Real, p-Adic and Adelic Noncommutative Scalar Solitons *

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Abstract

The actual interest in noncommutativity of space coordinates has emerged recently in string theory as a consequence of the properties of open string ending on D-branes.

This noncommutativity of coordinates x induces the Moyal star product $f \star g$ between (analytic) functions f and g. To investigate this subject in a systematic way, various noncommutative field theory models are introduced in the last few years. The basic equation in scalar field theory $\phi \star \phi = \phi$ has an infinite number of solitonic solutions, unlike the case with ordinary multiplication.

The simplest solution of the above equation in the two spatial dimensions is

$$\phi_0(x) = 2e^{-\frac{x_1^2+x_2^2}{\theta}} .$$

Introducing a new method, we reobtain this and other solutions. Also, we consider two p-adic generalizations of the above basic equation with the corresponding p-adic and adelic solutions.

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1 Introduction

Let us consider single scalar field theory in 2+1 dimensions with coordinates x^μ, $\mu = 0,1,2 \quad (i = 1,2)$. We suppose noncommutativity in the spatial directions $[x^1, x^2] = i\theta^{12} = i\theta$. The action [1] is

$$S = \int d^3x \left[\frac{1}{2}(\partial_0\phi)^2 - \frac{1}{2}\partial_i\phi\partial^j\phi - V(\phi)_\star\right], \tag{1}$$

where the potential

$$V(\phi)_\star = \frac{m^2}{2}\phi^2 + \sum_n \frac{a_n}{n}(\phi^n)_\star, \qquad (\phi^n)_\star = \phi\star\phi\star\cdots\star\phi, \tag{2}$$

is defined in terms of star product

$$(f\star g)(x) = e^{\frac{i}{2}\theta^{ij}\partial_{a_i}\partial_{b_j}} f(x+a)g(x+b)|_{a=b=0}, \qquad (\partial_{a_i} = \frac{\partial}{\partial a_i}). \tag{3}$$

Expanding the exponent and reordering the summation variables we get

$$(f\star g)(x) = \sum_{k=0}^{\infty}\sum_{q=0}^{\infty}\left(\frac{i\theta}{2}\right)^{k+q}\frac{(-1)^k}{k!q!}\partial_1^q\partial_2^k f(x)\partial_1^k\partial_2^q g(x). \tag{4}$$

¿From the property of the star product $\int d^2x f\star g = \int d^2x fg$ it is clear that in the quadratic part of the action the star product reduces to the usual one, so we omit $\star$ in the first two terms in the action S.

We are interesting in the existence of **solitons** in this theory, i.e in *time independent* solutions of the equation of motion when *energy* is finite. In the commutative limit $\theta = 0$ the Derrick theorem states that in a pure scalar field theory with $d \geq 2$ (d is a number of spatial dimensions) there are no stable finite energy solutions (solitons). The proof of the theorem is based on the rescaling of coordinates, but this argument fails in the presence of length scale $\sqrt{\theta}$.

We will consider large noncommutative limit $\theta \to \infty$ in the energy

$$E = \int d^2x[\frac{1}{2}(\partial_0\phi)^2 + \frac{1}{2}\partial_i\phi\partial^j\phi + V(\phi)_\star], \tag{5}$$

which in the static case reduces to the expression

$$E = \int d^2x [\frac{1}{2}\partial_i\phi\partial^j\phi + V(\phi)_\star] \,. \tag{6}$$

In terms of the dimensionless coordinates $y = \frac{x}{\sqrt{\theta}}$ the static energy becomes

$$E = \int d^2y [\frac{1}{2}\partial_i\phi\partial^j\phi + \theta V(\phi)_\star] \,, \tag{7}$$

where now integration and partial derivatives are with respect to y. If $\theta \to \infty$, which in our case means $\theta V \gg \partial_i\phi\partial^j\phi$, the expression for the energy reduces to the simple equation

$$E = \int d^2y \theta V(\phi)_\star \,. \tag{8}$$

The extremum of the energy corresponds to the solution of the equation

$$\frac{\partial V(\phi)_\star}{\partial\phi} = 0 \,. \tag{9}$$

Note that in a commutative case ($\theta = 0$) , the above equation is a polynomial one, so the solutions are only constants $\phi = \lambda_i \in \Lambda \,, \Lambda = \{\lambda_1, ..., \lambda_k\}$. Due to the presence of the derivatives in the definition of the star product, we expect here some nontrivial solitonic type solutions.

