Noncommutativity from strings by canonical methods *

B. Sazdović[†]
Institute of Physics, 11001 Belgrade, P.O.Box 57, Yugoslavia

Abstract

In this article we perform the canonical analysis of the open string in the presents of background antisymmetric tensor and vector fields. Treating boundary conditions as a constraints, we show that Dirac brackets between coordinates are nontrivial only at the strings end-points. As a consequence the D-brane world-volume, where the open strings endpoints lives, becomes a noncommutative one.

1 Introduction

The string theory is a leading candidate for unification of all fundamental forces of nature (electro-week, strong and gravitational) [1]. There is a hope that this theory will be able to solve the problem of ultra-violet divergences.

Recently the non-commutative geometry and its applications in string theory has been considered [2]. The connection between non-commutative and commutative Yang-Mills gauge fields and its role its string theory has been discussed in [3].

*Work supported in part by the Serbian Ministry of Science, Technology and Development, Yugoslavia

[†]e-mail address: sazdovic@phy.bg.ac.yu

Here, we are going to investigate the open string theory in background gravitational, antisymmetric tensor and $U(1)$ gauge fields. In the presents of D-branes, the hyperplane on which the open string points is constrained to lie, the antisymmetric tensor field lead to the non-commutativity of the D-brane manifold. This problem has been discussed in the literature [4], using the mode expansion of the string coordinate or discretization method.

2 Classical theory of free bosonic string

In the standard physical theories, the elementary objects are pointlike objects-elementary particles. The action of the relativistic particle, moving in D dimensional Minkowski space-time is proportional to the length of the world-line between initial and final points

$$S = -m \int_{s_i}^{s_f} ds = -m \int_{\tau_i}^{\tau_f} d\tau \sqrt{G_{\mu\nu}\dot{x}^\mu \dot{x}^\nu} = \int_{\tau_i}^{\tau_f} d\tau \mathcal{L} \, . \qquad (\dot{x}^\mu = \frac{dx^\mu}{d\tau}) \tag{1}$$

Here τ is an arbitrary parameter along the particle trajectory (world-line), $x^\mu(\tau)$ is the position of the particle ($\mu, \nu = 0, 1, ..., D-1$) and $G_{\mu\nu}$ is space-time metric tensor.

Let us instead of the particle introduce one dimensional elementary object, the string, parametrized by internal coordinate $\sigma \in [0, \pi]$. As an extended object, it can exists in two distinct topologies: open with end points at $\sigma = 0$ and $\sigma = \pi$, and closed with the condition $x^\mu(\sigma = \pi) = x^\mu(\sigma = 0)$. During the evolution in D dimensional space-time the string spans the two dimensional surface (world-sheet) with coordinates $x^\mu(\tau, \sigma)$, where τ is time-like evolution parameter.

In analogy with the point particle case, the action for the string is defined to be proportional to the area of the world-sheet Σ

$$S = \kappa \int_\Sigma d^2 A = \int_\Sigma d^2\xi \mathcal{L} \, , \tag{2}$$

where $\xi^\alpha = (\xi^0 = \tau, \xi^1 = \sigma)$ is a two dimensional vector. The string tension κ has the dimension of $(\text{length})^2$ and it is related to the Regge slope parameter α' by $\kappa = \frac{1}{2\pi\alpha'}$.

In terms of space-time metric tensor $G_{\mu\nu}(x)$, we can write the invariant interval as

$$ds^2 = G_{\mu\nu}(x)dx^\mu dx^\nu = G_{\mu\nu}(x)\partial_\alpha x^\mu \partial_\beta x^\nu d\xi^\alpha d\xi^\beta\,, \qquad (\partial_\alpha \equiv \frac{\partial}{\partial\xi^\alpha}) \tag{3}$$

so that the induced metric on the world-sheet is

$$G_{\alpha\beta}(x) = G_{\mu\nu}(x)\partial_\alpha x^\mu \partial_\beta x^\nu\,. \tag{4}$$

