2 \times S^2$ are complex manifolds. The torus T^6 is also the complex manifold.

The complex projective space $P^n(\mathbb{C})$ is also one typical example of complex manifolds. Moreover topology and geometry of this space are very reach and hence we explain here how one can define $P^n(\mathbb{C})$. Let $z = (z_0, \dots, z_n) \neq 0$ determines a complex line through origin. We introduce the equivalence relation $\sim$ between points with the condition

$$z \sim z' \quad \longleftrightarrow \quad z = cz', \;\; c = const \neq 0.$$

It implies all points $\neq 0$ of the same complex line are equivalent and determine one point of $P^n(\mathbb{C})$. Let $U_k \subset P^n(\mathbb{C})$ be a set of lines with $z_k \neq 0$. Then

$$\zeta_i^{(k)} = \frac{z_i}{z_k}$$

is well defined on U_k and it yields ordered multiplies of complex numbers

$$\zeta^{(k)} = (\zeta_0^{(k)}, \dots, \zeta_n^{(k)}) \quad \text{(we omit } \zeta_k^{(k)} = 1).$$

We use

$$\zeta_i^{(j)} = \frac{z_i}{z_k}\frac{z_k}{z_j} = \zeta_i^{(k)}(\zeta_j^{(k)})^{-1}$$

on $U_k \cap U_j$ to see finally $P^n(\mathbb{C})$ is the complex manifold. We call $(\zeta_0^{(k)}, \dots, \zeta_n^{(k)})$ *inhomogeneous coordinates* and $(z_0, \dots, z_n)$ *homogeneous coordinates.*

3.1 Equivalence of complex manifolds

One may ask now the question: how can we recognize some differences between two-dimensional torus T^2, $T^{'2}$ or all of them are the same, i.e.

one is the image of another one under some map.

We answer this question starting from more general point of view. Let M, N be two complex manifolds and $\varphi : M \longrightarrow N$ $1-1$ and *onto* map, such that φ, φ^{-1} are holomorphic. Then we say M and N are *equivalent* and write $M \sim N$.

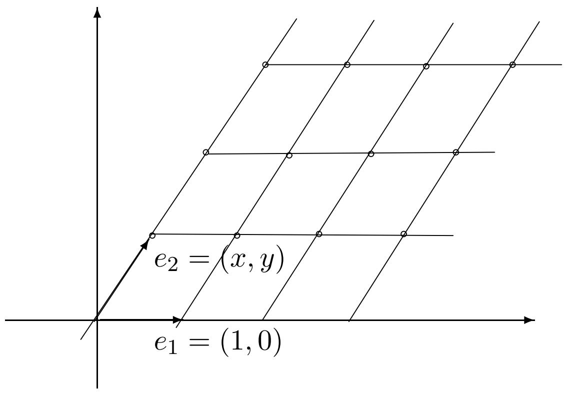

Figure 13: A lattice defined by two vectors

Let $\mathcal{L}$ be a lattice defined by vectors $e_1 = (1, 0)$ and $e_2 = (x, y)$, where $y > 0$ (see Fig. 13). Then

$$T_z = \mathbb{C}/\mathcal{L}, \quad z = x + iy$$

as a real manifold is diffeomorphic to T^2. It is possible to prove

$$T_z \sim T_{z'}$$

only if

$$z' = \frac{az+b}{cz+d},$$

where a, b, c, d are integers, such that

$$ad - bc = 1.$$

3.2 Cohomology for complex manifolds

Following the method of introducing cohomology groups for real manifolds it is clear we need, first of all, to give the notions of tangent and cotangent spaces, differential forms and the corresponding differential operators. Since complex numbers $z = x + iy$ have their conjugate $\bar{z} = x - iy$ and $z = \bar{z}$ only if $z \in \mathbb{R}$ we shall see that this phenomenon implies results of different type from these ones in the section 2.