Let us first consider the equation

$$F(\phi)_\star = 0 \,, \tag{10}$$

where $F(x)$ is an arbitrary analytic function and the star means that fields in F are multiplied using the star product. If we are able to solve the equation

$$\phi \star \phi = \phi \,, \tag{11}$$

then we have $\phi^n_\star = \phi$ and consequently $F(\lambda\phi)_\star = \sum_{n\geq 0} \frac{\lambda^n}{n!} F^{(n)}(0)\phi^n_\star = F(\lambda)\phi$. So, $\lambda_i\phi$ is a solution of the equation (10) if λ_i is a root of F. Similarly $\lambda_i\phi$ is a solution of (9) if λ_i is an extremum of $V(\lambda)$.

2 Simple nontrivial solution

We are looking for a simple non-trivial solution of the basic equation (11). Let us check that the expression

$$\phi_0(x) = 2e^{-\frac{x_1^2+x_2^2}{\theta}}\,, \tag{12}$$

is a solution of this equation. We will call it a *simple nontrivial solution*. Using the definition of the Hermite polynomials

$$H_n(\alpha) = (-1)^n e^{\alpha^2}\partial_\alpha^n e^{-\alpha^2}\,, \qquad (n = 0, 1, ...) \tag{13}$$

we have

$$\partial_\alpha^n e^{\frac{-\alpha^2}{\theta}} = (-1)^n \theta^{-\frac{n}{2}} e^{\frac{-\alpha^2}{\theta}} H_n\left(\frac{\alpha}{\sqrt{\theta}}\right)\,, \tag{14}$$

and according to (1.4), the star product can be written as

$$(\phi_0 \star \phi_0)(x) = K\left(-i, \frac{x_1}{\sqrt{\theta}}, \frac{x_2}{\sqrt{\theta}}\right) K\left(i, \frac{x_1}{\sqrt{\theta}}, \frac{x_2}{\sqrt{\theta}}\right) \phi_0^2(x)\,, \tag{15}$$

where

$$K(z, u, v) = \sum_{n=0}^{\infty} \frac{z^n}{2^n n!} H_n(u) H_n(v)\,. \tag{16}$$

With the help of the representation of Hermite polynomials

$$H_n(\alpha) = \frac{2^n}{\sqrt{\pi}} \int_{-\infty}^{\infty} dt e^{-t^2} (\alpha + it)^n\,, \tag{17}$$

we obtain

$$K(z, u, v) = \frac{1}{\pi} \int_{-\infty}^{\infty} dt e^{-t^2} \int_{-\infty}^{\infty} ds e^{-s^2 + 2z(u+it)(v+is)}\,, \tag{18}$$

After integration over s, the coefficient in front of $-t^2$ in the exponential turns into $1 - z^2$, so the integration over t gives the following result [2]

$$K(z, u, v) = \frac{1}{\sqrt{1-z^2}} e^{v^2 - \frac{(v-zu)^2}{1-z^2}}\,, \tag{19}$$

under the condition $Re(1-z^2) > 0$. Here $1-z^2 = 2 > 0$ so that

$$K(\mp i, u, v) = \frac{1}{\sqrt{2}} e^{\frac{u^2+v^2}{2} \mp iuv}\,, \tag{20}$$

and consequently

$$K\left(-i, \frac{x_1}{\sqrt{\theta}}, \frac{x_2}{\sqrt{\theta}}\right) K\left(i, \frac{x_1}{\sqrt{\theta}}, \frac{x_2}{\sqrt{\theta}}\right) = \frac{1}{2} e^{\frac{x_1^2+x_2^2}{\theta}} = \frac{1}{\phi_0}\,. \tag{21}$$

From equations (15) and (21) it follows that ϕ_0 is a solution of the (11).