Then using the expression for the world-sheet area $d^2A = d^2\xi\sqrt{-detG_{\alpha\beta}}$ the action takes the Nambu-Goto form

$$S_{NG} = \kappa \int_{s_i}^{s_f} d\tau \int_0^\pi d\sigma \sqrt{-detG_{\alpha\beta}} = \kappa \int_\Sigma d^2\xi \sqrt{-\dot{x}^2 x'^2 + (\dot{x}x')^2}\,, \tag{5}$$

where $x'^\mu = \frac{dx^\mu}{d\sigma}$ and $xy = G_{\mu\nu}x^\mu y^\nu$.

The nonlinearity of the action complicates the investigation of this theory. We can simplify the action by introducing new auxiliary variable, an intrinsic world-sheet metric $g^{\alpha\beta}$. Then, the equivalent but more convenient, Polyakov form of the action reads

$$S_G = \frac{\kappa}{2}\int_\Sigma d^2\xi\sqrt{-g}g^{\alpha\beta}\partial_\alpha x^\mu \partial_\beta x^\nu G_{\mu\nu}(x)\,, \tag{6}$$

where the index G indicate that the string moves in the space-time with metric tensor $G_{\mu\nu}$. The equation of motion for the variable $g^{\alpha\beta}$

$$T_{\alpha\beta} = G_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}g^{\gamma\delta}G_{\gamma\delta} = 0\,, \tag{7}$$

is equivalent to the requirement that energy-momentum tensor $T_{\alpha\beta}$ is zero. It is easy to check that

$$g_{\alpha\beta} = G_{\alpha\beta}\,, \tag{8}$$

is a solution of this equation, so that the intrinsic world-sheet metric is on-shell equivalent to the induced world-sheet metric. Since time derivatives of $g_{\alpha\beta}$ do not appear in Polyakov action we can substitute solution (8) back into (6) and obtain Nambu-Goto action. Therefore, both actions are classically equivalent.

It is possible to include other modes in the string action. With the usual requirement that the action is reparametrization invariant and has two world-sheet derivatives in each term, we can write

$$S = S_G + S_B + S_A + S_\Phi \,, \tag{9}$$

where

$$\begin{aligned} S_B &= \kappa \int_\Sigma d^2\xi \varepsilon^{\alpha\beta} B_{\mu\nu}(x) \partial_\alpha x^\mu \partial_\beta x^\nu \,, \\ S_A &= 2\kappa \int_{\partial\Sigma} A_\mu(x) dx^\mu \,, \\ S_\Phi &= \lambda \int_\Sigma d^2\xi \sqrt{-g}\Phi(x) R^{(2)} \,. \end{aligned} \tag{10}$$

Here $B_{\mu\nu} = -B_{\nu\mu}$ is antisymmetric background field, A_μ is $U(1)$ gauge field, Φ is dilaton field and $R^{(2)}$ is scalar curvature of the intrinsic world-sheet metric. The integration in the term S_A is over $\partial\Sigma$, the boundary of the world-sheet Σ.

Because in this article we will not be interested in dilaton field, we will omit S_Φ and after some arrangement of the S_A term the action becomes

$$S = \kappa \int_\Sigma d^2\xi \left[\frac{1}{2}\sqrt{-g} g^{\alpha\beta} G_{\mu\nu}(x) + \varepsilon^{\alpha\beta} F_{\mu\nu}(x) \right] \partial_\alpha x^\mu \partial_\beta x^\nu \,, \tag{11}$$

where $F_{\mu\nu} = B_{\mu\nu} + \partial_\mu A_\nu - \partial_\nu A_\mu$ is modified Born-Infeld field strength.