Let M be a complex manifold with local complex coordinates $(z^1, \ldots, z^n)$, where

$$z^k = x^k + iy^k \quad and \quad \bar{z}^k = x^k - iy^k.$$

We define

$$\begin{array}{ll} dz^k = dx^k + idy^k, & d\bar{z}^k = dx^k - idy^k \\ \frac{\partial}{\partial z^k} = \frac{1}{2}\left(\frac{\partial}{\partial x^k} - i\frac{\partial}{\partial y^k}\right), & \frac{\partial}{\partial \bar{z}^k} = \frac{1}{2}\left(\frac{\partial}{\partial x^k} + i\frac{\partial}{\partial y^k}\right), \end{array}$$

so that we have

$$df = \sum \frac{\partial f}{\partial z^k} dz^k + \sum \frac{\partial f}{\partial \bar{z}^k} d\bar{z}^k = \partial f + \bar{\partial} f.$$

Consequently, if $f(z)$ is a holomorphic function of a single variable then

$$\bar{\partial} f = \frac{\partial f}{\partial \bar{z}} d\bar{z} = 0.$$

Similarly we have for functions of more complex variables. If

$$\frac{\partial f}{\partial \bar{z}^k} = 0, \;\; k = 1, \ldots, n \quad or \quad \bar{\partial} f = 0$$

then a function f defined on $\mathbb{C}^n$ is *holomorphic.*

We are ready now to introduce the tangent and cotangent spaces

$$T_cM = \left\{\tfrac{\partial}{\partial z^j}\right\}, \quad \bar{T}_cM = \left\{\tfrac{\partial}{\partial \bar{z}^j}\right\},$$
$$T_c^*M = \{dz^j\}, \quad \bar{T}_c^*M = \{d\bar{z}^j\}.$$

One can check simply these spaces are independent of the particular local complex coordinates which are chosen as well as

$$T(M) \otimes \mathbb{C} = T_cM \oplus \bar{T}_cM,$$
$$T^*(M) \otimes \mathbb{C} = T_c^*M \oplus \bar{T}_c^*M.$$

Let $\Lambda^{p,q}$ be the set of complex exterior forms, whose elements ω are of the following type

$$\omega = \omega_{i_1 \ldots i_p \bar{j}_1 \ldots \bar{j}_q} dz^{i_1} \wedge \ldots \wedge dz^{i_p} \wedge d\bar{z}^{\bar{j}_1} \wedge \ldots d\bar{z}^{\bar{j}_q}.$$

The operator ∂ acts as

$$\partial : \Lambda^{p,q} \longrightarrow \Lambda^{p+1,q},$$

and similarly

$$\bar{\partial} : \Lambda^{p,q} \longrightarrow \Lambda^{p,q+1},$$

such that we have

$$\partial\omega = \tfrac{\partial\omega_{i_1 \ldots i_p \bar{j}_1 \ldots \bar{j}_q}}{\partial z^k} dz^k \wedge dz^{i_1} \wedge \ldots \wedge dz^{i_p} \wedge d\bar{z}^{\bar{j}_1} \wedge \ldots d\bar{z}^{\bar{j}_q},$$
$$\bar{\partial}\omega = \tfrac{\partial\omega_{i_1 \ldots i_p \bar{j}_1 \ldots \bar{j}_q}}{\partial \bar{z}^{\bar{k}}} d\bar{z}^{\bar{k}} \wedge dz^{i_1} \wedge \ldots \wedge dz^{i_p} \wedge d\bar{z}^{\bar{j}_1} \wedge \ldots d\bar{z}^{\bar{j}_q}.$$

It is simply to verify

$$\partial^2 = \bar{\partial}^2 = 0,$$
$$\partial\bar{\partial} + \bar{\partial}\partial = 0.$$

We introduce holomorphic forms as a generalization of holomorphic functions. So, if $\omega \in \Lambda^{p,0}$ and $\bar{\partial}\omega = 0$ then ω is *holomorphic.*

We combine ∂ and $\bar{\partial}$ to define the operator

$$d = \frac{1}{2}(\partial + \bar{\partial}).$$

Among all (p,q) forms we survey their distinguished classes: $\bar{\partial}$ *closed* $(\bar{\partial}\omega = 0)$ and $\bar{\partial}$ *exact* $(\omega = \bar{\partial}\alpha)$. Since $\bar{\partial}$ exact (p,q) forms are also $\bar{\partial}$ closed ones (recall $\bar{\partial}^2 = 0$) we may introduce *the Dolbeault cohomology groups* $H^{p,q}_{\bar{\partial}}(M)$ as a factor space

$$H^{p,q}_{\bar{\partial}}(M) \equiv \frac{\bar{\partial}\,\text{closed}\,(p,q)\,\text{forms}}{\bar{\partial}\,\text{exact}\,(p,q)\,\text{forms}}.$$

We notify for these groups the following facts:

- they posses the structure of a complex vector space;
- they measure global properties of the complex manifold;
- $H^{p,q}_{\bar{\partial}}(\mathbb{C}^n)$ are trivial.

