The Quantum Space of Stochastic Processes

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Stochastic reaction-diffusion processes are of both theoretical and experimental interest not only because they describe various mechanisms in physics and chemistry but they also provide a way of modelling phenomena like traffic flow, kinetics of biopolimerization, interface growth.

A stochastic process is described in terms of a master equation for the probability distribution $P(s_i, t)$ of a stochastic variable $s_i = 0, 1, 2...., n-1$ at a site $i = 1, 2, ...L$ of a linear chain. A state on the lattice at a time t is determined by the occupation numbers s_i and a transition to another configuration s'_i during an infinitesimal time step dt is given by the probability $\Gamma(s, s')dt$. The rates $\Gamma \equiv \Gamma^{ik}_{jl}$ $(i, j, k, l = 0, 1, ...n-1)$ are assumed to be independent from the position in the bulk. At the boundaries, i.e. sites 1 and L additional processes can take place with rates L^i_k and R^i_k. Due to probability conservation $\Gamma(s, s) = -\sum_{s' \neq s} \Gamma(s', s)$. The master equation for the time evolution of a stochastic system

$$\frac{dP(s,t)}{dt} = \sum_{s'} \Gamma(s, s')P(s', t) \tag{1}$$

is mapped to a Schroedinger equation for a quantum Hamiltonian in imaginary time

$$\frac{dP(t)}{dt} = -HP(t), \qquad H = \sum_j H_{j,j+1} + H^{(L)} + H^{(R)} \tag{2}$$

The ground state of this in general non-hermitean Hamiltonian corresponds to the stationary probability distribution of the stochastic dynamics. The mapping provides a connection with integrable quantum spin chains. Examples are the processes of particles hopping between lattice sites i, j with rates g_{ij} and a hard core repulsion (i.e. a site is emty or occupied by one particle). In the case of the symmetric exclusion process $g_{ij} = g_{ji}$ and the stochastic Hamiltonian is the $SU(2)$ symmetric spin 1/2 isotropic Heisenberg ferromagnet. The diffusion driven lattice gas (ASEP) of particles with rates $\frac{g_{i,i+1}}{g_{i+1,i}} = q \neq 1$ is mapped to a $SU_q(2)$-symmetric XXZ chain with anizotropy $\Delta = \frac{(q+q^{-1})}{2}$.

MATRIX PRODUCT GROUND STATES APPROACH - The stationary probability distribution, i.e. the ground state of the quantum Hamiltonian is expressed as a product of (or a trace over) matrices that form representation of a quadratic algebra determined by the dynamics of the process. (Derrida et. al.[1, 2] - ASEP with open boundaries, 3-species diffusion-type, reaction-diffusion processes.)

ANZATZ - Any zero energy eigenstate of a Hamiltonian with nearest neighbour interaction in the bulk and single site boundary terms can be written as a matrix product state with respect to a quadratic algebra

$$\Gamma^{ik}_{jl} D_i D_k = x_l D_j - x_j D_l$$

DIFFUSION - $\Gamma^{ik}_{ki} = g_{ik}$, $i, k = 0, 1 ... n-1$). We consider n-species diffusion process on a chain with L sites with nearest-neighbour interaction with exclusion. On successive sites the particles i and k exchange places with probability g_{ik}dt; particles number n_i in the bulk is conserved $\sum_{i=0}^{n-1} n_i = L$. One distinguishes closed systems (periodic boundary conditions) and open systems with boundary processes - at site 1 (left) and at site L (right) the particle i is replaced by the particle k with probabilities $L^i_k dt$ and $R^i_k dt$ respectively,

$$L^i_i = -\sum_{j=0}^{L-1} L^i_j, \qquad R^i_i = -\sum_{j=0}^{L-1} R^i_j \tag{3}$$

DIFFUSION ALGEBRA

$$g_{ik} D_i D_k - g_{ki} D_k D_i = x_k D_i - x_i D_k \tag{4}$$

where $i, k = 0, 1, ...n-1$ and x_i are c-numbers $\sum_{i=0}^{n-1} x_i = 0$. This is an algebra with INVOLUTION, hence hermitean D_i

$$D_i = D_i^+, \qquad g_{ik}^+ = -g_{ki} \qquad x_i = x_i^+ \tag{5}$$

(or $D_i = -D_i^+$, if $g_{ik} = g_{ki}^+$).

