Fuzzy-PP Waves

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Abstract

We present a noncommutative version of a plane-wave solution to the gravitational field equations. We start with a given classical solution, admittedly rather simple, and constructed an algebra and a differential calculus which supports the metric. In the particular solution presented as example the differential calculus differs in that the usual relation $(\theta^+)^2 = 0$ has become $(2\theta^+)^2 = i\bar{k}\mu f_a \theta^a \theta^-$ with f_a determined by the commutation relations of the algebra and therefore by the PP amplitude.

1 Motivation

A noncommutative generalization of a classical solution to Einstein's equation would require first of all a noncommutative version of a smooth manifold. Although noncommutative geometry has its roots in several attempts by physicists to overcome the problems of ultraviolet divergences in field theory it is to A. Connes that is due the first construction [1] of a smooth structure. The version of this which we shall use here [2] is a slight, even 'smoother', modification which permits use of a metric of Minkowski signature. This metric has also a locality property which is most easily expressible in terms of a 'frame' in the sense of Cartan. A smooth manifold V with a metric can be defined as a subspace of an euclidean space $\mathbb{R}^m$ of sufficiently high dimension m and the embedding can be chosen so that the induced metric coincides with that of the manifold. If the manifold is of dimension n then generically in this case m must satisfy $m \geq n(n-1)/2$. The embedding is equivalent to a projection

$$\mathcal{C}(\mathbb{R}^m) \to \mathcal{C}(V)$$

of the algebra of functions $\mathcal{C}(\mathbb{R}^m)$ onto the algebra $\mathcal{C}(V)$ of smooth functions on V and the latter can be formally defined in terms of generators and relations. This is convenient for the noncommutative generalizations.

Classical metrics have often been defined [3, 4] using embeddings and although it is possible in the present case to avoid this construction [5] it will perhaps in general be convenient. Other metrics with curvature have been proposed [6, 7, 8, 9] in the past but none has been found to be Einstein. The method we use is for the moment limited to what could be considered noncommutative analogues of parallelizable manifolds. Another popular approach has not this limitation but is restricted to metrics of euclidean signature; we refer to two recent books [10, 11] for an exhaustive description and reference to the original literature.

Within the general framework which we here consider, the principal difference between the commutative and noncommutative cases lies in the spectrum of the operators which we use to generate the noncommutative algebra which replaces the algebra of functions. This in turn depends not only on the structure of this algebra as abstract algebra but on the representation of it which we choose to consider. The ultimate aim is the construction of a noncommutative generalization of the theory of general relativity which will remain "smooth', but become essentially noncommutative, in regions where the commutative limit would be singular. The intuitive model we use as 'Vorbild' is based on the theory of 'blowing up' of singularities in algebraic geometry[12] as well as the resolution of core regions in solid-state physics. The present example is regular everywhere and we do not consider representation theory so the difference with the classical theory is minimal. We propose it rather is an example of the problems which arise when one attempts to introduce a noncommutative extension of riemannian curvature. For a discussion of this point we refer to Section 5.

2 The general formalism

For the purpose of the present exposition the expression 'noncommutative space-time' will designate a $*$-algebra $\mathcal{A}$ with trivial center and $n(=4)$ hermitian generators x^μ as well as a differential calculus $\Omega^*(\mathcal{A})$ over $\mathcal{A}$ which has certain 'parallelizability' properties; notably the $\mathcal{A}$-module $\Omega^1(\mathcal{A})$ is free. This will be discussed in more detail below. The x^μ will be referred to as 'position generators'. We shall suppose also that there is a set of $n(=4)$ antihermitian 'momentum generators' λ_α and a 'Fourier transform'

$$F : x^\mu \longrightarrow \lambda_\alpha = F_\alpha(x^\mu)$$

which takes the position generators to the momentum generators. Let ρ be a representation of $\mathcal{A}$ as an algebra of linear operators on some

Hilbert space. For every $k_\mu \in \mathbb{R}^4$ one can construct a unitary element $u(k) = e^{ik_\mu x^\mu}$ of $\mathcal{A}$ and one can consider the weakly closed algebra $\mathcal{A}_\rho$ generated by the image of the $u(k)$ under ρ. The momentum operators λ_α are also unbounded but using them one can construct also a set of 'translation' operators $\hat{u}(\xi) = e^{\xi^\alpha \lambda_\alpha}$ whose image under ρ belongs also to $\mathcal{A}_\rho$. In general $\hat{u}u \neq u\hat{u}$; if the metric which we introduce is the flat metric then we shall see that $[\lambda_\alpha, x^\mu] = \delta^\mu_\alpha$ and in this case we can write the commutation relations $\hat{u}u = qu\hat{u}$ with $q = e^{ik_\mu \xi^\mu}$; the 'Fourier transform' is the simple linear transformation

$$\lambda_\alpha = \frac{1}{i\kbar}\theta^{-1}_{\alpha\mu} x^\mu$$

for some symplectic structure $\theta^{\alpha\mu}$. As a measure of noncommmutativity, and to recall the many parallelisms with quantum mechanics, we use the symbol $\kbar$, which will designate the square of a real number whose value could lie somewhere between the Planck length and the proton radius. Using λ_α we construct the derivation

$$e_\alpha f = [\lambda_\alpha f].$$

That is, the derivations are related to the momenta as usual. In the flat case mentioned above we find that

$$e_\alpha x^\nu = \frac{1}{i\kbar}\theta^{-1}_{\alpha\mu}[x^\mu, x^\nu] = \delta^\nu_\alpha$$

provided we introduce the commutation relations

$$[x^\mu, x^\nu] = i\kbar\theta^{\mu\nu}$$

or equivalently the Poisson structure

$$\{x^\mu, x^\nu\} = \lim_{\kbar \to 0} \frac{1}{i\kbar}[x^\mu, x^\nu] = \theta^{\mu\nu}$$

in the commutative limit. Since the center is trivial the algebra can also be considered as the 'phase space' and F will be (implicitly) assumed to be an automorphism of $\mathcal{A}$:

$$\widehat{u_1 * u_2} = \hat{u}_1 \hat{*} \hat{u}_2.$$

We designate here for a moment the product with a star so we can place a hat on it. In the classical limit F is an extension to phase space of the ordinary Fourier transform. As a simple example, one can consider the two Pauli matrices $(x^\mu) = (\sigma^1, \sigma^2)$ as 'coordinates' for the algebra of 2×2 matrices and the linear combination $(\lambda_\alpha) = (-\sigma_2, \sigma_1) = \epsilon_{\mu\nu}\sigma^\nu$ as 'momenta'. The derivations $e_\alpha = \mathrm{ad}\,\lambda_\alpha$ almost behave as partial derivatives: $e_\alpha x^\mu = \delta^\mu_\alpha \sigma^3$. In this case there is no commutative limit.

In general the representation is not necessarily irreducible but we shall assume that it is a direct sum of an arbitrary number of copies of some representation which is irreducible. The subalgebra $\mathcal{A}'_\rho$ of those elements algebra $\mathcal{B}(\mathcal{H})$ of all bounded operators on $\mathcal{H}$ which commute with $\mathcal{A}_\rho$ is therefore nontrivial. There does not seem to be a direct role to be played by the $\mathcal{A}'_\rho$ except for the constraint that it must grow as the algebra becomes commutative if this limit is not to be singular. Typically then, as the algebra becomes increasingly commutative the representation becomes increasingly reducible.

We shall suppose that $\mathcal{A}$ has a commutative limit which is an algebra $\mathcal{C}(V)$ of smooth functions on a space-time V endowed with a globally defined moving frame θ^α and thus a metric. By parallelizable we mean that the module $\Omega^1(\mathcal{A})$ has a basis θ^α which commutes with the elements of $\mathcal{A}$. For all $f \in \mathcal{A}$

$$f\theta^\alpha = \theta^\alpha f. \tag{2.1}$$

This is the correspondence principle we use. We shall see that it implies that the metric components must be constants, a condition usually imposed on a moving frame. The frame θ^α allows one [2] to construct a representation of the differential algebra from that of $\mathcal{A}$. The frame elements belong in fact to the algebra $\mathcal{A}'_\rho$. Following strictly what one does in ordinary geometry, we shall introduce the set of derivations e_α to be dual to the frame θ^α, that is with

$$\theta^\alpha(e_\beta) = \delta^\alpha_\beta.$$

We define the differential exactly as did E. Cartan in the commutative case. If e_α is a derivation of $\mathcal{A}$ then for every element $f \in \mathcal{A}$ we define df by the constraint $df(e_\alpha) = e_\alpha f$. The construction of the differential calculus from the structure of the resulting module of 1-forms is essentially due to Connes [1]; in the present notation it is to be found elsewhere [2]. It follows from general arguments that the momenta λ_α must satisfy the consistency condition [13, 14]

$$2\lambda_\gamma \lambda_\delta P^{\gamma\delta}{}_{\alpha\beta} - \lambda_\gamma F^\gamma{}_{\alpha\beta} - K_{\alpha\beta} = 0. \tag{2.2}$$

The $P^{\gamma\delta}{}_{\alpha\beta}$ define the product π in the algebra of forms:

$$\theta^\alpha \theta^\beta = P^{\alpha\beta}{}_{\gamma\delta} \theta^\gamma \theta^\delta. \tag{2.3}$$

This product is defined to be the one with the least relations which is consistent with the module structure of the 1-forms. The $F^\gamma{}_{\alpha\beta}$ are related to the 2-form $d\theta^\alpha$ through the structure equations:

$$d\theta^\alpha = -\frac{1}{2} C^\alpha{}_{\beta\gamma} \theta^\beta \theta^\gamma.$$

In the noncommutative case the structure elements are defined as

$$C^\alpha{}_{\beta\gamma} = F^\alpha{}_{\beta\gamma} - 2\lambda_\delta P^{(\alpha\delta)}{}_{\beta\gamma}. \tag{2.4}$$

It follows that

$$e_\alpha C^\alpha{}_{\beta\gamma} = 0. \tag{2.5}$$

This must be imposed then at the classical level and can be used as a gauge-fixing condition.