Following the considerations in the section 2. for real manifolds to introduce the conjugate operators we need a metric, being compatible with the complex structure of M. Therefore, we introduce *a Hermitian metric* on M

$$ds^2 = g_{a\bar{b}}dz^a d\bar{z}^{\bar{b}},$$

assuming $(g_{a\bar{b}})$ is a Hermitian matrix. This metric allows to define an inner product of (p,q) forms and the operator $*$. Consequently, one may check that

$$\delta = (-1)^{np+n+1} * d* \equiv d^* = \partial^* + \bar{\partial}^*$$

is conjugate to d, and ∂^*, $\bar{\partial}^*$ respectively to $\partial, \bar{\partial}$.

The existence of three different differential operators $d, \partial, \bar{\partial}$ acting on the same spaces of forms implies also the existence of three kinds of Laplacians

$$\Delta = (d+\delta)^2,$$
$$\Delta' = 2(\partial + \partial^*)^2,$$
$$\Delta'' = 2(\bar{\partial} + \bar{\partial}^*)^2,$$

which makes this theory of complex manifolds more interesting.

We finish this section with answering the question: *how one can decompose any* (p,q) *form*? This question answers the complex version of Hodge theorem.

Theorem 3.1 *(*Hodge theorem - complex version*) Every* (p,q) *form* ω *has a unique orthogonal decomposition*

$$\omega = \alpha + \bar{\partial}\beta + \bar{\partial}^*\gamma,$$

where $\Delta''\alpha = 0$*,* β *is a* $(p,q-1)$ *form, and* γ *is a* $(p,q+1)$ *form.*

In particular, if $\bar{\partial}\omega = 0$ *then the last term vanishes and we again have a unique representative* α *for each cohomology class* $H^{p,q}_{\bar{\partial}}(M)$*.*

4 Kähler metrics

We recall from the previous considerations complex manifolds may be defined in terms of transition functions or a complex structure J. As we know in general case J is an endomorphism of the tangent vector bundle $T(M)$ satisfying the condition

$$J^2 = -id.$$

A manifold M endowed with an almost complex structure is called *an almost complex manifold*, and denoted by (M,J).
If g is a metric defined on M, compatible with J, i.e.

$$g(JX,JY) = g(X,Y), \quad X,Y \in T(M) \tag{5}$$

and ${}^{LC}\nabla$ the corresponding Levi-Civita connection, then according to properties of ${}^{LC}\nabla K$, where $K = g(\cdot, J\cdot)$, Gray and Hervella [12] have obtained sixteen different classes of almost Hermitian manifolds, i.e. almost complex manifolds endowed with an almost Hermitian metric g: a Riemannian metric which fulfills the condition (5). One of these classes is the class of Kähler manifolds, in some sense the simplest one, but nevertheless being one of very interesting. We devote this section just to Kähler spaces.

Let $ds^2 = g_{a\bar{b}} dz^a d\bar{z}^{\bar{b}}$ be a Hermitian metric (it means $(g_{a\bar{b}})$ is a Hermitian matrix. We associate to this metric the corresponding $(1,1)$ form

$$K = \frac{i}{2} g_{a\bar{b}} dz^a \wedge d\bar{z}^{\bar{b}},$$

which we call *the Kähler form.* The strightforward computation gives

$$\bar{K} = -\frac{i}{2} \bar{g}_{a\bar{b}} d\bar{z}^a \wedge dz^b = \frac{i}{2} g_{b\bar{a}} dz^b \wedge d\bar{z}^a = K, \quad \bar{z}^a = z^{\bar{a}}$$

and consequently K is a real form. Especially, if

$$dK = 0$$

then g is *the Kähler metric.* M is *a Kähler manifold* if it admits a Kähler metric.

One may find very easily the examples of Kähler spaces. Namely, if $\dim M = 2$ then M is a Kähler space. The condition g being the Kähler metric is possible to express in the terms of covariant derivative with respect to the Levi-Civita connection ${}^{LC}\nabla$. Namely, if g is a Riemannian metric satisfying the conditions

$$g(JX, JY) = g(X, Y), \quad {}^{LC}\nabla J = 0, \quad X, Y \in T(M),$$

then we call g a Kähler metric.

We notify some properties of Kähler spaces.