We can fix two parameter world-sheet reparametrization invariance and one parameter Weyl invariance. Then the $2D$ Minkowski metric becomes flat $g_{\alpha\beta} \to \eta_{\alpha\beta}$, and the expression for the action takes the form

$$S = \kappa \int_\Sigma d^2\xi \left[\frac{1}{2}\eta^{\alpha\beta} G_{\mu\nu}(x) + \varepsilon^{\alpha\beta} F_{\mu\nu}(x) \right] \partial_\alpha x^\mu \partial_\beta x^\nu \,. \tag{12}$$

Let us require that, according to the action principle, evolution from given initial configuration (at τ_i) to given final configuration (at τ_f) is such that the action (12) is stationary. As well as in the particle case this will give us the equation of motion, but for open strings we will obtain the new conditions from surface term: the boundary conditions. Besides the variables $g^{\alpha\beta}$, x^μ, $\dot{x}^\mu$ the action for strings also depends on x'^μ with no analogies in the particle case, which causes the boundary condition. We already solved the equation of motion for variable $g^{\alpha\beta}$, so we are now interested in variation with respect to x^μ. After partial integration with respect to τ and σ, the variation of the action becomes

$$\begin{aligned}\delta S = &\int d\sigma d\tau \Big(\frac{\partial \mathcal{L}}{\partial x^\mu} - \partial_\tau \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} - \partial_\sigma \frac{\partial \mathcal{L}}{\partial x'^\mu}\Big)\delta x^\mu \\ &+ \int d\sigma \Big(\frac{\partial \mathcal{L}}{\partial \dot{x}^\mu}\delta x^\mu\Big)/_{\tau=\tau_i}^{\tau=\tau_f} + \int d\tau \Big(\frac{\partial \mathcal{L}}{\partial x'^\mu}\delta x^\mu\Big)/_{\sigma=\sigma_i}^{\sigma=\sigma_f} = 0\,. \qquad (13)\end{aligned}$$

From the stationarity of the action all three terms must vanish separately. The first one gives us Euler-Lagrange equation of motion. The second one vanishes by action principle requirement that initial and final configurations are fixed, so that $\delta x^\mu/_{\tau=\tau_i,\tau_f} = 0$. The third term is a new one and it gives the condition

$$\frac{\partial \mathcal{L}}{\partial x'^\mu}\delta x^\mu/_{\sigma=\pi} - \frac{\partial \mathcal{L}}{\partial x'^\mu}\delta x^\mu/_{\sigma=0} = 0\,. \qquad (14)$$

For closed strings, the periodicity condition $x^\mu(\sigma=\pi) = x^\mu(\sigma=0)$ is sufficient to fulfill eq.(14). In the open string case the variations $\delta x^\mu/_{(\sigma=\pi)}$ and $\delta x^\mu/_{(\sigma=0)}$ are independent, so both terms in (14) must vanish separately. With the help of new notation

$$\gamma_\mu = \frac{\partial \mathcal{L}}{\partial x'^\mu} = \kappa(G_{\mu\nu}x'^\nu - 2F_{\mu\nu}\dot{x}^\nu)\,, \qquad (15)$$

we can write the boundary conditions at the endpoints of the open string in the form

$$\gamma_\mu/_{\sigma=0} = 0, \qquad \gamma_\mu/_{\sigma=\pi} = 0\,. \qquad (16)$$

For the small values of F we have $x'^{\mu}/_{\sigma=0,\pi} = 0$. This is Neumann boundary conditions, because normal derivative of x^μ is zero at the strings endpoints. In the case of $F \to \infty$ (or $G \to 0$) we obtain the Dirichlet boundary conditions. In this limit, the endpoints of the string are fixed. Therefore, the mixed conditions (16) are linear combinations of Neumann and Dirichlet.

From this point we will use the concept of Dp-branes. So, let us shortly introduce it. The p-branes are extended objects with p spatial dimensions which are solutions of the string equations. A special type of p-branes, with Dirichlet boundary conditions is known in the literature as a Dp-brane. In fact, it is defects in space-time where closed strings can end and open strings endpoints are forced to move on it.