3 General solution

There exist infinitely many solutions of the (11). Let us check that the expression

$$\phi_n(x) = (-1)^n L_n\left(2\frac{x_1^2+x_2^2}{\theta}\right)\phi_0(x)\,, \tag{22}$$

satisfies (11). Here $L_n(t)$ are the Laguerre polynomials defined as

$$L_n(\alpha) = \frac{1}{n!} e^{\alpha} \partial_\alpha^n (\alpha^n e^{-\alpha})\,. \qquad (n = 0, 1, ...) \tag{23}$$

With the help of the following representation

$$L_n(\alpha) = \frac{1}{n!} \partial_t^n \left(\frac{1}{1-t} e^{-\frac{\alpha t}{1-t}}\right) |_{t=0}\,, \tag{24}$$

instead of (22) we can write

$$\phi_n(x) = \frac{(-1)^n}{n!} \partial_t^n \left[\frac{1}{1-t}\phi_0[\theta(t), x]\right] |_{t=0}\,, \tag{25}$$

where $\phi_0[\theta(t), x]$ is the same expression as $\phi_0(x)$ in (12), just substituting

$$\theta(t) = \theta\frac{1-t}{1+t}\,, \tag{26}$$

instead of θ. In some sense, we expressed the ϕ_n in terms of ϕ_0, but with the t dependent parameter θ.

Now, we have

$$(\phi_n \star \phi_n)(x) = \frac{1}{(n!)^2}\partial_t^n \partial_s^n \left[\frac{1}{(1-t)(1-s)}\phi_0[\theta(t), x] \star \phi_0[\theta(s), x]\right] |_{t=s=0}\,, \tag{27}$$

where the star product is defined with respect to the parameter independent θ. Similarly as in the case with a simple nontrivial solution, using (4), (14) and (16) we find

$$\phi_0[\theta(t), x] \star \phi_0[\theta(s), x] = K\left(-i\sqrt{\frac{(1+t)(1+s)}{(1-t)(1-s)}}, \frac{x_1}{\sqrt{\theta(s)}}, \frac{x_2}{\sqrt{\theta(t)}}\right)$$

$$\times K\left(i\sqrt{\frac{(1+t)(1+s)}{(1-t)(1-s)}}, \frac{x_1}{\sqrt{\theta(t)}}, \frac{x_2}{\sqrt{\theta(s)}}\right)\ \phi_0[\theta(t), x]\, \phi_0[\theta(s), x]. \tag{28}$$

After some calculations using (17) we obtain

$$\phi_0[\theta(t), x] \star \phi_0[\theta(s), x] = \frac{(1-t)(1-s)}{1+ts}\phi_0[\theta(-ts), x]\,. \tag{29}$$

Note that because z is purely imaginary, the condition $Re(1-z^2) > 0$ is satisfied. Substituting this into (27) we find

$$(\phi_n \star \phi_n)(x) = \frac{1}{(n!)^2}\partial_t^n \partial_s^n \left[\frac{1}{1+ts}\phi_0[\theta(-ts), x]\right] |_{t=s=0}\,, \tag{30}$$

and using the formulae

$$\partial_t^n \partial_s^n \varphi(-ts) \mid_{t=s=0} = (-1)^n n! \partial_r^n \varphi(r) \mid_{r=0}\,, \tag{31}$$

we finally obtain

$$(\phi_n \star \phi_n)(x) = (-1)^n \frac{1}{n!}\partial_r^n \left[\frac{1}{1-r}\phi_0[\theta(r), x]\right] |_{r=0}\,, \tag{32}$$

which with help of (25) yields expression (11).

4 p-Adic aspects

Since 1987, p-adic numbers and adeles have been successfully applied in many topics of modern theoretical and mathematical physics (for a review, see e.g. [3, 4, 5]). In particular, p-adic string theory [3, 4], p-adic [6] and adelic [7] quantum mechanics, as well as p-adic and adelic quantum cosmology [8, 9], have been formulated and investigated. It is well known that combining quantum-mechanical and relativity principles one concludes that there exists a spatial uncertainty Δx in the form

$$\Delta x \geq \ell_0 = \sqrt{\frac{\hbar G}{c^3}} \sim 10^{-33} cm. \tag{33}$$

This uncertainty relation (33) may be taken as a reason to consider simultaneously noncommutative and p-adic aspects of spatial coordinates x^i when approaching to the Planck length ℓ_0. Hence we are interesting here in p-adic analogs of the above (real) noncommutative scalar solitons.

When we wish to consider basic properties of p-adic numbers it is convenient to start with the field of rational numbers Q, since Q is the simplest field of numbers of characteristic 0 and it contains all results of physical measurements. Any non-zero rational number can be expanded into two quite different forms of infinite series. The usual one is to the base 10, *i.e.*

$$\sum_{k=n}^{-\infty} a_k 10^k, \quad a_k = 0, \cdots, 9, \tag{34}$$

and the other one is to the base p (p is a prime number) and reads

$$\sum_{k=m}^{+\infty} b_k p^k, \quad b_k = 0, \cdots, p-1, \tag{35}$$

where n and m are some integers. These representations have the usual repetition of digits, but expansions are in the mutually opposite directions.