Finally, to complete the definition of the coefficients of the consistency condition (2.2) we introduce the special 1-form $\theta = -\lambda_\alpha \theta^\alpha$. In the commutative, flat limit

$$\theta \to i\partial_\alpha dx^\alpha.$$

It is referred to as a 'Dirac operator' [1]. As an (antihermitian) 1-form θ defines a covariant derivative on an associated $\mathcal{A}$-module with local gauge transformations given by the unitary elements of $\mathcal{A}$. The $K_{\alpha\beta}$ are related to the curvature of θ:

$$d\theta + \theta^2 = -\frac{1}{2} K_{\alpha\beta}\theta^\alpha\theta^\beta.$$

All the coefficients lie in the center $\mathcal{Z}(\mathcal{A}_{\bar{k}})$ of the algebra. With no restriction of generality we can impose the conditions

$$C^\epsilon{}_{\gamma\delta} = P^{\alpha\beta}{}_{\gamma\delta} C^\epsilon{}_{\alpha\beta}, \qquad K_{\gamma\delta} = P^{\alpha\beta}{}_{\gamma\delta} K_{\alpha\beta}. \tag{2.6}$$

Equation (2.4) is the correspondence principle which associates a differential calculus to a metric. On the left in fact the quantity $C^\alpha{}_{\beta\gamma}$ determines a moving frame, which in turn fixes a metric; on the right are the elements of the algebra which fix to a large extent the differential calculus. A 'blurring' of a geometry proceeds via this correspondence. It is evident that in the presence of curvature the 1-forms cease to anticommute. On the other hand it is possible for flat 'space' to be described by 'coordinates' which do not commute. The correspondence principle between the classical and noncommutative geometries can be also described as the map

$$\tilde{\theta}^\alpha \mapsto \theta^\alpha \tag{2.7}$$

with the product satisfying the condition

$$\tilde{\theta}^\alpha \tilde{\theta}^\beta \mapsto P^{\alpha\beta}{}_{\gamma\delta}\theta^\gamma\theta^\delta$$

The tilde on the left is to indicate that it is the classical form. The condition can be written also as

$$\tilde{C}^\alpha{}_{\beta\gamma} \mapsto C^\alpha{}_{\eta\zeta} P^{\eta\zeta}{}_{\beta\gamma}$$

or as

$$\lim_{\bar{k}\to 0} C^\alpha{}_{\beta\gamma} = \tilde{C}^\alpha{}_{\beta\gamma}.$$

A solution to these equations would be a solution to the problem we have set. It would be however unsatisfactory in that no smoothness condition has been imposed. This can at best be done using the inner derivations. We shall construct therefore the set of momentum generators. The procedure we shall follow is not always valid; a counter example has been constructed [15] for the flat metric on the torus.

We write $P^{\alpha\beta}{}_{\gamma\delta}$ in the form

$$P^{\alpha\beta}{}_{\gamma\delta} = \frac{1}{2}\delta^{[\alpha}_{\gamma}\delta^{\beta]}_{\delta} + i\hbar Q^{\alpha\beta}{}_{\gamma\delta}\,. \tag{2.8}$$

It will be convenient also to introduce the notation

$$G_{\alpha\beta} = K_{\alpha\beta} + \lambda_\gamma G^\gamma{}_{\alpha\beta}, \qquad G^\gamma{}_{\alpha\beta} = F^\gamma{}_{\alpha\beta} - 2i\hbar\lambda_\delta Q^{\gamma\delta}{}_{\alpha\beta} \tag{2.9}$$

so we can write (2.2) in the form

$$[\lambda_\alpha, \lambda_\beta] = G_{\alpha\beta} - i\hbar[\lambda_\gamma, \lambda_\delta]Q^{\gamma\delta}{}_{\alpha\beta}.$$

If we replace the commutator on the right-hand side by its value we obtain the equation

$$[\lambda_\alpha, \lambda_\beta] = G_{\alpha\beta} - i\hbar Q^{\gamma\delta}{}_{\alpha\beta}(K_{\gamma\delta} + G_{\gamma\delta}) + (i\hbar)^2 Q^{\eta\zeta}{}_{\alpha\beta}Q^{\gamma\delta}{}_{\eta\zeta}[\lambda_\gamma, \lambda_\delta].$$

This can be simplified by a redefinition

$$K_{\alpha\beta} - i\hbar Q^{\gamma\delta}{}_{\alpha\beta}K_{\gamma\delta} \mapsto K_{\alpha\beta}, \qquad F^\epsilon{}_{\alpha\beta} - i\hbar Q^{\gamma\delta}{}_{\alpha\beta}F^\epsilon{}_{\gamma\delta} \mapsto F^\epsilon{}_{\alpha\beta}$$

of the coefficients $K_{\alpha\beta}$ and $F^\gamma{}_{\alpha\beta}$. Iterating this proceedure one obtains the consistency condition in the form

$$[\lambda_\alpha, \lambda_\beta] = G_{\alpha\beta} \tag{2.10}$$

as well as the condition

$$Q^{[\gamma\delta]}{}_{\alpha\beta} = 0$$

on the coefficients. The condition that $P^{\gamma\delta}{}_{\alpha\beta}$ be an idempotent now implies a similar condition on $i\hbar Q^{(\gamma\delta)}{}_{(\alpha\beta)}$. Unless we are in a nonperturbative regime we can set then the latter equal to zero.

It is convenient to introduce the derivations

$$e_{\alpha\beta} = -2i\hbar Q^{\gamma\delta}{}_{\alpha\beta}\mathrm{ad}\,(\lambda_\gamma\lambda_\delta)$$

so that one can write the identity

$$[e_\alpha, e_\beta] = F^\gamma{}_{\alpha\beta} e_\gamma + e_{\alpha\beta}, \tag{2.11}$$

a condition which is again equivalent to vanishing torsion. The commutative limits of the extra derivations are calculated using the identity

$$e_{\alpha\beta} = -2i\bar{k}(2Q^{\gamma\delta}{}_{\alpha\beta}\lambda_\gamma e_\delta - Q^{\gamma\delta}{}_{\alpha\beta} e_\gamma e_\delta).$$

The first term on the right remains in the commutative limit as a vector field; the second term vanishes. The second term resembles a second derivative, but as a derivation. It is easy to see here the relation between on expression from noncommutative geometry and an effective commutative limit with higher-order derivative corrections.

The moving frame is dual to a set of derivations $\tilde{e}_\alpha$ which satisfy the commutation relations

$$[\tilde{e}_\alpha, \tilde{e}_\beta] = \tilde{C}^\gamma{}_{\alpha\beta} \tilde{e}_\gamma$$

If we use the correspondence principle and the expression (2.4) we find that this becomes

$$[e_\alpha, e_\beta] = F^\gamma{}_{\alpha\beta} e_\gamma - 4i\bar{k} Q^{\gamma\delta}{}_{\alpha\beta} \lambda_\delta e_\gamma \tag{2.12}$$

Equations (2.11) and (2.12) are identical to within corrections of order $\bar{k}$. The right-hand side of the latter is not a derivation but when corrected to be so becomes the right-hand side of the former.

From (2.11) we deduce the equation

$$e_{[\alpha} e^\mu_{\beta]} = e^\mu_\gamma G^\gamma{}_{\alpha\beta}. \tag{2.13}$$

Because of the symmetries of the indices $G_{\alpha\beta}$ can be considered as a field strength for a gauge potential which takes its values in the formal Lie algebra associated to the group of unitary elements of the associative algebra $\mathcal{A}$. Although the algebra is generally of infinite dimension the relation (2.2) acts as a finiteness condition and one can consider the λ_α as generators of a Lie algebra with commutation relations given [13, 16] by the bracket (2.10). The Jacobi identities with three momenta can be written as

$$\sum_{(\alpha\beta\gamma)} C^\delta{}_{\alpha\beta} G_{\gamma\delta} = 0. \tag{2.14}$$

These must be verified in each example.

It is necessary [17] to introduce a flip operation

$$\sigma : \ \Omega^1(\mathcal{A}_k) \otimes \Omega^1(\mathcal{A}_k) \to \Omega^1(\mathcal{A}_k) \otimes \Omega^1(\mathcal{A}_k).$$

to define the reality condition and the Leibniz rules. If we write

$$S^{\alpha\beta}{}_{\gamma\delta} = \delta^\beta_\gamma \delta^\alpha_\delta + i\bar{k} T^{\alpha\beta}{}_{\gamma\delta}$$

we find that a choice [13] of connection which is torsion-free, and satisfies all Leibniz rules is given by

$$\omega^\alpha{}_\beta = \frac{1}{2} F^\alpha{}_{\gamma\beta}\theta^\gamma + i\bar{k}\lambda_\gamma T^{\alpha\gamma}{}_{\delta\beta}\theta^\delta. \tag{2.15}$$

We can compare this with the expression (2.4) for the structure elements. The relation

$$\pi \circ (1 + \sigma) = 0$$

must hold [13, 14] to assure that the torsion be a bilinear map; it is bilinear only when it vanishes. In terms of the coefficients $P^{\alpha\beta}{}_{\gamma\delta}$ it can be written in the form

$$T^{\alpha\beta}{}_{\eta\zeta} P^{\eta\zeta}{}_{\gamma\delta} + Q^{(\alpha\beta)}{}_{\gamma\delta} = 0.$$

The symmetric part $T^{(\alpha\beta)}{}_{(\gamma\delta)}$, which is here arbitrary, will be fixed by the condition that the connection be metric. It follows that

$$\omega^\alpha{}_{\eta\zeta} P^{\eta\zeta}{}_{\beta\gamma} = \frac{1}{2} C^\alpha{}_{\beta\gamma}. \tag{2.16}$$

This is the usual relation between the Ricci-rotation coefficients and the Levi-Civita connection. Using it one can deduce (2.4) from (2.15).