We will suppose that the open string endpoints lies on some Dp-brane. Consequently, the $U(1)$ gauge field A lives on the Dp-brane because it is define on $\partial\Sigma$. We can rewrite the boundary conditions as

$$\gamma_i = \kappa(G_{ij}x'^j - 2F_{ij}\dot{x}^j)\,, \qquad \gamma_i/_{\sigma=0,\pi} = 0\,, \tag{17}$$

where x^i is position of the Dp-brane $(i, j = 1, 2, ...p)$.

3 Canonical analysis and boundary conditions as a constraints

It is very useful to perform the canonical analysis of the action (12), treating the conditions (17) as constraints of the theory. The canonical momenta are

$$\pi_\mu = \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} = \kappa(G_{\mu\nu}\dot{x}^\nu - 2\delta^i_\mu F_{ij}x'^j)\,. \tag{18}$$

In terms of the currents

$$j_{\pm\mu} = \pi_\mu + 2\kappa\delta^i_\mu P_{\pm ij}x'^j\,, \qquad (P_{\pm ij} = F_{ij} \pm \frac{1}{2}G_{ij}) \tag{19}$$

we can solve (18) as

$$\dot{x}^\mu = \frac{1}{2\kappa} G^{\mu\nu}(j_{+\nu} + j_{-\nu})\,. \tag{20}$$

The expression for the canonical hamiltonian is

$$H_c = \int d\sigma \mathcal{H}_c = \int d\sigma (t_- - t_+)\,, \qquad (t_\pm = \frac{\mp 1}{4\kappa} G^{\mu\nu} j_{\pm\mu} j_{\pm\nu}) \tag{21}$$

and boundary conditions takes the form

$$\gamma_i = P_{-i}{}^j j_{+j} + P_{+i}{}^j j_{-j}\,, \qquad \gamma_{0i} = \gamma_i/_{\sigma=0}\,, \qquad \gamma_{\pi i} = \gamma_i/_{\sigma=\pi}\,. \tag{22}$$

Starting with basic Poisson brackets $\{x^\mu(\sigma), \pi_\nu(\bar\sigma)\} = \delta^\mu_\nu \delta(\sigma - \bar\sigma)$, we can check the consistency conditions for the constraint (22). To simplify calculations, we put the conditions on the string background fields B_{ij} and $G_{\mu\nu}$, supposing that they are constant. Then all quantities with the opposite chirality commute and

$$\{j_{\pm\mu}, j_{\pm\nu}\} = \pm 2\kappa G_{\mu\nu} \delta'\,. \tag{23}$$

Note that A_i can be arbitrary vector field because its contribution disappear from the last equation as a consequence of Bianchi identity.

Poisson brackets between canonical hamiltonian and the currents are

$$\{H_c, j_{\pm i}(\sigma)\} = \mp j_{\pm i}{}'(\sigma)\,, \tag{24}$$

so they just change the sign of the j_+ and add one σ derivative. Therefore, with the help of the relation

$$\{H_c, \gamma_{ni}\} = \gamma_{n+1,i}\,, \tag{25}$$

we obtain an infinite set of the consistency conditions

$$\gamma_{ni} = \partial_\sigma^n [P_{+i}{}^j j_{-j} + (-1)^n P_{-i}{}^j j_{+j}]/_{\sigma=0} = 0\,. \qquad (n = 0, 1, ...) \tag{26}$$

Then we substitute this infinite set of the constraints at a point by one parameter dependent constraint

$$\Gamma_i(\sigma) = \sum_{n=0}^{\infty} \frac{\sigma^n}{n!} \gamma_{ni}(0) = [P_{+i}{}^j j_{-j}(\sigma) + P_{-i}{}^j j_{+j}(-\sigma)] \,. \tag{27}$$

It differs from the original expression γ_i just in sign in front of σ in j_+. We can check that $\gamma_i/_{\sigma=\pi}$ gives the same result and that final constraint (27) weakly commutes with the hamiltonian,

$$\{H_E, \Gamma_i\} = \Gamma_i{}' \,, \tag{28}$$

so that there are no more constraints.