PROBABILITY DISTRIBUTION

- periodic boundary conditions

$$P(s_1, ...s_L) = Tr(D_{s_1} D_{s_2} ... D_{s_L}) \tag{6}$$

- boundary processes

$$P(s_1, ...s_L) =< w|D_{s_1} D_{s_2} ... D_{s_L}|v > \tag{7}$$

and the vectors $|v>$ and $<w|$ are defined by

$$< w|(L^k_i D_k + x_i) = 0, \qquad (R^k_i D_k - x_i)|v >= 0 \tag{8}$$

THUS to find the stationary probability distribution one has to compute traces or matrix elements with respect to the vectors $|v>$ and $<w|$ of monomials of the form $D_{s_1}^{m_1} D_{s_2}^{m_2} .... D_{s_L}^{m_L}$. The problem to be solved is twofold -

Find a representation of the matrices D that is a solution of the quadratic algebra and match the algebraic solution with the boundary conditions.

The relations (4) allow an ordering of the elements D_k. Monomials of given order are the Poincare- Birkhoff-Witt (PBW) basis for polynomials of fixed degree as the probability distribution is due to the bulk conservation laws. We thus consider an associative algebra generated by an unit e and n elements D_k obeying $n(n-1)/2$ relations (4). The alphabetically ordered monomials

$$D_{s_1}^{n_1} D_{s_2}^{n_2} \ldots . D_{s_l}^{n_l}, \tag{9}$$

where $s_1 < s_2 < \ldots . s_l$, $l \geq 1$ and $n_1, n_2, \ldots . n_l$ are non-negative integers, are a linear basis in the algebra, the PBW basis.

PROPOSITION:

1. In the case of Lie-algebra type diffusion algebras the n generators D_i, and e can be mapped to the generators J_{jk} of $GL(n)$ and the mapping is invertible. The UEA generated by D_i belongs to the UEA of the Lie-algebra of $GL(n)$.

2. The multiparameter quantized noncommutative space can be realized equivalently as a q-deformed Heisenberg algebra of n oscillators depending on $n(n-1)/2 + 1$ parameters (in general on $n(n-1)/2 + n$ parameters). The UEA of the elements D_i in the case of a diffusion algebra with all coefficients x_i on the RHS of eq.(4) equal to zero belongs to the UEA of a multiparameter deformed Heisenberg algebra to which a consistent multiparameter $GL_q(n)$ quantization corresponds.

3. In an algebra with x-terms on the RHS of (4) only then is braid associativity satisfied if out of the coefficients x_i, x_k, x_l corresponding to a triple $D_i D_k D_l$ either one coefficient x is zero or two coefficients x are zero

and the rates are respectively related. The diffusion algebras in this case can be obtained by either a change of basis in the n-dimensional noncommutative space or by a suitable change of basis of the lower dimensional quantum space. The appearance of the nonzero linear terms in the RHS of the quantum plane relations leads to a lower dimensional noncommutative space and a reduction of the $GL_q(n)$ invariance.

REPRESENTATIONS of the DIFFUSION ALGEBRAS

A. Lie-algebra types

There are two such algebras. The one is with all rates equal, $g_{ij} = g_{ji} = g$ and corresponds to the n-species symmetric exclusion process. The PBW-basis of the algebra generated by the elements D is a subsystem of the basis of the UEA of $sl(n) \oplus u(1)$ which is the hidden symmetry algebra of a stochastic diffusion system with all rates equal. For algebras with only $g_{ij} \neq 0$ the algebraic relations read $g_{ij} D_i D_j = x_j D_i - x_i D_j$. The symmetry reduces from $SL(n) \otimes U(1)$ to $U(1)^{n-1} \otimes U(1)$.

B. The quantized noncommutative space of a diffusion model

1. Algebras with no x-dependent linear terms

The algebraic relations (4) without the x-terms on the RHS roughly parallel the Manin's multiparameter quantized space [3] with the n elements D_i viewed as its co-ordinates

$$g_{ik} D_i D_k - g_{ki} D_k D_i = 0 \tag{10}$$

A representation of the quantum space is obtained by identifying the monomials (9) with the states of n oscillators

$$a_0^{+n_0} a_1^{+n_1} ... a_{n-1}^{+n_{n-1}} |0> \tag{11}$$

Then the generators D_i correspond to n creation operators and the noncommutative space is equivalent to a multideformed oscillator algebra. It is convenient to consider the ratios