Using the vanishing-torsion condition one can write the expression for the commutator of the derivations as

$$[e_\alpha, e_\beta] - \omega^\gamma{}_{[\alpha\beta]} e_\gamma = -2i\hbar Q^{\gamma\delta}{}_{\alpha\beta}(e_\gamma e_\delta - \omega^\epsilon{}_{\gamma\delta} e_\epsilon). \qquad (2.17)$$

A second form of the correspondence principle is the map

$$\tilde{\omega}^\alpha{}_\beta \tilde{\theta} \mapsto \omega^\alpha{}_\beta \theta \qquad (2.18)$$

between the classical and noncommutative connection forms.

We shall suppose that $\mathcal{A}_\hbar$ has a metric, which we define as a bilinear map

$$g: \ \Omega^1(\mathcal{A}_\hbar) \otimes \Omega^1(\mathcal{A}_\hbar) \to \mathcal{A}_\hbar. \qquad (2.19)$$

In terms of the frame one can define the metric by the condition that

$$g(\theta^\alpha \otimes \theta^\beta) = g^{\alpha\beta}. \qquad (2.20)$$

The $g^{\alpha\beta}$ are taken to be deformed versions

$$g^{\alpha\beta} = \eta^{\alpha\beta} + i\hbar h^{\alpha\beta}$$

of the components of the standard euclidean or lorentzian metric $\eta^{\alpha\beta}$ on $\mathbb{R}^n$ which satisfy [18] the symmetry condition

$$P^{\alpha\beta}{}_{\gamma\delta} g^{\gamma\delta} = 0. \qquad (2.21)$$

Together with the reality condition

$$g^{\beta\alpha} + i\hbar T^{\alpha\beta}{}_{\gamma\delta} g^{\gamma\delta} = (g^{\beta\alpha})^*$$

this means that $h^{\alpha\beta} = 0$ and also that the flip satisfies the constraint

$$T^{\alpha\beta}{}_{\gamma\delta} \eta^{\gamma\delta} = 0.$$

We suppose then that the coefficients of the metric are the usual commutative ones. The various reality conditions [19, 18] imply also that

$$(Q^{\alpha\beta}{}_{\gamma\delta})^* = Q^{\alpha\beta}{}_{\gamma\delta} + o(i\hbar), \qquad (T^{\alpha\beta}{}_{\gamma\delta})^* = T^{\alpha\beta}{}_{\gamma\delta} + o(i\hbar).$$

The connection is compatible with the metric if

$$T^{\alpha\gamma}{}_{\delta\epsilon}g^{\epsilon\beta} + T^{\beta\gamma}{}_{\delta\epsilon}g^{\alpha\epsilon} + i\bar{k}T^{\beta\gamma}{}_{\epsilon\zeta}g^{\eta\zeta}T^{\alpha\epsilon}{}_{\delta\eta} = 0. \tag{2.22}$$

To first order this simplifies to the usual condition

$$T^{(\alpha\gamma}{}_{\delta}{}^{\beta)} = o(i\bar{k}). \tag{2.23}$$

The index was lifted here with the metric.

If we introduce formally the 'covariant derivative' $D_\alpha X^\beta$ of a 'vector' X^α by the formula

$$D_\alpha X^\beta = \omega^\beta{}_{\alpha\gamma}X^\gamma$$

and the covariant derivative $D_\alpha(X^\beta Y^\gamma)$ of the product of two such 'vectors' as

$$D_\alpha(X^\beta Y^\gamma) = D_\alpha X^\beta Y^\gamma + S^{\beta\delta}{}_{\alpha\epsilon}X^\epsilon D_\delta Y^\gamma,$$

since there is a 'flip' as the index on the derivation crosses the index on the first 'vector', then the condition (2.22) that the connection be metric can be written

$$D_\alpha g^{\beta\gamma} = 0$$

as usual. More satisfactory origins of the condition (2.22) can be given [20].

The 'position' commutation relations are of the form

$$[x^\mu, x^\nu] = i\bar{k}J^{\mu\nu}. \tag{2.24}$$

The right-hand side is not necessarily in the center. The system of operator equations we must solve consists of Equation (2.15), Equation (2.17), the metric compatibility equation (2.23) as well as the constraints on the coefficients given above. Using the same formal manipulations as above we can find the commutation relations for the position generators as solution to a set of differential conditions. From (2.24) we see that

$$i\bar{k}e_\gamma J^{\mu\nu} = [x^{[\mu}, e_\gamma^{\nu]}].$$

We choose the symplectic structure $\theta^{\alpha\beta}$ such that

$$\theta^{-1}_{\alpha\beta} = -i\bar{k}K_{\alpha\beta} + o(h^{\alpha\beta})$$

and such that

$$J^{\mu\nu} = e^{\mu}_{\alpha}e^{\nu}_{\beta}\theta^{\alpha\beta}. \tag{2.25}$$

This cannot be generally possible unless $\theta^{(\alpha\beta)} \neq 0$. It will allow us to define (formally) a divergence $D_{\mu}J^{\mu\nu}$ by setting

$$D_{\mu}J^{\mu\nu} = e^{\nu}_{\beta}D_{\alpha}\theta^{\alpha\beta} \tag{2.26}$$

and using the connection form (2.15). The condition

$$D_{\mu}J^{\mu\nu} = 0$$

establishes a link [21] between the symplectic structure and the connection. We cannot make any general statements concerning this condition in the present context but we shall see that the example we consider is such that it is satisfied.

It is tempting to suppose that to lowest order at least, in a semiclassical approximation, there is an analogue of Darboux's lemma and that it is always possible to choose generators which satisfy commutation relations of the form (2.24) with the right-hand in the center. However the example we shall examine in detail shows that this is not always the case. Having fixed the generators, the manifestations of curvature would be found then in the form of the frame. The two sets of generators x^{μ} and λ_{α} satisfy, under the assumptions we make, three sets of equations. The commutation relations (2.24) for the position generators x^{μ} and the associated Jacobi identities permit one definition of the algebra. The commutation relations for the momentum generators permit a second definition. The conjugacy relations assure that the two descriptions concern the same algebra. We shall analyze these identities later using the example to show that they have interesting non-trivial solutions.

The problem of gauge invariance and the algebra of observables is a touchy one on which we shall not dwell. It is obvious that not all of the elements of $\mathcal{A}$ are gauge invariant but not that all observables are gauge-invariant. A gauge transformation is a local rotation

$$\theta^\alpha \mapsto \theta'^\alpha = \Lambda^{-1}{}^\alpha_\beta \theta^\beta. \tag{2.27}$$

This formula is slightly misleading since we have written θ'^α as though it belonged to the module of 1-forms of the original calculus whereas in fact it is a frame for a different differential calculus. One should more properly use a direct-sum notation and consider Λ^α_β and its inverse as maps between the two terms; this would complicate considerably the formulae. The structure elements transform classically as

$$C'^\alpha{}_{\beta\gamma} = \Lambda^{-1}{}^\alpha_\delta C^\delta{}_{\epsilon\zeta}\Lambda^\epsilon_\beta \Lambda^\zeta_\gamma + \Lambda^{-1}{}^\alpha_\delta e_\epsilon \Lambda^\delta_{[\beta}\Lambda^\epsilon_{\gamma]}.$$

This is a familiar formula but it cannot be taken over as is to the noncommutative case; under the gauge transformation the calculus transforms also. This is apparent in the conditions (2.6). Within the context of the present calculi the nearest one can come is the transformation

$$C'^\alpha{}_{\beta\gamma} = \Lambda^{-1}{}^\alpha_\delta C^\delta{}_{\epsilon\zeta}\Lambda^\epsilon_\beta \Lambda^\zeta_\gamma + \delta C^\alpha{}_{\beta\gamma} \qquad \delta C^\alpha{}_{\beta\gamma} = -2e_\delta \Lambda^{-1}{}^\alpha_\epsilon \Lambda^\delta_\zeta \Lambda^\epsilon_\eta P'^{\zeta\eta}{}_{\beta\gamma}.$$

If P and P' are of the form considered here one can write

$$\begin{aligned} C'^\alpha{}_{\beta\gamma} = \Lambda^{-1}{}^\alpha_\delta C^\delta{}_{\epsilon\zeta}\Lambda^\epsilon_\beta \Lambda^\zeta_\gamma &- 2e_\delta \Lambda^{-1}{}^\alpha_\epsilon [\Lambda^\delta_\zeta, \Lambda^\epsilon_\eta] P'^{\zeta\eta}{}_{\beta\gamma} \\ &+ \Lambda^{-1}{}^\alpha_\delta e_\epsilon \Lambda^\delta_{[\beta}\Lambda^\epsilon_{\gamma]} + \Lambda^{-1}{}^\alpha_\delta e_\epsilon \Lambda^\delta_\zeta \Lambda^\epsilon_\eta P'^{(\zeta\eta)}{}_{\beta\gamma}. \end{aligned}$$

In the linear approximation, with $\Lambda^\alpha_\beta \simeq \delta^\alpha_\beta + \Lambda^{(1)}{}^\alpha_\beta$, the inhomogeneous term simplifies to the expression

$$-2e_\delta \Lambda^{-1}{}^\alpha_\epsilon \Lambda^\delta_\zeta \Lambda^\epsilon_\eta P'^{\zeta\eta}{}_{\beta\gamma} = e_{[\beta}\Lambda^{(1)}{}^\alpha_{\gamma]} + e_\delta \Lambda^{(1)}{}^\alpha_\epsilon P^{(\delta\epsilon)}{}_{\beta\gamma}. \tag{2.28}$$

Since

$$\theta'^\alpha(e_\beta) = \Lambda^{-1}{}^\alpha_\beta$$

in a certain formal way one can imagine that

$$e'_\alpha \sim \Lambda^\beta_\alpha e_\beta$$

except that the right-hand-side is not a derivation in general. It makes sense however to introduce $\bar\lambda'_\alpha = \Lambda^\beta_\alpha \lambda_\beta$ and to compare it with the elements λ'_β defined to be those which generate the derivations dual to θ'^α. This means that we must consider the identity

$$\Lambda^{-1}{}^\gamma_\beta \theta^\beta_\mu [\lambda'_\alpha, x^\mu] = \theta^\gamma_\mu [\Lambda^{-1}{}^\beta_\alpha \bar\lambda'_\beta, x^\mu]$$

as an equation for e'_α in terms of $\bar e'_\alpha = \mathrm{ad}\,\bar\lambda'_\alpha$ and solve. There results a rather complicated situation because of the change in Poisson structure which must accompany a gauge transformation in general. We shall attempt to elucidate this with the example.