The series (34) and (35) are convergent with respect to the usual absolute value $|\cdot|_\infty$ and p-adic absolute value $|\cdot|_p$, respectively. Allowing all possible combinations for digits, we obtain standard representation of real numbers (34) and p-adic numbers (35). R and Q_p exhaust all number fields which contain Q as a dense subfield. They have many distinct geometric and algebraic properties. Geometry of p-adic numbers is the nonarchimedean one. For much more information on p-adic numbers and p-adic analysis one can see,e.g. [3, 4, 10, 11].

Practically, there are two kinds of analysis on Q_p based on two different mappings: $Q_p \to Q_p$ and $Q_p \to C$. Both of them are used here. Elementary p-adic functions are defined by the same series as in the real case, but the region of convergence is different. For instance, $\exp x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges if $|x|_p < |2|_p$. Derivatives of p-adic valued functions are also defined as in the real case, but using p-adic valuation instead of the absolute value.

Usual complex-valued p-adic functions are: *(i)* an additive character $\chi_p(x) = \exp 2\pi i \{x\}_p$, where $\{x\}_p$ is the fractional part of $x \in Q_p$, *(ii)* a multiplicative character $\pi_s(x) = |x|_p^s$, where $s \in C$, and *(iii)* locally constant functions with compact support. There is well defined Haar measure and integration.

In the previous sections, spatial coordinates x_1, x_2 and the noncommutativity parameter θ are real variables. In the present section we are going to consider possible extensions of the obtained results when these variables become p-adic valued. One can introduce two types of the p-adic Moyal product. They are p-adic differential and integral analogs of the usual Moyal star product. While in the real case differential and integral forms of the Moyal product are equivalent, we shall see that their p-adic generalizations yield quite different expressions.

As a p-adic Moyal product in the differential form we take the same expression (3), where $f(x)$ and $g(x)$ are analytic p-adic valued functions, and $\sqrt{-1}$ is treated p-adically. Note that [3] p-adic exponential function $\exp x$ is defined by the same infinite power series as in the real case, but convergence is regarded with respect to the p-adic norm and it is for $\exp x$ restricted to the domain $|x|_p < |2|_p$. p-Adic derivatives are defined in the same usual way but using p-adic norm instead of the absolute value.

Exponential derivative operator in (3) acts formally in the same way in real and p-adic cases. Now we are interested in p-adic solution of the equation (11) in the form (12), where $|x^i|_p < \sqrt{|2\theta|_p}, \quad i = 1, 2$. Expanding exponential function with derivatives one obtains

$$(\phi_0 \star \phi_0)(x) = \sum_{n=0}^{\infty} A_n(x), \tag{36}$$

where

$$A_n(x) = 4\frac{i^n \theta^n}{2^n n!} \left(\frac{\partial}{\partial x_1} \frac{\partial}{\partial y_2} - \frac{\partial}{\partial x_2} \frac{\partial}{\partial y_1} \right)^n \exp \left(- \frac{x_1^2 + x_2^2 + y_1^2 + y_2^2}{\theta} \right) |_{y=x}. \tag{37}$$

Performing some calculations explicitly in (37) one has

$$A_n(x) = 4 \exp \left(- \frac{2(x_1^2 + x_2^2)}{\theta} \right) B_n(x), \tag{38}$$

where $B_n(x)$ are:

$$\begin{aligned} &B_{2k+1}(x) = 0, \quad k = 0, 1, 2, \cdots; \quad B_0(x) = 1, \quad B_2(x) = -1 + \frac{2}{\theta}(x_1^2 + x_2^2), \\ &B_4(x) = 1 - \frac{4}{\theta}(x_1^2 + x_2^2) + \frac{4}{2!\theta^2}(x_1^2 + x_2^2)^2, \\ &B_6(x) = -1 + \frac{6}{\theta}(x_1^2 + x_2^2) - \frac{12}{2!\theta^2}(x_1^2 + x_2^2)^2 + \frac{8}{3!\theta^3}(x_1^2 + x_2^2)^3, \ \cdots \end{aligned} \tag{39}$$