$$q_{ij} = \frac{g_{ij}}{g_{ji}}, \qquad i < j \tag{12}$$

as the set of $n(n-1)/2$ independent parameters and introduce further n real parameters r_i. The latter are at this stage auxiliary parameters needed for a consistent quantized phase space calculus [4] whose realization proceeds as follows. One starts with n classical oscillators A_i and A_i^+, obeying $[A_i, A_j^+] = \delta_{ij}$ for $i, j = 0, 1, ...n-1$ and defines $n_k = A_k^+ A_k$. A deformation of the Heisenberg algebra is achieved through the invertible maps:

$$\begin{aligned} a_i &= \prod_{k>i} q_{ik}^{\frac{n_k}{2}} \sqrt{\frac{r_i^{n_i+1} - 1}{(r_i - 1)(n_i + 1)}} A_i \\ a_i^+ &= \prod_{k>i} q_{ik}^{-\frac{n_k}{2}} \sqrt{\frac{r_i^{n_i+1} - 1}{(r_i - 1)n_i}} A_i^+ \end{aligned} \tag{13}$$

where

$$\frac{r_i^{n_i} - 1}{r_i - 1} = a_i^+ a_i \tag{14}$$

These deformed oscillators obey the following algebraic relations

$$\begin{aligned} a_i a_i^+ - r_i a_i^+ a_i &= 1 \\ a_i^+ a_j^+ - q_{ji} a_j^+ a_i^+ &= 0 \\ a_i a_j - q_{ji} a_j a_i &= 0 \\ a_i a_j^+ - q_{ji}^{-1} a_j^+ a_i &= 0 \end{aligned} \tag{15}$$

with $i < j$. The deformed oscillators can be arranged in bilinears to construct the $GL_q(n)$ generators $J_{ij} = a_i^+ a_j$, $i, j = 0, 1, ...n-1$. A consistent multiparameter $GL_q(n)$ quantization depending on $n(n-1)/2$ independent parameters is achieved with all auxiliary r_i set equal to one. The corresponding R-matrix satisfies an Yang-Baxter equation and is diagonal. If we want to relate such an R-matrix to a transition probability rates matrix we are lead to a trivial matrix. Therefore one should consider the case of all deformation parameters q_{ij} and r_i equal, i.e.

$$q_{ij} = q = r_i^{-1}, \qquad i < j, \qquad i, j = 0, 1, ...n-1 \tag{16}$$

A representation of the matrices D_i and D_i^T and the generators J_{ij} is provided by the action of the oscillators a_i^+ and a_i on the basis vectors $|n_i> \equiv |n_0 n_1 ... n_{n-1}>$ (with $n_i = 0, 1, 2...$)

$$a_i^+|n_j> = \prod_{k>i} q^{\frac{n_k}{2}} \frac{(1-q^{n_i+1})^{1/2}}{(1-q)^{1/2}} |n_j + \delta_{ij}> \tag{17}$$

$$a_i|0> = 0, \qquad\qquad i = 0, 1, ...n-1 \tag{18}$$

$$a_i|n_j> = \prod_{k>i} q^{-\frac{n_k}{2}} \frac{(1-q^{n_i})^{1/2}}{(1-q)^{1/2}} |n_j - \delta_{ij}> \tag{19}$$

The symmetry of the model is $GL_q(n)$.

2. Algebras with x-dependent linear terms

2.1. Algebras with one x-dependent linear term - Consider the quantum plane

$$D_i D_j = q_{ji} D_j D_i, \qquad i < j, \qquad q_{ij} q_{ji} = 1, \tag{20}$$

and shift one D_i, fixing $i \equiv i_1$ (it can be any one of the matrices D)

$$D_{i_1} = D'_{i_1} + x_{i_1}/(1 - q_{ji_1}) \tag{21}$$

Due to the shift the quommutators $D_{i_1} D_j = q_{ji_1} D_j D_{i_1}$ change and the algebra is

$$D_{i_1} D_j - q_{ji_1} D_j D_{i_1} = -x_{i_1} D_j \qquad\qquad i_1 \neq j \tag{22}$$

$$D_j D_k = q_{kj} D_k D_j \tag{23}$$

where j, k run the set of all the indices of the $n-1$ elements D obtained upon exclusion of the index of the shifted one. The algebraic relations (23) define a noncommutative space of dimension $n-1$. A representation is constructed in terms of $n-1$ deformed creation operators a_j^+ upon identifying $D_j = a_j^+$ for $j = 0, 1, ...n-2$. With the constraint $q_{jk} = q$, $j < k$ the diffusion algebra leads to a consistent one-parameter $GL_q(n-1)$ quantization with a corresponding R-matrix satisfying Yang-Baxter equation. The rescaled matrix D_{i_1} forms a $u(1)$ algebra. Thus the presence of one x-dependent term due to a boundary process reduces the $GL_q(n)$ invariance in the bulk to $GL_q(n-1) \otimes U(1)$ invariance.