Consider the expression (2.15) for the connection form. Under a change of basis (2.27) it transforms as a gauge potential. The condition that the transformed potential be also of the form (2.15) is a strong restriction on the matrix Λ^β_α. We shall attempt to elucidate this also with the example.

3 The classical plane wave

Plane-fronted gravitational waves with parallel rays (*pp*-waves) have a metric [22] which can be defined [5] by the moving frame $\tilde\theta^a = dx^a$, $x^a = (x, y)$ and

$$\tilde\theta^+ = \frac{1}{\sqrt{2}}(\tilde\theta^0 + \tilde\theta^3) = \tilde\Lambda^{-1}(du + \tilde h dv), \tilde\theta^- = \frac{1}{\sqrt{2}}(\tilde\theta^0 - \tilde\theta^3) = \tilde\Lambda dv. \quad (3.1)$$

The generators of $\mathcal{A}$ will be chosen $x^\mu = (u, v, x, y)$, with u and v null coordinates in the commutative limit. The $\tilde\Lambda$ is for the moment an arbitrary invertible element of $\mathcal{A}$. When $\tilde h = 0$ the space-time is Minkowski and the vector ∂_v is tangent to the bicharacteristics along the characteristic surfaces $u =$ constant. If $\tilde h = \tilde h(v)$ one can introduce an element $I\tilde h(v) \in \mathcal{A}$ such that $\partial_v I\tilde h(v) = \tilde h(v)$ and use it to define

$u' = u + I\tilde{h}(v)$. With $\tilde{\Lambda} = 1$ then one has $\tilde{\theta}^+ = du'$ and $\tilde{\theta}^- = dv$ The map defined by $(uv) \mapsto (u', v)$ does not respect the structure of $\mathcal{A}$ as an algebra but it does leave invariant the commutator: $[u', v] = [u, v]$. Consider the derivations e_α dual to the moving frame. In the commutative limit we have $e_1 = \partial_x$, $e_2 = \partial_y$ and

$$e_+ = \tilde{\Lambda}\partial_u, \qquad e_- = \tilde{\Lambda}^{-1}(\partial_v - \tilde{h}\partial_u).$$

The metric-compatible connection form is given by

$$\tilde{\omega}^\alpha{}_\beta = \begin{pmatrix} \tilde{\omega}^+{}_+ & 0 & \tilde{\omega}^+{}_b \\ 0 & \tilde{\omega}^-{}_- & 0 \\ 0 & \tilde{\omega}^a{}_- & 0 \end{pmatrix}$$

with

$$\tilde{\omega}^+{}_+ = \tilde{\Lambda}^{-1}\tilde{e}_+\tilde{h}\tilde{\Lambda}\tilde{\theta}^- + \tilde{\Lambda}^{-1}d\tilde{\Lambda}, \qquad \tilde{\omega}^+{}_a = \tilde{\Lambda}^{-1}\tilde{e}_a\tilde{h}\tilde{\theta}^-.$$

One of the expressions of the zero-torsion condition is the identity

$$\tilde{D}_{[\alpha}\tilde{D}_{\beta]}\tilde{x}^\mu = 0$$

which can also be written as

$$\tilde{e}_{[\alpha}\tilde{e}_{\beta]}\tilde{x}^\mu = \tilde{\omega}^\gamma{}_{[\alpha\beta]}\tilde{e}_\gamma x^\mu. \tag{3.2}$$

In writing this we have used the fact that the connection form commutes with the coordinates. This will no longer be true in general and extra terms can appear.

The is another natural frame which can be obtained by the following argument. Consider an arbitrary metric in null coordinates $x^\mu = (u, v, x^i)$. The hypersurfaces $u = u_0$ are assumed to be null and v to be the affine parameter along the corresponding bicharacteristics. The line element can be written in the form

$$ds^2 = 2dudv + g_{++}du^2 + 2g_{+i}dvdx^i + g_{ij}dx^i dx^j.$$

Since $(0,1,0,0)$ is a null vector we have $g_{--}=0$. We have also used the local coordinate freedom to set

$$g_{+-}=1, \qquad g_{-i}=0.$$

The remaining metric components are general functions of all coordinates. We now rescale

$$g_{\mu\nu}\mapsto \omega^{-2}g_{\mu\nu}, \qquad u\mapsto \omega^2 u, \qquad x^i\mapsto \omega x^i.$$

The line element becomes

$$ds^2 = 2dudv + \omega^2 g_{++}du^2 + 2\omega g_{+i}dudx^i + g_{ij}dx^i dx^j$$

with the metric components now functions of the form

$$g_{\mu\nu} = g_{\mu\nu}(\omega^2 u, v, \omega x^i).$$

As a singular limit when $\omega\to 0$ one optains [23, 24] the *pp*-wave with line element

$$ds^2 = 2dudv + g_{ij}(v)dx^i dx^j.$$

If the initial metric is described by a frame of the form

$$\theta^+ = du, \qquad \theta^- = dv + \theta^-_+ du + \theta^-_i dx^i, \qquad \theta^a = \theta^a_i dx^i$$

then the scaled metric is described by

$$\theta^+ = du, \qquad \theta^- = dv, \qquad \theta^a = \theta^a_i(v)dx^i. \tag{3.3}$$

We shall refer to (3.3) as the Rosen frame.

The dual frame is such that the only nontrivial commutation relations are given by

$$[e_-, e_a] = (e^{-1}\dot{e})^b{}_a e_b, \qquad \theta^a_i e^i_b = \delta^a_b. \tag{3.4}$$

The structure functions $C^\alpha{}_{\beta\gamma}$ are seen to be with

$$C^a{}_{-b} = -C^a{}_{b-} = (e^{-1}\dot{e})^b{}_a.$$

the only non-vanishing components.

The transformation from the Rosen frame to the previous frame is made using the transformation of coordinates

$$u = \tilde{u} - \frac{1}{2}A_{ab}(\tilde{v})\tilde{x}^a\tilde{x}^b, \qquad v = \tilde{v}, \qquad x^i = e^i_a(\tilde{v})\tilde{x}^a.$$

In the old coordinates the Rosen frame is of the form $\theta^\alpha = \Lambda^\alpha{}_\beta\tilde{\theta}^\beta$ with

$$\Lambda^\alpha{}_\beta = \begin{pmatrix} 1 & 0 & -F_{bc}\tilde{x}^c \\ 0 & 1 & 0 \\ 0 & (e^{-1}\dot{e})^a{}_c\tilde{x}^c & 1 \end{pmatrix}$$

The matrix $\Lambda^\alpha{}_\beta$ is a local Lorenz transformation if

$$g^{+a} = \Lambda^+{}_\alpha\Lambda^a{}_\beta g^{\alpha\beta} = \Lambda^a{}_- + \Lambda^+{}_b g^{ab} = (e^{-1}\dot{e} - F)^a{}_b\tilde{x}^c = 0.$$

That is, if

$$F = e^{-1}\dot{e}$$

Here the dot denotes derivative with respect to v. It should be noted that, because of the special form of h one obtains, this applies to the special case of plane waves.

The matrix $F^a{}_b(v)$ is an arbitrary real element of the algebra M_2 of complex 2×2 matrices, which we write

$$F = S + A$$

as the sum of a symmetric part and an antisymmetric part. The latter defines a rotation in the plane transverse to the null coordinates. The non-vanishing components of the connection form are given by

$$\omega^a{}_+ = -S^a{}_b\theta^b, \qquad \omega^-{}_a = S_{ab}\theta^b, \qquad \omega^a{}_b = A^a{}_b\theta^+.$$

The curvature 2-form has

$$\Omega^a{}_+ = -(\dot{S}^a{}_b + [A,S]^a{}_b)\theta^+\theta^b, \qquad \Omega^a{}_b = 0$$

as non-vanishing components. The non-vanishing components of the Riemann tensor are given by

$$R^a{}_{++b} = -(\dot{S} + [A,S])^a{}_b$$

as well as those obtained from symmetries. The only non-vanishing component of the Ricci tensor is

$$R_{++} = -\mathrm{Tr}(\dot{S} + [A, S]).$$

So the field equations reduce to the condition that the trace of a matrix be constant. The solution has three arbitrary functions; two dynamical ones in S and a gauge degree of freedom in A.

4 The PP algebra

We shall consider the two classical forms mentioned in the previous section now in succession. The commutator (3.2) is generally written as

$$\tilde{e}_{[\alpha}\tilde{e}_{\beta]} = \tilde{C}^\gamma{}_{\alpha\beta}\tilde{e}_\gamma$$

The noncommutative generalization will involve essentially finding an equivalent of this equation. It will be a commutation relation of the form (2.17) and better expressed in terms of the momentum generators. This will fix the algebra as well the calculus. That is, the correspondence principle which we shall actually use is a modified version of the map

$$\tilde{e}_\alpha \mapsto \lambda_\alpha$$

which is, of course, that introduced by Bohr. The commutation relations for the $\tilde{e}_\alpha$ must be 'lifted' to commutation relations between the λ_α. For this we notice that the L^2-completion of the algebra of functions on the space-time can be identified with $\mathcal{H} = L^2(\tau, \xi, \eta, \zeta)$. We define then for smooth $\psi \in \mathcal{H}$

$$(\lambda_\alpha \psi)(\tau, \xi, \eta, \zeta) = \tilde{e}_\alpha \psi(\tau, \xi, \eta, \zeta). \tag{4.1}$$

Once we have implemented the correspondence principle then this set of momentum generators will satisfy the consistency Equation (2.2), however with $K_{\alpha\beta} = 0$. To obtain a non-zero value for the central extension one must modify the representation. We do this in our analysis of another metric [25]; here we restrict our considerations to formal algebras.