The constraints $\Gamma_i(\sigma)$ are of the second class

$$\{\Gamma_i(\sigma), \Gamma_j(\bar\sigma)\} = \kappa G^{eff}_{ij} \delta'(\sigma - \bar\sigma) \equiv \Delta_{ij} \,, \tag{29}$$

where the effective metric tensor

$$G^{eff}_{ij} = -4(P_\pm P_\mp)_{ij} = (G - 4FG^{-1}F)_{ij} \,, \tag{30}$$

is known in the literature as the open string metric.

To eliminate physically unimportant degrees of freedom we should use Dirac brackets defined as

$$\{X, Y\}^\star = \{X, Y\} - \{X, \Phi_i\}(\Delta^{-1})^{ij}\{\Phi_j, Y\} \,. \tag{31}$$

Here the inverse of Δ_{ij},

$$\int d\sigma_1 \Delta_{ik}(\sigma, \sigma_1)(\Delta^{-1})^{kj}(\sigma_1, \bar\sigma) = \delta_i^j \delta(\sigma - \bar\sigma) \,, \tag{32}$$

can be expressed as

$$(\Delta^{-1})^{kj}(\sigma, \bar\sigma) = \frac{-1}{\kappa} (G^{eff})^{ij} \Delta(\sigma - \bar\sigma) \,, \tag{33}$$

where $\Delta(\sigma)$ is solution of the equation $\Delta'(\sigma) = \delta(\sigma)$. After some calculations we obtain that Dirac brackets between coordinates

$$\{x^i, x^j\}^\star = 2\theta^{ij}\Delta(\sigma + \bar{\sigma}), \tag{34}$$

are proportional to the F dependent non-commutativity parameter $\theta^{ij} = \frac{1}{4\kappa}(P_\pm^{-1} F P_\mp^{-1}) = \frac{-1}{\kappa} F (G^{eff})^{-1}$, or explicitly

$$\theta^{ij} = \frac{1}{4\kappa}\left(\frac{1}{F + \frac{G}{2}} F \frac{1}{F - \frac{G}{2}}\right)^{ij}. \tag{35}$$

If we subtract τ independent part of the string center of mass $x^i_{cm} = \frac{1}{\pi}\int_0^\pi d\sigma x^i$, which we denote by $\bar{x}^i$ and call center of the string we obtain $\widetilde{x}^i = x^i - \bar{x}^i$. For the new variable we have

$$\{\widetilde{x}^i, \widetilde{x}^j\}^\star = 2\theta^{ij}\left[\Delta(\sigma + \bar{\sigma}) - \frac{1}{2}\right] \equiv \theta^{ij}\begin{cases} -1 & \sigma = 0 = \bar{\sigma} \\ 1 & \sigma = \pi = \bar{\sigma} \\ 0 & \text{otherwise} \;. \end{cases} \tag{36}$$

The non trivial Dirac brackets appear only at the endpoints of the string

$$\{\widetilde{x}^i(0), \widetilde{x}^j(0)\}^\star = -\{\widetilde{x}^i(\pi), \widetilde{x}^j(\pi)\} = \theta^{ij}. \tag{37}$$

Consequently, because open string endpoints are forced to move on Dp-brane, Dp-brane world volume becomes noncommutative manifold.

4 Canonical quantization and noncommutative coordinates

Let us perform the quantization of the matter fields $\widetilde{x}^i$ substituting them by operators $\widehat{x}^i$. We also replace the Dirac brackets (37) by the commutators

$$[\widehat{x}^i, \widehat{x}^j] = i\theta^{ij}. \qquad (i, j = 1, 2, ...p) \tag{38}$$

We are interesting just in the string endpoints, and because their commutation relations are differ only in sign, for definiteness we will use the sign of $\sigma = 0$ case.