- The three different kinds of Laplacians, introduced on Hermitian manifold, coincide, i.e.

  $$\Delta = \Delta' = \Delta''.$$

  Consequently, every harmonic $(p, 0)$ form is holomorphic and vice versa.

- The forms

$$K,\; K\wedge K, \dots, \underbrace{K\wedge\dots\wedge K}_{n \text{ times}}$$

  are all non-zero and harmonic.

- K, $K\wedge K, \dots, K\wedge\dots\wedge K$ define cohomology classes in $H^p(M,\mathbb{R})$ for $p = 2,\dots,2n$; so K defines an element of $H^2(M,\mathbb{R})$, $K\wedge K$ defines an element of $H^4(M,\mathbb{R})$, etc.

  All these cohomology classes are non-trivial.

- The volume form is given as $V = \frac{1}{n!}\underbrace{K\wedge\dots\wedge K}_{n \text{ times}}$.

- The Betti numbers are positive, i.e. $b_{2p} \geq 1$.

Among examples of complex manifolds, which we have considered in previous section some of them admit a Kähler metric, but not all of them. So, the complex projective space $P^n(\mathbb{C})$ is a Kähler manifold and all of its cohomology classes are generated by scalar multiplies of forms $K,\dots,K\wedge\dots\wedge K$. The Fubini-Study metric on $P^n(\mathbb{C})$ is given by the Kähler form

$$\begin{aligned} K &= \tfrac{i}{2}\partial\bar\partial \lg(1+\textstyle\sum z^\alpha \bar z^\alpha) \\ &= \tfrac{i}{2}\tfrac{dz^\alpha\wedge d\bar z^\beta}{1+\sum z^\gamma \bar z^\gamma}[\delta_{\alpha\beta}(1+\textstyle\sum z^\gamma\bar z^\gamma) - \bar z^\alpha z^\beta]. \end{aligned}$$

The manifold $S^3\times S^3$ does not admit any Kähler metric as $b_2 = 0$.

The complex dimension $h^{p,q}$, so called *the Hodge number*, of the Dolbeault cohomology group $H^{p,q}_{\bar\partial}(M)$

$$h^{p,q} = \dim_{\mathbb{C}} H^{p,q}_{\bar\partial}(M)$$

is one of important topological invariants of M. Since for physicists, interested in string theory, $\dim_{\mathbb{C}} M = 3$ is the most important, as we pointed out at the beginning we give the Hodge diamond in this case

$$
\begin{array}{ccccccc}
 & & & h^{3,3} & & & \\
 & & h^{3,2} & & h^{2,3} & & \\
 & h^{3,1} & & h^{2,2} & & h^{1,3} & \\
h^{3,0} & & h^{2,1} & & h^{1,2} & & h^{0,3} \\
 & h^{2,0} & & h^{1,1} & & h^{0,2} & \\
 & & h^{0,1} & & h^{0,1} & & \\
 & & & h^{0,0} & & &
\end{array}
$$

The Hodge and Betti numbers are connected by the equation

$$b_n = \sum_{p+q=n} h^{p,q}. \tag{6}$$

The Hodge numbers possess also the properties

$$h^{p,q} = h^{q,p}, \tag{7}$$

as ω is harmonic if and only if $\bar{\omega}$ is harmonic;

$$h^{p,q} = h^{n-p,n-q}. \tag{8}$$

These properties define symmetries of the Hodge diamond. Moreover the relations (6) and (7) imply b_{2n+1} is even. One can use (7) and (8) to see that we have only 6 independent Hodge numbers if $\dim_{\mathbb{C}} M = 3$.

5 Ricci flat Kähler metrics

In the previous section we have considered the properties of Kähler spaces in terms of Kähler form. But to introduce special class of Kähler spaces, so called Calabi-Yau spaces we need to study the curvatures of a Kähler space.

Let g be a Kähler metric. Then one can compute non-vanishing Christoffel symbols of its Levi-Civita connection

$$\Gamma^j_{kl} = g^{j\bar{m}} g_{k\bar{m},l}, \quad \Gamma^{\bar{j}}_{\bar{k}\bar{l}}, \tag{9}$$

as well as non-vanishing components of the curvature

$$R^j{}_{kl\bar{m}} = -\Gamma^j_{kl,\bar{m}} \tag{10}$$

and those related by symmetry and complex conjugation. It follows from here also

$$R^j{}_{kl\bar{m}} = R^j{}_{lk\bar{m}}. \tag{11}$$

We can combine (9) and (10) to obtain the components of the Ricci tensor

$$\rho_{j\bar{k}} = -\frac{\partial^2(\lg \det g)}{\partial z^j \partial \bar{z}^k}.$$

We introduce the Ricci form as

$$\rho := i\rho_{j\bar{k}} dz^j \wedge d\bar{z}^k.$$

Since $d\rho = 0$, it follows the Ricci form is closed and defines an element $[\rho]$ in $H^2(M)$. One can compute that the cohomology class $[\rho]$ is independent of the Kähler metric.