According to the correspondence principle as expressed by (2.7) we introduce a frame θ^α which has the classical frame as a limit and is as 'near' to it in form as possible for all values of $\hbar$. We shall assume in this example that the noncommutative frame has the same functional form as its limit except for a possible conformal factor in the u-direction. When $h = 0$, we have noticed, the space-time is flat; the noncommutative version is also flat. We introduce the frame $dx^\mu = e^\mu_\alpha \theta^\alpha$ with

$$e^\mu_\alpha = \begin{pmatrix} 1 & -h & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

This defines also the differential of the generators of the algebra in terms of the basis of the module of 1-forms. It defines therefore the conjugacy relations

$$\begin{aligned} &[\lambda_+, u] = 1, &&[\lambda_+, v] = 0, \\ &[\lambda_-, u] = -h, &&[\lambda_-, v] = 1, \end{aligned} \tag{4.2}$$

as well as

$$[\lambda_a, x^b] = \delta^b_a.$$

From (4.2) and the reality conditions it follows that

$$h^* = h.$$

The derivations satisfy the commutation relations

$$\begin{aligned} &[\tilde{e}_+, \tilde{e}_-] = -\tilde{\partial}_+ \tilde{h} \tilde{e}_+, &&[\tilde{e}_-, \tilde{e}_a] = \tilde{\partial}_a \tilde{h} \tilde{e}_+, \\ &[\tilde{e}_+, \tilde{e}_a] = 0, &&[\tilde{e}_a, \tilde{e}_b] = 0. \end{aligned}$$

The frame is the noncommutative form of (3.1) which we write in the same way as

$$\theta^+ = \frac{1}{\sqrt{2}}(\theta^0 + \theta^3) = (du + hdv), \qquad \theta^- = \frac{1}{\sqrt{2}}(\theta^0 - \theta^3) = dv,$$
$$\theta^a = dx^a. \tag{4.3}$$

The problem is to find an algebra with a set of 4 generators x^μ satisfying commutation rules as well as a Fourier transformation to a new set λ_α which satisfies the conditions (2.10) and (4.2). We use the second form of the correspondence principle, the map (2.18) and the metric-compatible connection form $\omega^\alpha{}_\beta$. In the classical case the geometry is completely determined (to within integration constants) by (3.2). We seek an analogous equation in the noncommutative case. This means that we must rewrite the noncommutative version of (3.2) in terms of the momenta and compare it with the Equation (2.10).

We start from Equation (2.10) and impose the condition that the torsion vanish, which leads to the relation (2.11). According to the correspondence principle the form of the $C^\alpha{}_{\beta\gamma}$ as functionals of h are determined in the classical limit. A stronger condition is to assume that they have the same functional form when $\kbar \neq 0$. The non-vanishing structure elements are given then by

$$C^+{}_{\alpha\beta} = e_{[\alpha} h \delta^-_{\beta]} - 2i\kbar e_\gamma h Q^{\gamma -}{}_{\alpha\beta}, \tag{4.4}$$

The linear-connection coefficients are defined in terms of the structure elements by the formula (2.16), which can be written in the form

$$\omega^\alpha{}_{[\beta\gamma]} = C^\alpha{}_{\beta\gamma} - i\kbar \omega^\alpha{}_{(\eta\zeta)} Q^{\eta\zeta}{}_{\beta\gamma}.$$

The cases which we shall consider are such however that the second term vanishes and so we have the usual formula for the Levi-Civita connection. The associated covariant derivative is defined by the actions $D\theta^\alpha = -\omega^\alpha{}_\beta \theta^\beta$ with the right-hand side defined using (2.15).

¿From the quantization condition we deduce that the expression for $C^\alpha{}_{\beta\gamma}$ as given by (4.4) must be identical to that given in the

Equations (2.4):

$$F^{\alpha}{}_{\beta\gamma} - 4i\bar{k}\lambda_\delta Q^{\alpha\delta}{}_{\beta\gamma} = -2e_\delta h P^{\delta-}{}_{\beta\gamma}\delta^{\alpha}_{+}.$$

The Jacobi identities (2.13) with two momentum and one position operator yield the constraints

$$[G_{\alpha\beta}, u] = C^{+}{}_{\alpha\beta} - hC^{-}{}_{\alpha\beta},$$

$$[G_{\alpha\beta}, v] = 0,$$

$$[G_{\alpha\beta}, x^a] = 0.$$

¿From these we conclude that

$$G_{\alpha\beta} = G_{\alpha\beta}(\lambda_+).$$

We must impose the 'Bohr-Ehrenfest' condition

$$\lim_{\bar{k}\to 0} h = \tilde{h} \tag{4.5}$$

to fix the classical limit. Equations (2.10) should be read as an equation for h in terms of the commutator $[\lambda_a, \lambda_+]$ if one believes that the noncommutative structure determines the gravitational field. If they are read from right to left then they are equations for the structure of the algebra in terms of the gravitational field.

To obtain a general solution to this problem one must consider in greater detail representations like (4.1) but with a central extension. Here we shall consider a simple case where a formal algebra is sufficient and where the 'Fourier transformation' is local, as it is in the flat case. We shall use the notation

$$z = i\bar{k}\mu\lambda_+, \qquad \bar{z} = i\bar{k}\mu\lambda_-, \qquad f_\alpha = i\bar{k}\mu K_{\alpha+}, \qquad \bar{f}_\alpha = i\bar{k}\mu K_{\alpha-}.$$

We have then

$$f_- = -\bar{f}_+, \qquad g^{ab} f_a \bar{f}_b = 0.$$

We must choose the momenta such that the 'geometric' expression (4.4) for the structure elements can be equated to the 'algebraic expression (2.4). The second term on the right of Equation (4.4) vanishes.

This yields the equations

$$F^{+}{}_{+-} - 4i\bar{k}\lambda_\delta Q^{\delta+}{}_{+-} = -e_+ h, \tag{4.6}$$

$$F^{+}{}_{+a} - 4i\bar{k}\lambda_\delta Q^{\delta+}{}_{+a} = 0, \tag{4.7}$$

$$F^{+}{}_{-a} - 4i\bar{k}\lambda_\delta Q^{\delta+}{}_{-a} = e_a h, \tag{4.8}$$

$$F^{+}{}_{ab} - 4i\bar{k}\lambda_\delta Q^{\delta+}{}_{ab} = 0. \tag{4.9}$$

We can choose

$$F^{\alpha}{}_{+a} = 0, \qquad F^{\alpha}{}_{ab} = 0, \qquad Q^{\alpha\beta}{}_{+a} = 0, \qquad Q^{\alpha\beta}{}_{ab} = 0$$

to satisfy (4.7) and (4.9). To obtain a simple solution to the remaining two equations we can suppose $h = h(z)$ to be a functional of the generator z. For example, if we choose also

$$F^{\alpha}{}_{+-} = 0, \qquad Q^{\alpha\beta}{}_{+-} = 0$$

then to solve (4.6) we can set

$$h = \alpha z + \frac{1}{2} z^2.$$

The μ is a mass scale introduced to keep the physical dimensions correct and α is a dimensionless parameter. Equation (4.8) is also satisfied provided one chooses

$$F^{+}{}_{-a} = \alpha f_a, \qquad Q^{++}{}_{a-} = \frac{1}{4}\mu f_a.$$

The frame is calculated from the expressions (4.3). To find h as a function of the position generators we must solve the conjugacy relations (4.2). The only expression of interest is given by

$$\lambda_+ = -K_{+a} x^a, \qquad z = i\bar{k}\mu\lambda_+ = J^{-1}_{+a} x^a$$

which one deduces from the commutation relations

$$[\lambda_+, u] = -K_{+a}[x^a, u] = -i\bar{k} K_{+a} J^{a+} = 1.$$

The expression for h in terms of the position generators is therefore

$$h = h_a x^a + \frac{1}{2} h_{ab} x^a x^b$$

with

$$h_a = \alpha\theta^{-1}_{+a}, \qquad h_{ab} = h_a h_b.$$

The anticommutation relations which determine the structure of the algebra of forms is the usual one except for the modified relation

$$(\theta^+)^2 = \frac{1}{4} i\bar{k}\mu f_a \theta^a \theta^-. \tag{4.10}$$

The standard Minkowski metric remains hermitian and symmetric. The relations which determine the structure of the algebra can be written as

$$[\lambda_+, \lambda_-] = 0, \qquad [\lambda_+, \lambda_a] = K_{+a},$$
$$[\lambda_-, \lambda_a] = K_{+a}, \quad [\lambda_a, \lambda_b] = 0.$$

In terms of the position generators this becomes

$$[x^a, x^b] = 0, \qquad [x^a, u] = i\bar{k}\theta^{a+} + \theta^{a-}h,$$
$$[v, x^b] = i\bar{k}\theta^{-b}, \quad [u, v] = 0.$$

In the ‘quasi-commutative’ limit the line element is given by

$$ds^2 = 2dudv + 2hdv^2 - dx^2 - dy^2.$$

In the Rosen frame it is of the form

$$ds^2 = 2dudv - (\theta^1)^2 - (\theta^2)^2.$$

This form of the metric has suscitated some interest in the string-theory community [24]. It has a certain similarity to the Kasner metric, which has also [25] a noncommutative extension.

5 Curvature

Although not a satisfactory object, we define the curvature exactly as in the commutative case, as the left-linear map $\mathrm{Curv} = -D^2$ from $\Omega^1(\mathcal{A})$ into $\Omega^2(\mathcal{A}) \otimes \Omega^1(\mathcal{A})$, with associated Ricci map from $\Omega^1(\mathcal{A})$ into $\Omega^1(\mathcal{A})$ given by

$$\mathrm{Ric}(\theta^+) = -\Delta h \theta^-, \qquad \mathrm{Ric}(\theta^-) = 0, \qquad \mathrm{Ric}(\theta^a) = 0.$$

The Laplace operator

$$\Delta h = g^{ab} e_a e_b h$$

is with respect to the induced metric on the wave fronts. The only non-vanishing component of the Ricci tensor is therefore

$$R^+{}_- = -\Delta h = -g^{ab} e_a e_b h$$

An expression for the function h in terms of λ_+ was given for which

$$R^+{}_- = -\frac{1}{4}\mu g^{ab} f_a f_b.$$

It follows that the metric is Ricci-flat only if $f_a = 0$ in which case it is also flat.