2.2. Algebras with two x-dependent linear terms - A diffusion algebra with two c-numbers x always contains one (and only one) relation of the form

$$g_{ij}D_iD_j - g_{ji}D_jD_i = x_jD_i - x_iD_j \tag{24}$$

This can be mapped to one of the quommutators of the Heisenberg algebra, namely

$$a_ia_i^+ - r_ia_i^+a_i = 1 \tag{25}$$

by a simultaneous shift of a_i and a_i^+. THUS to make the correspondence with the quantum plane one identifies ONE of the GENERATORS of a diffusion algebra with TWO x-terms with a rescaled ANNIHILATION OPERATOR. Therefore one considers a quantum plane of DIMENSION $n-1$ and the reduction to a diffusion algebra proceeds as follows: Since $i < j$, choose $i = 0$ and $j = n-1$ and rewrite (24) in the form

$$D_{n-1}D_0 - \frac{g_{0,n-1}}{g_{n-1,0}}D_0D_{n-1} = \frac{x_0}{g_{n-1,0}}D_{n-1} - \frac{x_{n-1}}{g_{n-1,0}}D_0 \tag{26}$$

Then one identifies $D_{n-1} = D_0^+$ and singles out of the corresponding $n-1$ relations the pair a_0^+ and a_0 to define

$$D_0 = \frac{x_0}{g_{n-1,0}}\left(\frac{1}{1-r_0} + \frac{a_0^+}{\sqrt{1-r_0}}\right) \tag{27}$$

$$D_{n-1} = -\frac{x_{n-1}}{g_{n-1,0}}\left(\frac{1}{1-r_0} + \frac{a_0}{\sqrt{1-r_0}}\right)$$

which satisfies (24) with

$$r_0 = \frac{g_{0,n-1}}{g_{n-1,0}} \tag{28}$$

The rest of the generators D_k, $k = 1, 2...., n-2$ are to be identified with the remaining $n-2$ creation operators a_k^+

$$D_k = a_k^+ \tag{29}$$

for k different from the fixed index $i = 0$. The diffusion algebra is obtained from the deformed Heisenberg commutation relations (15)

$$D_{n-1}D_0 - \frac{g_{0,n-1}}{g_{n-1,0}}D_0D_{n-1} = \frac{x_0}{g_{n-1,0}}D_{n-1} - \frac{x_{n-1}}{g_{n-1,0}}D_0$$

$$\begin{aligned} D_0D_k - q_kD_kD_0 &= -\frac{x_0}{g_k}D_k \\ D_kD_{n-1} - q_kD_{n-1}D_k &= \frac{x_{n-1}}{g_k}D_k \\ D_kD_l - q_{kl}^{-1}D_lD_k &= 0 \end{aligned} \tag{30}$$

where $k = 1, 2...n-2$ and

$$\begin{aligned} q_k &= \frac{g_{k0}}{g_{0k}} = \frac{g_{n-1,k}}{g_{k,n-1}} \\ g_k &= g_{0k} = g_{k,n-1} \end{aligned} \tag{31}$$

$$g_{0k} - g_{k0} = g_{k,n-1} - g_{n-1,k} = g_{0,n-1} - g_{n-1,0} \tag{32}$$

The last relation in (30) with the constraint

$$\frac{g_{kl}}{g_{lk}} = q \qquad k < l \tag{33}$$

leads to a one-parameter $GL_q(n-2)$ quantization with R-matrix satisfying the Yang-Baxter equation. The matrices D_0 and D_{n-1} and the unit generate the deformed UEA of $GL_q(2)$. The symmetry is reduced to $GL_q(n-2) \otimes GL_q(2)$.

SUMMARY - The quadratic algebras of the n-species symmetric and assymetric diffusion processes define in general the comodule structure of $GL_q(n)$ and reveal the hidden symmetries of the processes. The time evolution operator governing the stochastic dynamics can thus be mapped to integrable $GL_q(n)$ quantum spin chain Hamiltonian. The most important constraint of the symmetry is the charge conservation interpreted as particle number conservation. Boundary processes lead to a reduction of the bulk $GL_q(n)$ symmetry to either $GL_q(n-1)$ or $GL_q(n-2)$ thus breaking the particle number conservation law.

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