We call $c_1 = [\rho]$ *the first Chern class.* We consider here only the first Chern class on the tangent bundle. One may introduce the Chern characteristic classes on any vector bundle (see for example [14] for more details).

If $c_1 \neq 0$ on M then there cannot exist a Ricci flat metric.

The problem of prescribing the Ricci flat curvature on a compact Kähler manifold was open a long period. Calabi [7] has proved $c_1 = 0$ implies uniqueness and 20 years later Yau [19] has proved the same condition implies also existence of Ricci flat metric. More precisely, we have the following theorem.

Theorem 5.1 *Given a complex compact manifold with $c_1 = 0$ and any Kähler metric $g_{j\bar{k}}$ with Kähler form $K = ig_{j\bar{k}}dz^j \wedge d\bar{z}^{\bar{k}}$, then there exists a unique Ricci flat Kähler metric $\hat{g}_{j\bar{k}}$ whose Kähler form is in the same cohomology class as K.*

So, we are ready to define Calabi-Yau manifold.

Definition 3 *The Calabi-Yau manifold is a compact Kähler manifold with $c_1 = 0$ in $H^2(M)$.*

We point out the Ricci flat metric as unique is not important. Let us recall the condition $c_1 = 0$ is equivalent to the existence of a nonvanishing holomorphic n-form ω.

As we know the Kähler space $S^2 \times S^2 \times S^2$ has $b_3 = 0$ and consequently it may not be endowed with harmonic 3-forms. This implies $S^2 \times S^2 \times S^2$ cannot admit a Ricci flat Kähler metric.

5.1 Symmetries of the Hodge numbers of Calabi-Yau spaces M^3

We present a consideration to 3-dimensional Calabi-Yau spaces as they are important for superstring theory.

Since $c_1 = 0$, the Hodge numbers get an extra symmetry. One can use the covariantly constant three form ω ($\nabla\omega = 0$) to see that the dual of a harmonic $(p, 0)$ form is also harmonic form of type $(0, 3-p)$. Therefore:

$$\text{if } c_1 = 0, \quad \text{then} \quad h^{p,0} = h^{0,3-p}. \tag{12}$$

We have in mind that lowering an index with a Hermitian metric changes its type.

Property (12) reduces the six independent Hodge numbers to just four which one can take to be $h^{0,0}$, $h^{1,0}$, $h^{1,1}$, $h^{2,1}$. However $h^{0,0}$ is just the dimension of the space of constant functions and hence $h^{0,0} = 1$. It is also simple to compute $h^{1,0}$ using the Weitzenbock identity which makes possible to express the Laplacian on p forms in terms of the covariant derivative ∇ and curvature of the metric:

$$\Delta_d = -\nabla^2 + \text{curvature terms}.$$

For a one form, the curvature term involves only the Ricci tensor. This implies that on a Ricci flat manifold, every harmonic one form must be covariantly constant and for this reason non-vanishing. But, a manifold with $\aleph \neq 0$ does not admit any nowhere vanishing 1 forms. Consequently $\aleph \neq 0$ implies $b_1 = 0$ and hence $h^{1,0} = 0$. But we use also $\aleph \neq 0$ to see $\pi_1(M)$ is finite. Hence there exists a compact simply connected covering space.

So we conclude the cohomology of a Calabi-Yau space with $\aleph \neq 0$ is characterized by two integers $h^{1,1}$ and $h^{2,1}$, such that we have $\aleph = 2(h^{1,1} - h^{2,1})$.