6 Embeddings

In this section we would like to consider the PP-wave metrics as perturbations of flat metrics and also consider them as noncommutative embeddedings [14, 26]. To distinguish them we shall place a prime on perturbed quantities. Suppose that $(x^\pm, \lambda_a)$ commute and consider the case

$$\phi = 1, \qquad \Lambda = 1.$$

The commuting generators can be considered as the coordinates of an $\mathbb{R}^4$. If $h = 0$ one can set $\lambda_a = -K_{a\pm} x^\pm$ to define an embedding $\mathbb{R}^2 \hookrightarrow \mathbb{R}^4$. Consider now the 8-dimensional phase-space associated to the $\mathbb{R}^4$ obtained by adding 4 extra variables $(x^a, \lambda_\pm)$ and 'quantize' this space by imposing the commutation relations

$$[\lambda_\alpha, x^\mu] = i\bar{k}\delta^\mu_\alpha$$

We write the corresponding relations Equation (4.2) in the form

$$[\lambda'_\alpha, x'^\mu] = i\bar{k} e'^\mu_\alpha$$

and we consider the latter as a perturbation

$$e'^\mu_\alpha = \delta^\mu_\alpha + i\bar{k} f^\mu_\alpha$$

of the former. If we perturb

$$x'^{\mu} = x^{\mu} + i\bar{k}\xi^{\mu}, \qquad \lambda'_{\alpha} = \lambda_{\alpha} + i\bar{k}l_{\alpha}$$

both the position and the momentum generators we see that the relation

$$f^{\mu}_{\alpha} = [\lambda_{\alpha}, \xi^{\mu}] + [l_{\alpha}, x^{\mu}]$$

holds. The induced perturbation of the frame is given by (2.27) with

$$\Lambda^{\alpha}_{\beta} \simeq \delta^{\alpha}_{\beta} - i\bar{k}f^{\alpha}_{\beta}$$

no longer necessarily a gauge transformation.

If λ_{α} and its perturbed equivalent are to generate differential calculi then they both must satisfy (2.2) with in general different coefficients. We set

$$P'^{\alpha\beta}{}_{\gamma\delta} = P^{\alpha\beta}{}_{\gamma\delta} + i\bar{k}Q^{\alpha\beta}{}_{\gamma\delta}\,,$$

$$F'^{\alpha}{}_{\gamma\delta} = F^{\alpha}{}_{\gamma\delta} + i\bar{k}\phi^{\alpha}{}_{\gamma\delta}, \qquad K'_{\gamma\delta} = K_{\gamma\delta} + i\bar{k}\kappa_{\gamma\delta}$$

and we shall restrict our attention to the special case

$$P^{\alpha\beta}{}_{\gamma\delta} = \frac{1}{2}\delta^{[\alpha}_{\gamma}\delta^{\beta]}_{\delta}, \qquad F^{\alpha}{}_{\gamma\delta} = 0$$

which is a perturbation of flat space. Comparing then the two sets of equations we find the relation

$$\frac{1}{2}e_{[\alpha}l_{\beta]} = \phi^{\gamma}{}_{\alpha\beta}\lambda_{\gamma} + \kappa_{\alpha\beta} - 2i\bar{k}Q^{\gamma\delta}{}_{\alpha\beta}l_{\gamma}\lambda_{\delta} + Q^{\gamma\delta}{}_{\alpha\beta}e_{\delta}l_{\gamma}$$

If the metric is a PP-wave then

$$i\bar{k}f^{\mu}_{\alpha} = \begin{pmatrix} 0 & -h & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The embedding ‘space’ described by an algebra $\mathcal{B}$ has a natural differential calculus with a module of 1-forms of rank 8 freely generated by the elements dy^i, with

$$y^i = (\theta^{\mu\alpha}\lambda_{\alpha}, x^{\nu})$$

A basis of the 1-forms of the embedded 'space' $\mathcal{A}$ would have four elements θ^α and would have to be in some sense a restriction of the 1-forms over $\mathcal{B}$. Associated to the inclusion of $\mathcal{A}$ in $\mathcal{B}$ there is an inclusion map

$$\iota^* : \Omega^1(\mathcal{A}) \hookrightarrow \Omega^1(\mathcal{B})$$

between the respective module of 1-forms. If θ^α is a frame and f an element of $\mathcal{A}$ then we must have the relation

$$0 = \iota^*(f\theta^\alpha - \theta^\alpha f) = [\iota^* f, \theta^\alpha_i] dy^i.$$

This implies then the relations

$$[\iota^* f, \theta^\alpha_i] = 0.$$

There are $2n^2$ unknowns and an equal number of relations.

We started with four generators λ_α of an algebra $\mathcal{A}$ and an unknown functional h of a second set of four generators x^μ of the same algebra. Consider the subalgebra $\mathcal{B}$ generated by the x^a. With the choice we have made for the form of the matrix $K_{\alpha\beta}$ we can identify $\mathcal{B}$ as a function algebra and therefore by the commutation relations (4.2) we can conclude that the $\lambda_\pm$ are functions of the x^a. Likewise we identify the λ_a as functions of the $x^\pm$. The four functions constitute an embedding of $\mathcal{A}$ as a subalgebra of a 'flat' algebra $\mathcal{B}$ with twice the generators. Because of the simple structure of the problem, two of these functions are just linear transformations.

To find an expression for h as a function of the position generators we must determine the 'Fourier' transform; we must in particular find $\lambda_+(x^\mu)$. This is determined by the conjugacy relations (4.2). We have given h as a polynomial in one momentum generator; to find its expression in terms of position generators we must formally solve Equations (4.2). We start with an Ansatz of the form of a perturbation

$$\lambda_\alpha = -K_{\alpha\mu} x^\mu + l_\alpha$$

for the ‘Fourier transform’ F of Section 2. We know that in this case

$$\lim_{h\to 0} l_\alpha = 0.$$

We find then from Equations (4.2) that

$$[l_+, x^\mu] = 0, \qquad [l_-, x^\mu] = -h\delta^\mu_+, \qquad [l_a, x^\mu] = 0$$

As a solution we shall choose $l_+ = 0$, $l_a = 0$ and

$$l_- = -\frac{1}{2} i\hbar\mu(\alpha + \frac{2}{3} i\hbar\mu\beta\lambda_+)\lambda_+^2.$$

Generically the transformation from the coordinate generators to the momentum generators is nonlocal.

7 Discussion

It would seem that it is possible to find a noncommutative geometry, of Minkowski signature in ‘dimension’ four. The example we have given here is not a solution to the Einstein field equations. We have started with a given classical metric, admittedly rather degenerate, and constructed an algebra and a differential calculus which supports the metric. The algebra is just that of noncommutative $\mathbb{R}^4$; the differential calculus differs in that the usual relation $(\theta^+)^2 = 0$ is replaced by (4.10) with f_a determined by the commutation relations of the algebra.

Our calculations have been all carried out on the level of formal algebra. The real problem lies in fact in finding a representation of the quadratic algebra (2.17). We have given a certain number of necessary conditions for such a representation to exist; they are not necessarily sufficient. To obtain a concrete solution we must find a representation in terms of operators on a Hilbert space. To each representation corresponds a trace and therefore an action.

The Riemann tensor is of null type. Its relation with $J^{\alpha\beta}$ would be expressed in terms of the action of $J^{\alpha\beta}$ on the principle null direction.

Let l_α be this vector. We must compare $g^{\alpha\beta}l_\beta$ with $J^{\alpha\beta}l_\beta$. In the limit $\hbar \to 0$ we recuperate the commutative plane-wave solution plus in addition a symplectic structure $J^{\alpha\beta}$. The relation of J to the Riemann tensor in general is a subject of interest which we leave also to a future publication.

Appendix I

For completeness we give here a brief review of plane-fronted metrics using the 'principal null vector' approach, a formalism which is a bit more elaborate than is essential but which explicitly identifies, from the beginning, the important vector $l = \tilde{e}_+$, with components k^α. In special cases l can be identified as a principal null vector of the Riemann or Weyl (conformal) curvature tensors. We shall consider n-metrics since to do so is no more complicated than 4-metrics ; when $n = 4$ these metrics are included in the Kerr-Schild class of metrics [22]. We are considering only classical commutative geometry here so we can omit, without ambiguity, the tilde notation used in previous discussions of classical metrics.

The line element is given in the frame formalism by

$$ds^2 = \theta^- \otimes \theta^+ + \theta^+ \otimes \theta^- + \eta_{ab}\theta^a \otimes \theta^b, \tag{7.1}$$

The class of metrics to be considered will be called *plane-fronted*, and can be defined in local coordinates x^α by a frame

$$\theta^\alpha = (\delta^\alpha_\beta + hk^\alpha k_\beta)dx^\beta, \tag{7.2}$$

where the k^α are constants given by

$$k^\alpha = \delta^\alpha_+, \tag{7.3}$$

so that $\partial_\beta(k^\alpha) = 0$. Furthermore $\eta_{\alpha\beta}k^\alpha k^\beta = 0$, $\eta_{\alpha\beta}k^\beta = k_\alpha = \delta^-_\alpha$, where, in the natural coordinate basis $(x^+ = u, x^- = v, x^a)$, the flat metric η takes the form

$$\eta_{\alpha\beta}dx^\alpha \otimes dx^\beta = 2duddv + \eta_{ab}dx^a dx^b,$$

and h is a function. Hence the line element (7.1) of a *plane-fronted metric* g is given in Kerr-Schild form by

$$ds^2 = 2du \otimes dv + \eta_{ab} dx^a \otimes dx^a + 2h(dv)^2.$$

Note that the vector field $k^\alpha e_\alpha = e_+$ is a null vector with respect to both η and g.