We conclude that expansion (36) is divergent since it contains infinitely many divergent subseries like

$$1 - 1 + 1 - 1 + 1 - \cdots \tag{40}$$

Fortunately this subseries can be easily made convergent using expression

$$\sum_{k=0}^{\infty} (-1)^k q^{2k} = \frac{1}{1 + q^2} \tag{41}$$

and its derivatives in the point $q = 1$. This kind of summation attaches the sum $1/2$ to all divergent subseries and using this procedure one gets that

$\phi_0(x) = 2\exp\left(-\frac{x_1^2+x_2^2}{\theta}\right)$ is a solution of the equation $(\phi_0 \star \phi_0)(x) = \phi_0(x)$, where $\star$ denotes the Moyal product defined by (3). It is worth noting that this treatment of the equation $(\phi_0 \star \phi_0)(x) = \phi_0(x)$ does not depend on whether x and θ are real or p-adic. In other words, the above equation $(\phi_0 \star \phi_0)(x) = \phi_0(x)$ and its solutions are number field invariant. Since the sum of a divergent series depends on the way of summation it follows that rearrangement in summation to get (4) from (3) removed divergences in the real case of $(\phi_0 \star \phi_0)(x) = \phi_0(x)$ in previous sections. However when $z = \pm\sqrt{-1}$ the series (16) is not convergent in the p-adic case and such procedure is not suitable for p-adic generalization.

In order to introduce the corresponding integral version of the p-adic Moyal product let us recall that its integral form in the real case is

$$(f \star g)(x) = \int dkdk' \exp\left(\frac{2\pi i}{h}(kx + k'x - \frac{\theta^{ij}}{2}k_ik'_j)\right)\tilde{f}(k)\tilde{g}(k'), \tag{42}$$

where $\tilde{f}(k)$ and $\tilde{g}(k')$ are Fourier transforms ($\tilde{f}(k) = \int \exp(-\frac{2\pi i}{h}kx)f(x)dx$) of $f(x)$ and $g(x)$, respectively. The equation (42) is derived by the Weyl quantization prescription of the functions on noncommutative coordinates $\{\hat{x}^i\}$, which satisfy canonical relation

$$[\hat{x}^i, \hat{x}^j] = i\frac{h}{2\pi}\theta^{ij}. \tag{43}$$

The expression (42) can be rewritten in the form

$$(f \star g)(x) = \int dkdk'\chi_\infty(-kx - k'x + \frac{\theta^{ij}}{2}k_ik'_j)\tilde{f}(k)\tilde{g}(k'), \tag{44}$$

where $\chi_\infty(u) = \exp(-2\pi iu)$ is the additive character in the real case, and it is also taken $h = 1$.

Defining the integral p-adic Moyal product as p-adic generalization of (44) we have

$$(f \star g)(x) = \int dkdk'\chi_p(-kx - k'x + \frac{\theta^{ij}}{2}k_ik'_j)\tilde{f}(k)\tilde{g}(k'), \tag{45}$$

where p-adic additive character is $\chi_p(u) = \exp(2\pi i\{u\}_p)$ and $\{u\}_p$ is rational part of p-adic number u. Note that $f, g, (\chi_p)$ are real (complex) functions of p-adic variables and $\tilde{f}(k) = \int \chi_p(kx)f(x)dx$. This p-adic integration is well defined by the Haar additive measure.

The equation $(\varphi \star \varphi)(x) = \varphi(x)$ now reads

$$\int dkdk'\chi_p(-kx - k'x + \frac{\theta^{ij}}{2}k_ik_j')\tilde{\varphi}(k)\tilde{\varphi}(k') = \varphi(x), \tag{46}$$

where for simplicity we take $i, j = 1, 2$. According to the rules of integration [3] we find the following solution

$$\varphi_\nu(x) = \Omega(p^\nu|x_1|_p)\ \Omega(p^{-\nu}|x_2|_p) \tag{47}$$

of (46) with restriction $\left|\frac{\theta^{ij}}{2}\right|_p \leq 1$, where $\nu \in Z$ and

$$\Omega(p^\nu|x|_p) = \begin{cases} 1, & |x|_p \leq p^{-\nu}, \\ 0, & |x|_p > p^{-\nu}. \end{cases} \tag{48}$$