5.2 The holonomy group of a Calabi-Yau space M^3

We are familiar with the notion of a holonomy group. As we know the holonomy group of a Levi-Civita connection ${}^{LC}\nabla = \nabla(g)$ is in general $SO(6)$, when g is a Riemannian metric. But, if the metric g is Kähler, then the complex structure is preserved and the holonomy group is contained in $U(3)$. If the metric is Kähler and Ricci flat, then the holonomy group is even more restricted. Namely, the Lie algebra of the holonomy group is given by parallel transport around infinitesimal loops. The change in a vector is then given by the curvature tensor by Ambrose-Singer theorem [3]. The change for a Kähler metric is $R^j{}_{kl\bar{m}} v^l v^{\bar{m}}$ which are generators of $U(3)$. We use (11) to compute the trace of these generators

$$\mathrm{tr} R^j{}_{kl\bar{m}} v^l v^{\bar{m}} = R^k{}_{kl\bar{m}} = R^k{}_{lk\bar{m}} = \rho_{l\bar{m}}.$$

Consequently the holonomy group is contained in $SU(3)$ if and only if the metric is Ricci flat and Kähler.

This problem of relations between curvature tensors and holonomy groups from representation theory point of view has been considered by S. Salamon [16].

5.3 Examples

We checked already that $S^2 \times S^2 \times S^2$ does not admit Ricci flat Kähler metric. We finish this section with some examples of 3-dimensional Kähler spaces. One of them does not admit Ricci flat metric, second one admits metric of this type.

- $P^3(\mathbb{C})$. The complex projective spaces admit Kähler metrics, given by the Kähler form

$$K = \frac{i}{2}\partial\bar{\partial}\lg(1 + \sum_{\alpha=0}^{3} z^\alpha \bar{z}^\alpha)$$

  and have the smallest possible Hodge numbers $h^{p,q} = \delta_{p,q}$. We conclude that $P^3(\mathbb{C})$ does not admit a Ricci flat Kähler metric using the equality $h^{3,0} = 0$, which implies there is no holomorphic three form and hence $c_1 \neq 0$.

- We define the submanifold $Y_{4,5}$ of $P^4(\mathbb{C})$ by the equation

$$\sum_{i=1}^{5} z_i^5 = 0,$$

  where z_i are homogeneous coordinates. If $y_i = z_i/z_5$, $i = 1, \ldots, 4$, then the holomorphic three form

$$\frac{1}{y_4^4} dy_1 \wedge dy_2 \wedge dy_3$$

is non-singular and non-vanishing in all points of $Y_{4,5}$. Consequently $Y_{4,5}$ has $c_1 = 0$ and admits a Ricci flat Kähler metric. Moreover one can show that $h^{2,1} = 101$ and $h^{1,1} = 1$ and consequently $\aleph = -200$. These relations imply some interesting consequences. So, we use $h^{1,1} = 1$ to see there is a one dimensional space of cohomology classes of the Kähler form K, and consequently there is an one complex dimensional space of harmonic $(1,1)$ forms. Since the Kähler form is real, this implies one has only an one real dimensional space of possible cohomology classes for K.

The equality $h^{2,1} = 101$ implies that there is a 101 (complex) dimensional space of complex structures on this manifold. This yields a 203 (real) dimensional space of Ricci flat metrics. It is interesting, while not one of these metrics is known explicitly, the different complex structures on this manifold are known explicitly. So, let

$$f(z^i) = A_{ijklm} z^i z^j z^k z^l z^m, \quad i, \dots, m = 1, \dots, 5$$

be any homogeneous polynomial in $\mathbb{C}^5$. We define a smooth submanifold M_A in $P^4(\mathbb{C})$ by the equation $f = 0$, provided $df \neq 0$. Every two different tensors A, A' yield diffeomorphic submanifolds M_A, $M_{A'}$, while they have different complex structures only unless the tensors A, A' are related by a $GL(5, \mathbb{C})$ transformation. Therefore, it is simple to find the number of inequivalent complex structures. We use that there is a 126 (complex) dimensional space of symmetric tensors A and $\dim GL(5, \mathbb{C}) = 25$ to obtain $126 - 25 = 101$ inequivalent complex structures.

6 Geometry and topology of Kähler and Calabi-Yau spaces

We devote the last section to a presentation of some results related to geometry and topology of Kähler and Calabi-Yau spaces obtained by group of mathematicians from Belgrade. So we pay the attention on relations between groups of transformations and the first Chern class, furthermore some other geometrical (conharmonic curvature tensor) and topological invariances (Hirzebruch signature, arithmetic genus) and finally the characterization of complex space forms in terms of the shape operator for every sufficiently small tube.