The Levi-Civita connection 1-forms, in components with respect to the above bases $(e_\alpha, \theta^\alpha)$ are given by

$$\omega^\alpha{}_\beta = (k^\alpha \partial_\beta h - \partial^\alpha h k_\beta) k_\gamma dx^\gamma = (k^\alpha \partial_\beta h - \partial^\alpha h k_\beta)\theta^-.$$

The components of the curvature 2-forms, relative to the same bases, are given by

$$R^\alpha{}_{\beta\gamma\delta} = k_\gamma k_\beta \partial_\delta \partial^\alpha h - k_\delta k_\beta \partial_\gamma \partial^\alpha h + k^\alpha k_\delta \partial_\beta \partial_\gamma h - k^\alpha k_\gamma \partial_\beta \partial_\delta h.$$

Consequently

$$R^\alpha{}_{\beta\gamma\delta} k^\delta = k_\gamma k_\beta \partial_\delta h^\alpha k^\delta - k^\alpha k_\gamma \partial_\beta \partial_\delta h k^\delta,$$

and the Ricci tensor has components

$$R_{\beta\delta} = R^\alpha{}_{\beta\alpha\delta} = k_\alpha k_\beta \partial_\delta \partial^\alpha h - k_\delta k_\beta \partial_\alpha \partial^\alpha h + k^\alpha k_\delta \partial_\beta \partial_\alpha h.$$

Hence

$$R_{\beta\delta} k^\delta = 0 \quad \text{iff} \quad \partial_\beta \partial_\delta h k^\delta k^\beta = 0.$$

The results above can be used to prove the following theorems which illustrate some of the properties of the class of metrics being considered here [22, 27, 28, 29, 30].

Theorem 7.1 *Let D_α denote the Levi-Civita covariant derivative (components with respect to the moving frame). Then*

1. *The following equality holds*

$$D_\alpha k_\beta = k_\alpha k_\beta \partial_\gamma h k^\gamma. \tag{7.4}$$

2. $k^\alpha D_a k_\beta = 0$, *and* $k^\alpha e_\alpha$ *is tangent to affinely parametrized null geodesics.*

3. $D_{[\alpha} k_{\beta]} = D_\alpha k^\alpha = 0$; $\quad D_\alpha k_\beta = D_{(\alpha} k_{\beta)}$ *and* $D_\alpha k_\beta D^\alpha k^\beta = 0$.

This justifies the teminology 'plane-fronted', since these conditions applied to vacuum Lorentzian 4-metrics of the form considered here, define the so-called *plane-fronted metrics*

Theorem 7.2 $D_a k_\beta = 0$ *if and only if*

$$\partial_+ h = \partial_\beta h k^\beta = 0. \tag{7.5}$$

Once again using the previously mentioned Lorentzian 4-metric terminology, when this equation is satisfied, the plane fronted metrics are said to be *plane-fronted with parallel rays (pp waves).*

Theorem 7.3 *Consider* plane-fronted *metrics. Two useful results are the following.*

1. $R^\alpha{}_{\beta\gamma\delta} k^\delta = 0$ *if and only if*

$$\partial_\alpha \partial_\beta h k^\beta = F k_\alpha, \tag{7.6}$$

for some function F, and then the Ricci tensor must take the form

$$R_{\alpha\beta} = k_\alpha k_\beta (2F - \partial_\alpha h^\alpha). \tag{7.7}$$

2. *Let* $Q_{\alpha\beta} = k_\alpha q_\beta - k_\beta q_\alpha$ *where* $k^\alpha q_\alpha = 0$ *and the* q_α *are also constants. That is,* $\partial_\beta q_\alpha = 0$. *Then* $Q_{\alpha\beta} k^\beta = 0$ *and, if f is any function,*

$$D_\alpha (f Q_{\beta\gamma}) = Q_{\beta\gamma} (\partial_\alpha f - h k_\alpha \partial_\beta f k^\beta + f k_\alpha \partial_\beta h k^\beta). \tag{7.8}$$

Hence when $f Q_{\beta\gamma}$ *is non-zero* $D_\alpha (f Q_{\beta\gamma}) = 0$ *if and only if the equation*

$$\partial_\alpha \log |f| + k_\alpha \partial_\beta h k^\beta = 0, \tag{7.9}$$

admits a non zero solution f.

By using the results above it is a straight-forward matter to obtain further results about particular classes of solutions of Einstein's equations (see further comment below).

Theorem 7.4 *1. In any dimension greater than two, plane-fronted metrics are Ricci flat, $R_{\alpha\beta} = 0$, if and only if the two equations*

$$\partial_\alpha \partial_\beta k^\beta h = F k_\alpha, \qquad \partial_\alpha h^\alpha = 2F$$

for some function F, are satisfied.

Hence it follows from the above results that when the latter two equations are satisfied, the Weyl conformal tensor satisfies the equation $C^\alpha_{\beta\gamma\delta} k^\delta = 0$ and, in the four dimensional Lorentzian case, is said to be of Petrov type N with $k^\alpha e_\alpha$ being a repeated principal null vector.

2. Using the local coordinate expressions further it follows immediately that plane-fronted metrics are Ricci flat if and only if the function h is given by

$$h = F(v)u + H(v, x^a), \tag{7.10}$$

where F is any function of v and the function H satisfies the (Laplace-type) equation

$$\eta^{ab} \partial_a \partial_b H = 0. \tag{7.11}$$

3. In the four dimensional Lorentzian case, writing $z = x + iy$, it follows from these equations that plane fronted metrics g is Ricci flat if and only if

$$H = K(v, z) + c.c., \qquad h = F(v)u + H(v, x^a). \tag{7.12}$$

for any complex K holomorphic in z.

As an immediate corollary follows the

Theorem 7.5 *pp wave metrics are Ricci flat if and only if*

$$\partial_\beta h k^\beta = 0, \qquad \eta^{ab} e_a e_b h = 0.$$

In the four-dimensional Lorentzian case, using the notation of Theorem 7.4, it therefore follows that Ricci flat *pp* waves *are determined by an arbitrary complex function K where*

$$h = H(v, x^a) = K(v, z) + c.c. \tag{7.13}$$

In summary

1. Functions $h = h(v)$ give flat metrics as do functions linear in the coordinates x^α.
2. It is clear from the form of the Ricci tensor given above that is is a straightforward matter to produce plane-fronted (or more particularly *pp*-wave) metrics which are solutions of the Einstein field equations with matter sources such as a massless scalar field ($R_{\alpha\beta} = \kappa e_\alpha(\varphi) e_\beta(\varphi)$, with say $\varphi = \varphi(v)$) and gauge fields (for example the null electromagnetic field with components $F_{\alpha\beta} = k_\alpha p_\beta - k_\beta p_\alpha$, with $k_\alpha p^\alpha = 0$, so that $F_{\alpha\beta}F^{\alpha\beta} = 0$ and the energy momentum tensor is (modulo sign conventions) given by $k_\alpha k_\beta (p_\gamma p^\gamma)$).
3. One final specialization made in the literature which is relevant here is from *pp* waves to *plane waves.*. This corresponds to making particular choices of solutions of the equations above. In the vacuum case , the solutions of Equation (7.11) given by

$$h = H_{ab}(v) x^a x^b, \qquad \eta^{ab} H_{ab} = 0,$$

are *vacuum plane wave solutions*. This terminology can be justified by computing the Weyl tensor and noting the planar symmetry. In addition, the notions of amplitude and polarisation can be introduced.

Appendix II

We group here some of the calculations used in deriving the formulae used in the differential calculus. The noncommutative version of the first frame which we use is given by

$$\theta^+ = \Lambda^{-1}(\phi du + h dv), \qquad \theta^- = \Lambda dv, \qquad \theta^a = dx^a.$$

It follows that

$$du = \phi^{-1}(\Lambda\theta^+ - h\Lambda^{-1}\theta^-), \qquad dv = \Lambda^{-1}\theta^-$$

and that the differentials are given by

$$d\theta^+ = e_\gamma(\Lambda^{-1}\phi)\phi^{-1}\Lambda\theta^\gamma\theta^+ - e_\gamma(\Lambda^{-1}\phi)\phi^{-1}h\Lambda^{-1}\theta^\gamma\theta^-$$

$$+ e_\gamma(\Lambda^{-1}h)\Lambda^{-1}\theta^\gamma\theta^-,$$

$$d\theta^- = e_\gamma\Lambda\Lambda^{-1}\theta^\gamma\theta^-.$$

We expand

$$\Lambda = 1 + \omega, \qquad \phi = 1 + f$$

and retain only terms first order in ω and f. This yields

$$-\frac{1}{2}C^+{}_{\alpha\beta} = e_\gamma(f - \omega)P^{\gamma +}{}_{\alpha\beta} -$$

$$(e_\gamma fh - e_\gamma h + e_\gamma h\omega + \omega e_\gamma h)P^{\gamma -}{}_{\alpha\beta},$$

$$-\frac{1}{2}C^-{}_{\alpha\beta} = e_\gamma\omega P^{\gamma -}{}_{\alpha\beta}.$$

The expression (4.4) follows from this. If we can set $Q^{--}{}_{\alpha\beta} = 0$ then we have

$$C^+{}_{\alpha\beta} = -e_{[\alpha}(f - \omega)\delta^+_{\beta]} - 2i\bar{k}e_+(f - \omega)Q^{++}{}_{\alpha\beta} -$$

$$-2i\bar{k}e_-(f - \omega)Q^{-+}{}_{\alpha\beta}$$

$$+ (e_{[\alpha}fh - e_{[\alpha}h + e_{[\alpha}h\omega + \omega e_{[\alpha}h)\delta^-_{\beta]}$$

$$+ 2i\bar{k}(e_+fh - e_+h + e_+h\omega + \omega e_+h)Q^{+-}{}_{\alpha\beta},$$

$$C^-{}_{\alpha\beta} = -e_{[\alpha}\omega\delta^-_{\beta]} - 2i\bar{k}e_+\omega Q^{+-}{}_{\alpha\beta}.$$