To investigate the algebra of functions depending on the operators $\widehat{x}^i$, let us first introduce prescription which uniquely assign the operator $\widehat{f}(\widehat{x})$ to any function $f(x)$. For simplicity we will chose $p = 2$. If the Fourier transformation $\widetilde{f}(k)$ of the given function $f(x)$ is

$$\widetilde{f}(k) = \int d^2 x f(x) e^{ikx} , \tag{39}$$

then we define the corresponding operator $\widehat{f}(\widehat{x})$ as

$$\widehat{f}(\widehat{x}) = \frac{1}{(2\pi)^2} \int d^2 k \widetilde{f}(k) e^{-ik\widehat{x}} . \tag{40}$$

This prescription is known as Weyl or symmetric ordering prescription.

The *Moyal star* product of the functions is defined to be isomorphic to the operator multiplication, which means that

$$\widehat{f}(\widehat{x}) \cdot \widehat{g}(\widehat{x}) = \widehat{h}(\widehat{x}) \qquad \sim \qquad f(x) \star g(x) = h(x) , \tag{41}$$

where the corresponding functions are connected as in (39)-(40).

Using Baker-Hausdorff formula it is easy to show that

$$(f \star g)(x) = e^{\frac{i}{2}\theta^{ij}\partial_{a_i}\partial_{b_j}} f(x+a) g(x+b)|_{a=b=0} , \tag{42}$$

where we used the notation $\partial_{a_i} = \frac{\partial}{\partial a_i}$.

From the isomorphism with the operator multiplication, it is clear that the star product is

$$\textit{noncommutative} \qquad\qquad f \star g \neq g \star f , \tag{43}$$

and

$$\textit{associative} \qquad (f \star g) \star h = f \star (g \star h) . \tag{44}$$

If the functions f and g vanish rapidly enough at infinity then one can integrate by parts to show that

$$\int d^2x f \star g = \int d^2x g \star f = \int d^2x f g \, . \tag{45}$$

5 Conclusion

We considered the boundary conditions contributions to the open string theory in the presents of background antisymmetric field B_{ij}. Because the string lagrangian depends on x'^μ, the action principle, beside equation of motion, leads to the boundary conditions. We used canonical approach and treated these conditions as a constraints. The consistency conditions give us the infinite set of the constraints at the string endpoints, which we substitute with one σ-dependent constraint Γ_i. The constraints Γ_i are SCC, so we introduce the Dirac brackets to eliminate unphysical degrees of freedom. It turns out that only endpoints of the gauge invariant part of the coordinates satisfies the nontrivial Dirac brackets. Consequently, Dp-brane world volume, the manifold where the open string endpoints forced to move, becomes a noncommutative one.

After canonical quantization of the matter fields $\widetilde{x}^i$, the variables on the Dp-brane depend on the operators $\widehat{x}^i$. It is shown that it is possible to use ordinary functions instead the operator dependent ones, if the operator product is substitute by Moyal star product.

References

[1] M. B. Green, J. H. Schwarz and E. Witten, Superstring Theory (Cambridge University Press, Cambridge, 1987); J. Polchinski, String theory (Cambridge University Press, Cambridge, 1998).

[2] A. Connes, M. R. Douglas and A. Schwarz, JHEP **02** 003 (1998).

[3] N. Seiberg and E. Witten, JHEP **09**, 032 (1999).

[4] T.Lee, Phys. Rev. **D62** (2000) 024022; C. S. Chu and P. M. Ho, Nucl. Phys. **B550** (1999) 151; C. S. Chu and P. M. Ho, Nucl. Phys. **B568** (2000) 447; F. Ardalan, H. Arfaei and M. M. Sheikh-Jabbari JHEP **02** (1999) 016; F. Ardalan, H. Arfaei and M. M. Sheikh-Jabbari Nucl. Phys. **B576** (2000) 578.