We recall a map $f : (\tilde{M}, \tilde{\nabla}) \longrightarrow (M, \nabla)$ of manifolds with symmetric connections is called *projective* if for each geodesic γ of $\tilde{\nabla}$, $f \circ \gamma$ is a reparametrisation of a geodesic of ∇ (see Fig. 14). If a transformation s of M preserves geodesics and the affine character of the parameter on each geodesic, then s is called *an affine transformation* of the connection ∇ or simply of the manifold M.

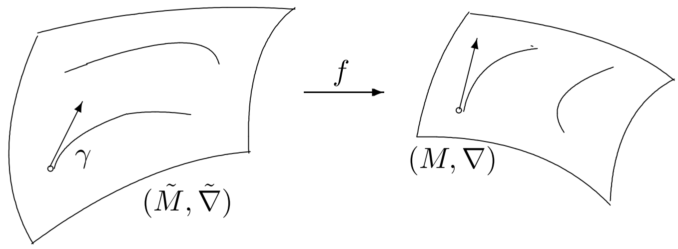

Figure 14: A projective map

Theorem 6.1 [5] *Let (M, g) be a Calabi-Yau space of complex dimension $n > 1$ and the scalar curvature $\tau = const$. Then:*

1. *the group of projective diffeomorphisms of Levi-Civita connection $\nabla(g)$ coincides with the group of affine diffeomorphisms of the same connection;*

2. *the identity component of the group of holomorphically projective diffeomorphisms of Levi-Civita connection coincides with the identity component of its group of isometries.*

Corollary 6.2 [5] *Let (M, g) be an irreducible Calabi-Yau space of complex dimension $n > 1$ and the scalar curvature $\tau = const$. Then the group of projective diffeomorphisms of the Levi-Civita connection $\nabla(g)$ coincides* *with its group of isometries.*

Studying the conformal transformations of Kähler manifolds and connections with torsion M. Prvanović [15] has introduced the complex conharmonic curvature tensor H. Let c_2 be the second Chern class determined by the form γ_2. We have in the following theorem a characterization of flat space in terms of H and γ_2.

Theorem 6.3 [6] *Let M be a compact Kähler manifold of complex dimension $n > 1$. If $H = 0$ then $\omega^{n-2}c_2(M) = \int \gamma_2 \wedge K^{n-2} \leq 0$ with equality sign if and only if M is a flat space.*

Let $\tau(M)$ and $a(M)$ be Hirzebruch signature and arithmetic genus of a complex surface M. Then we can use these topological invariances and H to characterize also a flat complex space form.

Theorem 6.4 [6] *Let M be a compact Kähler surface with $H = 0$. Then*

$$\aleph(M) \leq \tfrac{3}{2}\tau(M),$$
$$a(M) \leq 0,$$

with equality sign if and only if M is a flat complex space form.

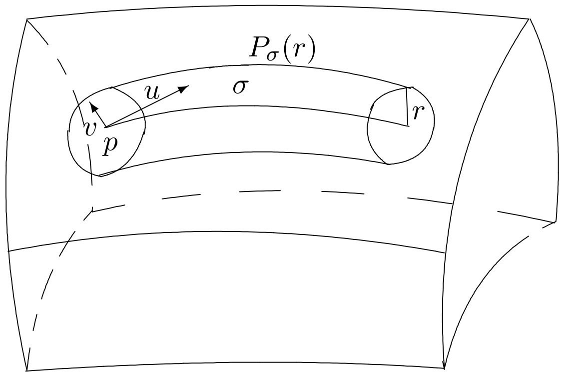

Figure 15: A tube about geodesic

Let $P_\sigma(r)$ be a tube of radius r about geodesic σ on M tangent to the unit vector field u. Further, consider the special point $p = \exp_{\sigma(t)}(rv),\ v \in \sigma(t)^\perp,\ v(\sigma(t)) = Ju(\sigma(t))$ of $P_\sigma(r)$ and suppose that $\gamma : s \mapsto \exp_{\sigma(t)} sv$ is the unit speed geodesic connecting $\sigma(t)$ and p.

Theorem 6.5 [9] *Let (M, g, J) be a Kähler manifold of* $\dim M \geq 4$. *Then with the convention made above M is a complex space form if and only if for every sufficiently small tube $P_\sigma(r)$, $(J\frac{\partial}{\partial s})(p)$ is an eigenvector of the shape operator S^σ for all $m \in M$ and all geodesics through m.*

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