If we set $\phi = 1$ then for general Λ we obtain, with $\omega_\gamma = \Lambda^{-1} e_\gamma \Lambda$,

$$-\frac{1}{2} C^+{}_{\alpha\beta} = -\omega_\gamma P^{\gamma +}{}_{\alpha\beta} + \Lambda^{-1} e_\gamma h \Lambda^{-1} P^{\gamma -}{}_{\alpha\beta},$$

$$-\frac{1}{2} C^-{}_{\alpha\beta} = \Lambda \omega_\gamma \Lambda^{-1} P^{\gamma -}{}_{\alpha\beta}.$$

It follows then that

$$C^+{}_{\alpha\beta} = -\Lambda^{-1} e_{[\alpha} h \Lambda^{-1} \delta^-_{\beta]} - 2i\hbar \Lambda^{-1} e_\gamma h \Lambda^{-1} Q^{\gamma -}{}_{\alpha\beta} +$$

$$+\omega_{[\alpha} \delta^+_{\beta]} + 2i\hbar \omega_\gamma Q^{\gamma +}{}_{\alpha\beta},$$

$$C^-{}_{\alpha\beta} = -\Lambda \omega_{[\alpha} \Lambda^{-1} \delta^+_{\beta]} - 2i\hbar \Lambda \omega_\gamma \Lambda^{-1} Q^{\gamma -}{}_{\alpha\beta}.$$

If we retain terms up to second-order in $i\hbar$ we can write

$$C^+{}_{\alpha\beta} = -\Lambda^{-1} e_{[\alpha} h \Lambda^{-1} \delta^-_{\beta]} - 2i\hbar e_\gamma h Q^{\gamma -}{}_{\alpha\beta} + \omega_{[\alpha} \delta^+_{\beta]}, \quad C^-{}_{\alpha\beta} = -\omega_{[\alpha} \delta^-_{\beta]}.$$

If we equate this time the two expressions for the structure elements we obtain

$$F^+{}_{+-} - 4i\hbar \lambda_\delta Q^{\delta +}{}_{+-} =$$

$$-\Lambda^{-1} e_+ h \Lambda^{-1} - 2i\hbar e_\delta h Q^{\delta -}{}_{+-} - \omega_-, \tag{7.1}$$

$$F^+{}_{+a} - 4i\hbar \lambda_\delta Q^{\delta -}{}_{+a} = -2i\hbar e_\delta h Q^{\delta -}{}_{+a} - \omega_a, \tag{7.2}$$

$$F^+{}_{-a} - 4i\hbar \lambda_\delta Q^{\delta -}{}_{-a} = \Lambda^{-1} e_a h \Lambda^{-1} - 2i\hbar e_\delta h Q^{\delta -}{}_{-a}, \tag{7.3}$$

$$F^+{}_{ab} - 4i\hbar \lambda_\delta Q^{\delta +}{}_{ab} = -2i\hbar e_\delta h Q^{\delta -}{}_{ab}, \tag{7.4}$$

$$F^-{}_{+-} - 4i\hbar \lambda_\delta Q^{\delta -}{}_{+-} = -\omega_+, \tag{7.5}$$

$$F^-{}_{+a} - 4i\hbar \lambda_\delta Q^{\delta -}{}_{+a} = 0, \tag{7.6}$$

$$F^-{}_{-a} - 4i\hbar \lambda_\delta Q^{\delta -}{}_{-a} = +\omega_a, \tag{7.7}$$

$$F^-{}_{ab} - 4i\hbar \lambda_\delta Q^{\delta -}{}_{ab} = 0. \tag{7.8}$$

According to the correspondence principle as expressed by (2.7) we introduce a frame θ^α which has the classical frame as a limit and is as 'near' to it in form as possible for all values of $\hbar$. We shall assume in this example that the noncommutative frame has the same functional form as its limit except for a possible conformal factor in the u-direction. When $h = 0$, we have noticed, the space-time is flat; the noncommutative version is also flat. We introduce the frame $dx^\mu = e^\mu_\alpha \theta^\alpha$ with

$$e^\mu_\alpha = \begin{pmatrix} \phi & -h & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

This defines also the differential of the generators of the algebra in terms of the basis of the module of 1-forms. It defines therefore the conjugacy relations

$$\begin{aligned} &[\lambda_+, u] = \phi^{-1}\Lambda, && [\lambda_+, v] = 0, \\ &[\lambda_-, u] = -\phi^{-1}h\Lambda^{-1}, && [\lambda_-, v] = \Lambda^{-1}, \end{aligned} \tag{7.9}$$

as well as

$$[\lambda_a, x^b] = \delta^b_a.$$

The order is important since *a priori* nothing commutes. We have here kept a subgroup of the $GL(4, \mathbb{R})$ which acts on the frame and split it into a product of a factor which would change the metric in the classical limit and a part which would not. We must therefore add the condition that

$$\lim_{k \to 0} \phi = 1$$

to keep contact with the classical *pp*-solution. We shall in a future publication discuss [31] the different actions of the two factors in the definition of a Seiberg-Witten map [32]. From (4.2) and the reality conditions it follows that

$$h^* = h, \qquad \phi^* = \phi.$$

The problem is to find an algebra with a set of 4 generators x^μ satisfying commutation rules as well as a Fourier transformation to a new set λ_α which satisfies the conditions (2.10) and (4.2). We use the second form of the correspondence principle, the map (2.18) and the metric-compatible connection form $\omega^\alpha{}_\beta$. In the classical case the geometry is completely determined (to within integration constants) by (3.2). We seek an analogous equation in the noncommutative case. This means that we must rewrite the noncommutative version of (3.2) in terms of the momenta and compare it with the Equation (2.10).

We start from Equation (2.10) and impose the condition that the torsion vanish, which leads to the relation (2.11). According to the correspondence principle the form of the $C^\alpha{}_{\beta\gamma}$ as functionals of h are determined in the classical limit. A stronger condition is to assume that they have the same functional form when $\hbar \neq 0$. The non-vanishing structure elements are given then by

$$
\begin{aligned}
C^+{}_{\alpha\beta} &= -e_{[\alpha}(f-\omega)\delta^+_{\beta]} - 2i\hbar e_\gamma(f-\omega)Q^{\gamma+}{}_{\alpha\beta} \\
&\quad + (e_{[\alpha}fh - e_{[\alpha}h + e_{[\alpha}h\omega + \omega e_{[\alpha}h)\delta^-_{\beta]} \\
&\quad + 2i\hbar(e_\gamma fh - e_\gamma h + e_\gamma h\omega + \omega e_\gamma h)Q^{\gamma-}{}_{\alpha\beta}, \quad (7.10)
\end{aligned}
$$

$$
C^-{}_{\alpha\beta} = -e_{[\alpha}\omega\delta^+_{\beta]} - 2i\hbar e_\gamma \omega Q^{\gamma-}{}_{\alpha\beta}. \tag{7.11}
$$

The linear-connection coefficients are defined in terms of the structure elements by the formula (2.16), which can be written in the form

$$
\omega^\alpha{}_{[\beta\gamma]} = C^\alpha{}_{\beta\gamma} - i\hbar\omega^\alpha{}_{(\eta\zeta)}Q^{\eta\zeta}{}_{\beta\gamma}.
$$

The cases which we shall consider are such however that the second term vanishes and so we have the usual formula for the Levi-Civita connection. The associated covariant derivative is defined by the actions $D\theta^\alpha = -\omega^\alpha{}_\beta\theta^\beta$ with the right-hand side defined using (2.15).

¿From the quantization condition we deduce that the expression for $C^\alpha{}_{\beta\gamma}$ as given by (4.4) must be identical to that given in the Equations (2.4):

$$
\begin{aligned}
F^{\alpha}{}_{\beta\gamma} - 4i\kappa\lambda_\delta Q^{\alpha\delta}{}_{\beta\gamma} = &-2e_\delta(\Lambda^{-1}\phi)\phi^{-1}\Lambda P^{\delta+}{}_{\beta\gamma}\delta^\alpha_+ \\
&- 2\Lambda^{-1}\phi e_\delta(\phi^{-1}h)\Lambda^{-1}P^{\delta-}{}_{\beta\gamma}\delta^\alpha_+ \\
&- 2e_\delta\Lambda\Lambda^{-1}P^{\delta-}{}_{\beta\gamma}\delta^\alpha_-.
\end{aligned} \tag{7.12}
$$

The Jacobi identities (2.13) with two momentum and one position operator yield the constraints

$$
\begin{aligned}
&[G_{\alpha\beta}, u] = \phi^{-1}(\Lambda C^+{}_{\alpha\beta} - h\Lambda^{-1}C^-{}_{\alpha\beta}), \\
&[G_{\alpha\beta}, v] = \Lambda^{-1}C^-{}_{\alpha\beta}, \\
&[G_{\alpha\beta}, x^a] = 0.
\end{aligned}
$$

¿From the last of these we conclude that

$$G_{\alpha\beta} = G_{\alpha\beta}(\lambda_+, \lambda_-).$$

We must impose the 'Bohr-Ehrenfest' condition

$$\lim_{\kappa\to 0} h = \tilde{h} \tag{7.13}$$

to fix the classical limit. Equations (2.10) should be read as an equation for h in terms of the commutator $[\lambda_a, \lambda_+]$ if one believes that the noncommutative structure determines the gravitational field. If they are read from right to left then they are equations for the structure of the algebra in terms of the gravitational field.

If one wishes to describe the algebra in terms of the position generators one needs to have the values of the commutators, that is, of the $J^{\mu\nu}$. A simple calculation yields

$$e_\alpha J^{\mu\nu} = \delta^{[\mu}_\lambda \delta^{\nu]}_+ \delta^-_\alpha [x^\lambda, h].$$

All components of $J^{\alpha\beta}$ are constant except possibly for J^{-+}, which must satisfy the equation

$$e_- J^{-+} = -\frac{1}{i\kappa}[v, h] = -J^{-a}e_a h = -\omega^+{}_{-a}J^{-a} = -\omega^+{}_{\alpha\beta}\theta^{\alpha\beta}.$$

One can write this using the covariant derivative introduced in Section 2. The additional terms due to 'twisting' vanish in the present case and one obtains the expression

$$D_\mu J^{\mu\beta} = 0.$$

This will be the case if the connection is symplectic [21, 33].

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