Note that the equation $[\Omega(p^\nu|x_1|_p)\ \Omega(p^{-\nu}|x_2|_p)]\star[\Omega(p^\nu|x_1|_p)\ \Omega(p^{-\nu}|x_2|_p)] = \Omega(p^\nu|x_1|_p)\ \Omega(p^{-\nu}|x_2|_p)$ is also valid when $\star$ product is replaced by the ordinary one. Since the Fourier transform of $\Omega(p^\nu|x|_p)$ is $\Omega(p^{-\nu}|k|_p)$ we have $\tilde{\varphi}_\nu(k) = \Omega(p^{-\nu}|k_1|_p)\Omega(p^\nu|k_2|_p)$. It means that when $\nu = 0$ then function $\Omega(|x|_p)$ has its Fourier transform $\Omega(|k|_p)$, and it resembles the same property of the Gaussian $\exp(-\pi x^2)$ in the real case. In such sense function $\Omega(|x|_p)$ is a p-adic analogue of $\exp(-\pi x^2)$. From the function $\varphi_\nu(x) = \Omega(p^\nu|x_1|_p)\Omega(p^{-\nu}|x_2|_p)$ and its Fourier transform it follows that p-adic solitonic solution is in the region $|x_1| \leq p^{-\nu}, \quad |x_2| \leq p^\nu, \quad |k_1| \leq p^\nu, \quad |k_2| \leq p^{-\nu}$.

5 Adelic aspects

An adele x [11] is an infinite sequence

$$x = (x_\infty, x_2, \cdots, x_p, \cdots), \tag{49}$$

where $x_\infty \in R$ and $x_p \in Q_p$ with the restriction that for all but a finite set S of primes p we have $x_p \in Z_p = \{x \in Q_p : |x|_p \leq 1\}$. One can use componentwise addition and multiplication. It is useful to present the ring of adeles $\mathcal{A}$ in the following form:

$$\mathcal{A} = \cup_S \mathcal{A}(S), \tag{50}$$

$$\mathcal{A}(S) = R \times \prod_{p \in S} Q_p \times \prod_{p \notin S} Z_p, \tag{51}$$

where Z_p is the ring of p-adic integers. $\mathcal{A}$ is also locally compact topological space with well-defined Haar measure. There are mainly two kinds of analysis over $\mathcal{A}$, which generalize those over R and Q_p.

Adelic approach gives possibility to treat real and all p-adic aspects of a quantum system simultaneously and as essential parts of a more complete description. Adelic quantum mechanics was formulated [7] and successfully applied to some simple and solvable quantum models. Here we use adelic approach to noncommutative scalar solitons.

According to the previous sections the equation $(\phi \star \phi)(x) = \phi(x)$ can be considered as real as well as p-adic. The $\star$ product may be realized in both differential and integral form. Unlike the real case, p-adic solutions of this equation depend on differential and integral formulations. Hence we have two adelic versions of adelic aspects of $(\phi \star \phi)(x) = \phi(x)$, which are related to differential and integral realizations of $\star$.

When $\star$ product has the differential (Moyal) form, we have adelic valued solutions

$$\phi(x) = (\phi_\infty(x_\infty), \phi_2(x_2), \cdots, \phi_p(x_p), \cdots), \tag{52}$$

where $\phi_\infty(x_\infty)$ and $\phi_p(x_p)$ are real and p-adic valued functions of the form (22), respectively, and $|\phi_p(x_p)| \leq 1$ for all but a finite set S of prime numbers p. Possible region of x_p is determined by convergence of the exponential functions and it is $|x_p|_p \leq \sqrt{|2\theta|_p}$. Parameter θ as a characteristic of a physical system has to be rational, since a rational number can be treated at the same time as real as p-adic.

If the $\star$ product has integral form then equation $(\varphi \star \varphi)(x) = \varphi(x)$ has real valued solutions of adelic variable x. These solutions have the following form:

$$\varphi(x) = \varphi_\infty(x_\infty) \prod_{p \in S} [\Omega(p^{\nu_p}|x_p^1|_p)\ \Omega(p^{-\nu_p}|x_p^2|_p)] \prod_{p \notin S} [\Omega(|x_p^1|_p)\ \Omega(|x_p^2|_p)], \quad (53)$$

where S can be arbitrary finite set of primes p and $\nu_p \in Z$. In (53) function $\varphi_\infty(x_\infty)$ may be any of the found real solutions (22) of the equation $(\varphi \star \varphi)(x) = \varphi(x)$.

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